Nonvanishing of central values of L-functions for modular forms on type IV symmetric domain

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1 Introduction (Elliptic modular case)

For an even positive integer $l \in 2\mathbb{Z}_{>0}$, let $S_l(\mathbf{SL}_2(\mathbb{Z}))$ denote the set of all the holomorphic cusp forms on $\mathbf{SL}_2(\mathbb{Z})$ of weight l; as usual, the $\mathbb{C}$-vector space $S_l(\mathbf{SL}_2(\mathbb{Z}))$ is endowed with the Petersson inner product defined by

$$\langle \phi, \phi_1 \rangle := \int_{\mathbf{SL}_2(\mathbb{Z})\backslash \mathfrak{h}_1} \phi(\tau)\, \overline{\phi_1(\tau)} (\mathrm{Im}\tau)^l \, \mathrm{d}\mu_{\mathfrak{h}_1}(\tau)$$

for ϕ, $\phi_1 \in S_l(\mathbf{SL}_2(\mathbb{Z}))$, where $\mathrm{d}\mu_{\mathfrak{h}_1}(\tau) = y^{-2}\,\mathrm{d}x\mathrm{d}y$ is the volume element associated with the Poincaré metric of the upper-half plane $\mathfrak{h}_1 := \{\tau = x + \mathrm{i}y \mid x \in \mathbb{R},\ y \in \mathbb{R}_{>0}\}$. (The imaginary unit of $\mathbb{C}$ is denoted by i.) It is well-known that $S_l(\mathbf{SL}_2(\mathbb{Z}))$ is a finite dimensional $\mathbb{C}$-vector space whose dimension d_l is numerically computable by an explicit dimension formula; in this article, only its asymptotic behavior $d_l = \frac{l}{12} + o(1)$, $l \to \infty$ is of relevance (if any). The important arithmetic information of modular form $\phi \in S_l(\mathbf{SL}_2(\mathbb{Z}))$ is encoded in its Fourier coefficients $\{a_\phi(n) \mid n \in \mathbb{Z}_{>0}\}$, which fits in the q-expansion:

$$\phi(\tau) = \sum_{n=1}^{\infty} a_\phi(n)\, q^n, \quad q := e^{2\pi \mathrm{i}\tau}.$$

The Hecke operators $T(n)\,(n \in \mathbb{Z}_{>0})$ on $S_l(\mathbf{SL}_2(\mathbb{Z}))$ defined as

$$[T(n)\phi]\,(\tau) = n^{l-1} \sum_{ad=n, 0 \leqslant b < d} \phi\left(\frac{a\tau + b}{d}\right) d^{-l}, \quad \phi \in S_l(\mathbf{SL}_2(\mathbb{Z})),$$

form a commuting family of self-adjoint operators on the finite dimensional Hilbert space $S_l(\mathbf{SL}_2(\mathbb{Z}))$. Thus, by linear algebra, we can find an orthogonal basis $\mathscr{F}_l$, which diagonalizes the operators $T(n)\,(n \in \mathbb{Z}_{>0})$ simultaneously. By examining the action of the Hecke operators on ϕ in terms of the Fourier coefficients $a_\phi(n)$, we see that $a_\phi(1) \neq 0$ and the eigenvalue of $T(n)$ on $\phi \in \mathscr{F}_l$ is $a_\phi(n)/a_\phi(1)$. Thus, we can choose $\mathscr{F}_l$ in such a way that $a_\phi(1) = 1$ for all $\phi \in \mathscr{F}_l$; if this condition is met, $\mathscr{F}_l$ is said to be Hecke normalized. Set

$$A_\phi(n) := \left(\frac{(4\pi)^{-(l-1)}\,\Gamma(l)}{\langle \phi, \phi \rangle}\right)^{1/2} n^{(1-l)/2} a_\phi(n)$$

for any non zero element $\phi \in \mathscr{F}_l$; we could say that the numbers $A_\phi(n)$ resembles to the Dirichlet characters in that they pssesses the following two properties:

(i) (Asymptotic orthogonality) For any $m, n \in \mathbb{Z}_{>0}$,

$$\frac{1}{l} \sum_{\phi \in \mathscr{F}_l} A_\phi(n)\, \overline{A_\phi(m)} = \delta_{m,n} + O_\epsilon\left(\tfrac{(mn)^{1/4+\epsilon}}{\sqrt{l}}\right) \quad (l \to \infty).$$

(ii) (Asymptotic boundedness) For any $\epsilon > 0$,

$$A_\phi(n) \ll_\epsilon (l\,n)^\epsilon, \quad \phi \in \mathscr{F}_l,\ n \in \mathbb{Z}_{>0},$$
$$l^{-\epsilon} \ll_\varepsilon A_\phi(1) \ll_\epsilon l^\epsilon, \quad \phi \in \mathscr{F}_l.$$

Property (i) is a consequence of Petersson's formula, which evaluates the quantity on the left-hand side of the equality in terms of the Bessel function and the Kloosterman sum. Property (ii) follows from the Rankin-Selberg formula, which identifies $\|f\|^2\,(4\pi)^{-(l-1)}\,\Gamma(l)^{-1}$ with a positive constant multiple of $\mathrm{Res}_{s=l}L(\phi\times\bar{\phi},s)$; we then invoke Deligne's estimate $a_\phi(n) = O_\epsilon(n^{(l-1)/2+\epsilon})$ and the estimate

$$l^{-\epsilon} \ll_\varepsilon \mathrm{Res}_{s=l}L(\phi\times\bar{\phi},s) \ll_\epsilon l^\epsilon, \quad \phi\in\mathscr{F}_l$$

due to Hoffstein-Lockhart and Iwaniec.

The Hecke's L-function of the Hecke eigenform $\phi \in \mathscr{F}_l$ is initially defined in terms of Dirichlet series of its Fourier coefficients or the Euler product:

$$L(\phi,s) := \sum_{n=1}^{\infty}\frac{a_\phi(n)}{n^s} = \prod_{p:\text{primes}}(1-a_\phi(p)\,p^{-s}+p^{-2s+2l-1})^{-1},$$

which is absolutely convergent on $\mathrm{Re}(s) \gg 0$. Owing to the integral representation

$$\Lambda(\phi,s) := \Gamma_{\mathbb{C}}(s)\,L(\phi,s) = \int_0^\infty \phi(\mathrm{i}y)y^s\,\frac{\mathrm{d}y}{y} \quad (\mathrm{Re}(s)\gg 0)$$

with $\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s)$, the completed L-function $\Lambda(\phi,s)$ has a holomorphic continuation to $\mathbb{C}$ satisfying the functional equation

$$\Lambda(\phi,l-s) = (-1)^{l/2}\,\Lambda(\phi,s).$$

In particular, $L(\phi,l/2) = 0$ unless $l/2$ is even. Suppose $l/2$ is even. The central value $L(\phi,l/2)$ is of some arithmetic interest. Using Petersson's formula and the approximate functional equation (Appr FE), one can prove the asymptotic formula

$$\spadesuit:\quad \frac{1}{l}\sum_{\phi\in\mathscr{F}_l} L\,(\phi,l/2)\;|A_\phi(1)|^2 \sim 1 \quad (l\to+\infty)$$

for their "harmonic" average[1], which yields nonvanishing L-values in large weights. Indeed, $\spadesuit$ also can be deduced from an exact formula of the average proved in [11], in which neither Petersson's formula nor Apprx FE is used. Since $l^{-\epsilon} \ll_\epsilon A_\phi(1) \ll_\epsilon l^\epsilon$ $(\forall\epsilon>0)$ and $\#(\mathscr{F}_l) \asymp l$, the asymptotic formula is consistent with the Lindelöf hypothesis

$$(\forall\epsilon>0)\quad L(\phi,l/2) = O_\epsilon(l^\epsilon), \quad \phi\in\mathscr{F}_l$$

in weight aspect, which is a consequence of the generalized Riemanian hypothesis for $L(\phi,s)$. The convexity bound $L(\phi,l/2) = O_\epsilon(l^{\frac{1}{2}+\varepsilon})$ for $\phi\in\mathscr{F}_l$ is proved by the functional equation and Stirling's formula; any bound by $O_\epsilon(l^{\theta+\epsilon})$ with the exponent $\theta\in(0,1/2)$ is called a subconvexity bound. Given the Lindelöf hypothesis is still far out of reach, to persue a smaller subconvexity exponent θ is a major business in this context. For this, a common approach is to study the asymptotic for the higher harmonic moments

$$\frac{1}{l}\sum_{\phi\in\mathscr{F}_l} M(\phi)L\,(\phi,l/2)^n\,|A_\phi(1)|^2 \quad (n=2,3,\dots)$$

[1] $|A_\phi(1)|^2$ coincides with so called the harmonic weight.

with suitably designed mollifier $M(\phi)$, whose proof, however, is considerably hard compared to the proof of ♠. There is another direction of generalization for ♠ if one only wants to have nonvanishing central L-values but for more general automorphic L-functions. Namely, regarding $\mathbf{PGL}(2) \cong \mathrm{SO}(2,1)$ as the first layer of the "tower" $\mathrm{SO}(2,m)$ $(m = 1, 2, \dots)$, we may seek a formula analogous to ♠ for higher degree Euler products of modular forms on $\mathrm{SO}(2,m)$. This problem was first proposed by the author's talk in 6-th Fukuoka Number Theory Symposium and a partial result was reported there. The author thanks the organizers of the Fukuoka Number Theory Symposium of this year for giving him an occasion to deliver a follow-up talk on this topic. In this write-up, we state an $\mathrm{SO}(2,m)$ counterpart of the formula ♠ in a complete and refined form omitting all proofs; for full account, we refer to [22] and [23].

2 Main result I (Siegel modular case)

For details, we refer to [22]. The symplectic group with similitude is defined by $\mathbf{GSp}_2 := \{g \in \mathbf{GL}_4 \mid {}^t g \left[\begin{smallmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{smallmatrix}\right] g = \nu(g) \left[\begin{smallmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{smallmatrix}\right]\}$ with $\nu(g) \in \mathbf{GL}_1$ the similitude norm of g; $\mathbf{Sp}_2$ is the kernel of the rational characer ν. Let $\mathfrak{h}_2 := \{Z = X + \mathrm{i}Y \in \mathrm{M}_2(\mathbb{C}) \mid {}^t Z = Z,\, Y \gg 0\}$ be the Siegel upper-half space. The space $S_l(\mathbf{Sp}_2(\mathbb{Z}))$ of holomorphic Siegel cusp forms on $\mathbf{Sp}_2(\mathbb{Z})$ of weight $l \geqslant \mathbb{Z}_{>0}$ is a finite dimensional $\mathbb{C}$-vector space; $\dim_{\mathbb{C}} S_l(\mathbf{Sp}_2(\mathbb{Z}))$ is explicitly known by Igusa's dimension formula, which tells us that $\dim_{\mathbb{C}} S_l(\mathbf{Sp}_2(\mathbb{Z})) \asymp l^3$ and that $S_l(\mathbf{Sp}_2(\mathbb{Z})) = \{0\}$ unless $l \geqslant 10$. We endow $S_l(\mathbf{Sp}_2(\mathbb{Z}))$ with the inner product defined by

$$\langle \Phi | \Phi_1 \rangle = \int_{\mathbf{Sp}_2(\mathbb{Z}) \backslash \mathfrak{h}_2} \Phi(Z) \overline{\Phi_1(Z)} \, (\det \mathrm{Im}(Z))^l \mathrm{d}\mu_{\mathfrak{h}_2}(Z), \quad \Phi, \Phi_1 \in S_l(\mathbf{Sp}_2(\mathbb{Z})),$$

where $\mathrm{d}\mu_{\mathfrak{h}_2}(Z) = \det(Y)^{-3} \, \mathrm{d}X \, \mathrm{d}Y$ is the invariant volume element on $\mathfrak{h}_2$. Let $\Phi(Z)$ be a non-zero element of $S_l(\mathbf{Sp}_2(\mathbb{Z}))$ with $l \in \mathbb{Z}_{\geqslant 10}$ and

$$\Phi(Z) = \sum_{T \in \mathcal{Q}^+} a_\Phi(T) \, \exp(2\pi \mathrm{i} \mathrm{tr}(TZ)), \quad Z \in \mathfrak{h}_2$$

its Fourier expansion, where $\{a_\Phi(T) \mid T \in \mathcal{Q}^+\}$ is the set of Fourier coefficients of Φ, which is indexed by $\mathcal{Q}^+$, the set formed by all the positive definite elements in

$$\mathcal{Q} := \left\{ T = \begin{bmatrix} b & a/2 \\ a/2 & c \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

The unimodular group $\mathbf{SL}_2(\mathbb{Z})$ acts on the $\mathbb{Z}$-lattice $\mathcal{Q}$ as

$$\mathcal{Q} \times \mathbf{SL}_2(\mathbb{Z}) \ni (T, \delta) \longmapsto \delta T {}^t\delta \in \mathcal{Q}.$$

From the modularity of $\Phi(Z)$, one can obtain the modular invariance of the Fourier coefficients, i.e.,

$$a_\Phi(\delta T {}^t\delta) = a_\Phi(T), \quad \delta \in \mathbf{SL}_2(\mathbb{Z}),\ T \in \mathcal{Q}^+.$$

Let $D < 0$ be a negative fundamental discriminant. Then the set

$$\mathcal{Q}^+_{\mathrm{prim}}(D) := \left\{ \begin{bmatrix} b & \frac{a}{2} \\ \frac{a}{2} & c \end{bmatrix} \in \mathcal{Q}^+ \;\middle|\; a^2 - 4bc = D,\ (a, b, c) = 1 \right\}$$

is preserved by the action of $\mathbf{SL}_2(\mathbb{Z})$, and the orbit space $\mathbf{SL}_2(\mathbb{Z}) \backslash \mathcal{Q}^+_{\mathrm{prim}}(D)$ is in a natural bijective correspondence with the ideal class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$, which is a finite abelian group; the ideal class corresponding to the $\mathbf{SL}_2(\mathbb{Z})$-orbit of $T \in \mathcal{Q}^+_{\rm prim}(D)$ is denoted by $[T]$. Let χ be a character of $\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$. Define

$$R(\Phi, D, \chi) := \sum_{T \in \mathbf{SL}_2(\mathbb{Z})\backslash \mathcal{Q}^+_{\rm prim}(D)} a_\Phi(T)\, \chi([T]).$$

Set

$$\omega^{\Phi}_{l,D,\chi} := c_{l,D}\, \frac{|R(\Phi, D, \chi^{-1})|^2}{\langle \Phi | \Phi \rangle} \times \begin{cases} 1 & (\chi^2 = \mathbf{1}), \\ 2 & (\chi^2 \neq \mathbf{1}), \end{cases}$$

where

$$c_{l,D} := \frac{\sqrt{\pi}}{4}(4\pi)^{3-2l}\Gamma\left(l - \frac{3}{2}\right)\Gamma(l-2) \times \left(\frac{|D|}{4}\right)^{\frac{3}{2}-l} \frac{4}{w_D\, h_D}$$

with $h_D := \#(\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$ and $w_D := \#(\mathbb{Q}(\sqrt{D})^\times_{\rm tor})$.

Next let us recall some known facts on the spinor L-function, which can be associated to our $\Phi(Z)$ only when it is a joint eigenfunction of all the Hecke operators. Fix such a Φ for a while. Let $\mathbb{A}$ denote the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_{\mathbf{f}}$ the subring of finite adels. Since $\mathbf{GSp}_2(\mathbb{A}) = \mathbf{GSp}_2(\mathbb{Q})\mathbf{GSp}_2(\mathbb{R})^0\mathbf{GSp}_2(\widehat{\mathbb{Z}})$, from $\Phi(Z)$, we can form a function $\tilde{\Phi}(g)$ on the adelization $\mathbf{GSp}_2(\mathbb{A})$ in such a way that

$$\tilde{\Phi}(\gamma g_\infty g_{\mathbf{f}}) = \nu(g_\infty)^{l/2} \det(CZ + D)^{-l}\Phi((A\mathsf{i} + D)(C\mathsf{i} + D)^{-1}),$$
$$\gamma \in \mathbf{GSp}_2(\mathbb{Q}),\ g_\infty = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathbf{GSp}_2(\mathbb{R})^0,\ g_{\mathbf{f}} \in \mathbf{GSp}_2(\widehat{\mathbb{Z}}).$$

Note that $\widetilde{\Phi}$ is invariant by the action of the center of $\mathbf{GSp}_2(\mathbb{A})$. Let π_Φ denote the automorphic cuspdidal representation of $\mathbf{GSp}_2(\mathbb{A})$ generated by the $\widetilde{\Phi}$. It is known that π_Φ is irreducible; as such, it is decomposed to a restricted tensor product to irreducible smooth representations $\pi_{\Phi,p}$ of $\mathrm{GSp}_2(\mathbb{Q}_p)$ for $p < \infty$ and a holomorphic discrete series representation $\pi_{\Phi,\infty}$ of weight l of $\mathbf{GSp}_2(\mathbb{R})$, i.e., $\pi_\Phi \cong \bigotimes_v \pi_{\Phi,v}$. Note $l \geqslant 10$. Let p be a prime; since $\widetilde{\Phi}$ is right $\mathbf{GSp}_2(\mathbb{Z}_p)$-invariant, the representation $\pi_{\Phi,p}$ contains a non-zero $\mathbf{GSp}_2(\mathbb{Z}_p)$-fixed vectors, i.e., $\pi_{\Phi,p}$ is an unramified representation. The unramified representations are parametrized by Satake parameters, which will be recalled next briefly. Let $\mathbf{B}$ be the Borel subgroup of $\mathbf{GSp}_2$, which consists of all the matrices of the form

$$\left[\begin{smallmatrix} A & 0 \\ 0 & \lambda\, {}^tA^{-1} \end{smallmatrix}\right] \left[\begin{smallmatrix} 1_2 & B \\ 0 & 1_2 \end{smallmatrix}\right], \quad A \in \mathbf{GL}_2,\ \lambda \in \mathbf{GL}_1,\ B = {}^tB \in \mathrm{Mat}_2 \tag{2.1}$$

with A being an upper-triangular unipotent matrix of degree 2. The set of elements of the form (2.1) with $A = 1_2$ (resp. A being diagonal and $B = 0$) is denoted by $\mathbf{U}$ (resp. by $\mathbf{T}$). Then $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$ and $\mathbf{B} = \mathbf{TU}$ is a Levi decomposition. The three involutions on the complex torus $(\mathbb{C}^\times)^2$

$$(a, b) \mapsto (b, a), \quad (a, b) \mapsto (a^{-1}, b), \quad (a, b) \mapsto (a, b^{-1})$$

generate a subgroup $W \subset \mathrm{Aut}((\mathbb{C}^\times)^2)$ isomorphic to the dihedral group of order 8, which is a realization of the Weyl group of the root system of type C_2. For any $y = (a, b)$ in the quotient set $(\mathbb{C}^\times)^2/W$, let $I_p(y)$ be the smooth representation of $\mathbf{GSp}_2(\mathbb{Q}_p)$ parabolically induced from a character $\chi_{\alpha,\beta}$ of the Borel subgroup $\mathbf{B}(\mathbb{Q}_p)$

$$\chi_{\alpha,\beta}(\mathrm{diag}(t_1, t_2, \lambda t_1^{-1}, \lambda t_2^{-1})n) = |t_1|_p^{-\alpha+\beta}|t_2|_p^{-\alpha-\beta}|\lambda|_p^{\alpha}, \quad (t_1, t_2, \lambda) \in (\mathbb{Q}_p^\times)^3,\ u \in \mathbf{U}(\mathbb{Q}_p),$$

where $\alpha := \mathrm{ord}_p(a)$ and $\beta := \mathrm{ord}_p(b)$. Note that $\chi_{-3,1}$ is the modulus character of $\mathbf{B}(\mathbb{Q}_p)$. Then it is known that the smooth representation $I_p(y)$ is of finite length and contains a unique unramified irreducible subquotient to be denoted by $\pi_p^{\rm ur}(y)$. Another realization of $\pi_p^{\rm ur}(y)$ is obtained as the smallest subspace of smooth functions on $\mathbf{GSp}_2(\mathbb{Q}_p)$ that is invariant by the right translations by $\mathbf{GSp}_2(\mathbb{Q}_p)$ and contains the spherical function $\omega_p(y)$: $\mathbf{GSp}_2(\mathbb{Q}_p) \to \mathbb{C}$ (a la Harish-Chandra and Satake) defined by

$$\omega_p(y;g) = \int_{\mathbf{GSp}_2(\mathbb{Z}_p)} \chi_{\alpha-3,\beta+1}(t(kg))\mathrm{d}k, \quad g \in \mathbf{GSp}_2(\mathbb{Q}_p),$$

where $t(g) \in \mathbf{T}(\mathbb{Q}_p)/\mathbf{T}(\mathbb{Z}_p)$ is uniquely defined by demanding $g \in t(g)\mathbf{U}(\mathbb{Q}_p)\mathbf{GSp}_2(\mathbb{Z}_p)$. The map $y \mapsto \pi_p^{\rm ur}(y)$ yields a bijection form $(\mathbb{C}^\times)^2/W$ onto the set of all the equivalence classes of smooth irreducible unramified representations of $\mathbf{GSp}_2(\mathbb{Q}_p)$. Let Y_p denote the set of $y = (a,b) \in \mathbb{C}^2$ such that $\omega_p(y)$ is positive type, or equivalently $\pi_p^{\rm ur}(y)$ is unitarizable. Set $[Y_p] := Y_p/W$. Since the local representation $\pi_{\Phi,p}$ is irreducible, unramified and unitarizable, there exists a unique point $y_p(\Phi) := (a_p, b_p) \in [Y_p]$, refereed to as the Satake parameter of Φ at p, such that $\pi_{\Phi,p}$ is equivalent to $\pi_p^{\rm ur}(a_p, b_p)$. The spinor L-function attached to Φ (or to π_Φ) is initially defined by the Euler product of degree 4

$$L(s,\pi_\Phi) := \prod_{p<\infty} (1-a_pp^{-s})^{-1}(1-b_pp^{-s})^{-1}(1-a_p^{-1}p^{-s})^{-1}(1-b_p^{-1}p^{-s})^{-1}, \quad \mathrm{Re}(s) > 5/2,$$

which is known to be absolutely convergent on the half-plane $\mathrm{Re}(s) > 5/2$ due to the unitarity of π_Φ. The completed L-function for $L(s,\pi_\Phi)$ is defined as

$$\Lambda(s,\pi_\Phi) = \Gamma_{\mathbb{C}}(s+1/2)\Gamma_{\mathbb{C}}(s+l-3/2) \times L(s,\pi_\Phi).$$

The basic properties of the spinor L-functions are listed below; (1) and (2) are due to Andrianov ([1], [2]), and (3) is proved independently by Oda ([16]) and by Evdokimov ([6]).

(1) The completed spinor L-function $\Lambda(s,\pi_\Phi)$ admits a meromorphic continuation to $\mathbb{C}$ admitting possible simple poles at $s = \frac{3}{2}, \frac{-1}{2}$, and satisfying the functional equation

$$\Lambda(1-s,\pi_\Phi) = (-1)^l\Lambda(s,\pi_\Phi).$$

(2) For any $T \in \mathcal{Q}^+_{\rm prim}(D)$, the Dirichlet series

$$Z_{\Phi,T}(s) := \sum_{n=1}^{\infty} \frac{a_\Phi(n\,T)}{n^s}$$

is absolutely convergent on $\mathrm{Re}(s) > l+1$ and equals to

$$L\left(s-l+\tfrac{3}{2},\pi_\Phi\right) \times \sum_{\lambda \in \widehat{\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))}} \frac{\lambda([T])^{-1}w_D^{-1}}{L(s-l+2,\lambda^{-1})}\, R(\Phi,D,\lambda).$$

(3) The function $\Lambda(s,\pi_\Phi)$ is entire unless l is even, in which case $s = \frac{3}{2}$ is a pole if and only if Φ is a Saito-Kurokawa lifting $\mathrm{SK}(f)$ from a Hecke-eigen cusp form $f \in S_{2l-2}(\mathbf{SL}_2(\mathbb{Z}))$; if this is the case,

$$L(s,\pi_\Phi) = \zeta\left(s-\tfrac{1}{2}\right)\zeta\left(s+\tfrac{1}{2}\right)L(s,f).$$

Remark. (1) Due to the sign of the functional equation, $L\left(\frac{1}{2},\pi_\Phi\right) = 0$ unless l is even.
(2) When $\Phi = \mathrm{SK}(f)$ with $f \in S_{2l-2}(\mathbf{SL}_2(\mathbb{Z}))$ being Hecke eigen form, then $L(\frac{1}{2},f) = 0$, so that

$$L\left(\tfrac{1}{2},\pi_{\mathrm{SK}(f)}\right) = \zeta(0)\,L'\left(\tfrac{1}{2},f\right).$$

Theorem 2.1. *Let D be a negative fundamental discriminant and χ a character of the ideal class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$. There exists a constant $C = C(D) > 1$, independent of χ, such that as $l \in 2\mathbb{Z}_{>0}$ grows to infinity,*

$$\sum_{\Phi \in \mathscr{F}_l} L(1/2, \pi_\Phi)\, \omega^{\Phi}_{l,D,\chi^{-1}} = 2\, P(l, D, \chi) + O(C^{-l})$$

with $\mathscr{F}_l$ being any orthonormal basis of $S_l(\mathbf{Sp}_2(\mathbb{Z}))$, and

$$P(l, D, \chi) := \begin{cases} L(1, \eta_D)\, (\psi(l-1) - \log(4\pi^2)) + L'(1, \eta_D) & (\chi = \mathbf{1}), \\ L(1, \chi) & (\chi \neq \mathbf{1}), \end{cases}$$

where $\psi(s) := \Gamma'(s)/\Gamma(s)$ is the di-gamma function and η_D is the Kronecker character, and $L(s, \chi)$ is the Hecke L-function of the idele class character of $\mathbb{Q}(\sqrt{D})^\times$ induced by χ.

To describe our second theorem, which is a refinement of Theorem 2.1, we need additional notation and definitions. Let $\mathscr{H}_p$ denote the Hecke algebra for $(\mathbf{GSp}_2(\mathbb{Q}_p), \mathbf{GSp}_2(\mathbb{Z}_p))$, i.e., the covolution algebra of all those $\mathbb{C}$-valued functions on $\mathbf{GSp}_2(\mathbb{Q}_p)$ that is bi-$\mathbf{GSp}_2(\mathbb{Z}_p)$-invariant and is compactly supported on $\mathbf{GSp}_2(\mathbb{Q}_p)$. The spherical Fourier transform of $f \in \mathscr{H}_p$ is defined by

$$\widehat{f}(y) := \int_{\mathbf{GSp}_2(\mathbb{Q}_p)} f(g) \omega_p(a^{-1}, b^{-1}; g)\, \mathrm{d}g, \quad y = (a, b) \in [Y_p]$$

with $\mathrm{d}g$ being the Haar measure on $\mathbf{GSp}_2(\mathbb{Q}_p)$ such that $\mathrm{vol}(\mathbf{GSp}_2(\mathbb{Z}_p)) = 1$. The Fourier inversion formula is known to be described as

$$f(g) = \int_{Y_p/W} \widehat{f}(y)\, \omega_p(y; g)\, \mathrm{d}\mu_p^{\mathrm{Pl}}(y), \quad f \in \mathscr{H}_p,\ g \in \mathbf{GSp}_2(\mathbb{Q}_p),$$

where $\mathrm{d}\mu_p^{\mathrm{Pl}}(y)$ is the Plancherel measure, which is a Radon measure on $[Y_p]$ supported on the tempered locus $[Y_p^0] := U(1)^2/W$. For $y = (a, b) \in (\mathbb{C}^\times)^2/W$ and an irreducible smooth unramified representation of $\mathbf{GL}_2(\mathbb{Q}_p)$ of Satake parameter $B_p = \mathrm{diag}(c, c^{-1}) \in \mathbf{GL}_2(\mathbb{C})$, set $A_p := \mathrm{diag}(a, b, a^{-1}, b^{-1}) \in \mathbf{Sp}_4(\mathbb{C})\ (= {}^L\mathbf{PGSp}_4)$ and

$$L(s, \pi_p^{\mathrm{ur}}(y) \times \sigma_p) := \det(1_8 - (A_p \otimes B_p)\, p^{-s})^{-1},$$
$$L(s, \pi_p^{\mathrm{ur}}(y); \mathrm{Ad}) := \det(1 - \rho_{10}(A_p) p^{-s})^{-1},$$

where ρ_{10} is the 10 dimensional representation of $\mathbf{Sp}_4(\mathbb{C})$ on its Lie algebra. For a class group character $\chi \in \widehat{\mathrm{Cl}}(\mathbb{Q}(\sqrt{D})$, viewing it as a character of the idele class group of $\mathbb{Q}(\sqrt{D})^\times$, we form its automorphic induction $\mathcal{AI}(\chi) = \bigotimes_v \mathcal{AI}_p(\chi)$ to $\mathbf{GL}_2(\mathbb{A})$, which is an irreducible automorphic representation of $\mathbf{GL}_2(\mathbb{A})$ such that its completed L-function (á la Jacquet-Langlands) coincides with the completion of $L(s, \chi)$. It is known that $\mathcal{AI}(\chi)$ is *not* cuspidal if and only of $\chi = \mathrm{N}_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}} \circ \chi_0$ for some Hecke character χ_0 of $\mathbb{Q}^\times$, in which case $\mathcal{A}I(\chi) = \chi_0 \boxplus \eta_D \chi_0$.

For each $l \in (2\mathbb{Z})_{\geqslant 10}$, let $S_l^{\#}(\mathbf{Sp}_2(\mathbb{Z})) := \mathrm{SK}(S_{2l-2}(\mathbf{SL}_2(\mathbb{Z}))$ be the image of the Saito-Kurokawa lifting, which is a linear subspace of $S_l(\mathbf{Sp}_2(\mathbb{Z}))$ stable under the action of Hecke operators. Fix an orthonormal basis $\mathscr{F}_l^{\#}$ of $S_l^{\#}(\mathbf{Sp}_2(\mathbb{Z}))$ and extend it to an orthonormal basis $\mathscr{F}_l$ of the total space $S_l(\mathbf{Sp}_2(\mathbb{Z}))$. Set $\mathscr{F}_l^{\flat} := \mathscr{F}_l - \mathscr{F}_l^{\#}$. For a finite set S of prime numbers, set $[Y_S] := \prod_{p \in S} [Y_p]$ and $[Y_S^0] := \prod_{p \in S} [Y_p^0]$.

Theorem 2.2. *Let $D < 0$ be a negative fundamental discriminant and S a finite set of odd prime numbers prime to D. For any $\alpha \in C([Y_S])$, as $l \in 2\mathbb{Z}_{>0}$ grows to infinity,*

$$\frac{1}{(\log l)^{\delta(\chi=\mathbf{1})}} \sum_{\Phi \in \mathscr{F}_l^\flat} \alpha(y_S(\Phi))\, L(1/2, \pi_\Phi)\, \omega^{\Phi}_{l,D,\chi^{-1}} \longrightarrow 2\mathbf{\Lambda}_S^{\chi}(\alpha) \times \begin{cases} L(1,\eta_D) & (\chi = \mathbf{1}), \\ L(1,\chi) & (\chi \neq \mathbf{1}), \end{cases}$$

where $y_S(\Phi) = \{y_p(\Phi)\}_{p\in S} \in [Y_S]$ is the set of Satake parameter of Φ, and $\mathbf{\Lambda}_S^\chi$ is a Radon measure on $[Y_S]$ supported on $[Y_S^0]$ such that

$$\mathbf{\Lambda}_S^\chi(\alpha) := \prod_{p\in S} \frac{\zeta_p(1)^{-1}\zeta_p(2)\zeta_p(4)}{L(1, \mathcal{AI}(\chi)_p)} \int_{[Y_p^0]} \alpha_p^{\mathrm{ur}}(x)\, \frac{L\left(\frac{1}{2}, \pi_p^{\mathrm{ur}}(y) \times \mathcal{AI}(\chi)_p\right) L\left(\frac{1}{2}, \pi_p^{\mathrm{ur}}(y)\right)}{L(1, \pi_p^{\mathrm{ur}}(y), \mathrm{Ad})}\, \mathrm{d}\mu_p^{\mathrm{Pl}}(y).$$

Corollary 2.3. *Let $D < 0$ be a negative fundamental discriminant, and S be a finite set of odd prime numbers prime to D. Let U be a measurable subset of $[Y_S^0]$ such that $\mu_S^{\mathrm{Pl}}(S) > 0$ and $\mu_S^{\mathrm{Pl}}(\partial U) = 0$. Then, there exists $l_0 \in \mathbb{Z}_{>0}$ with the following property: For any $l \in 2\mathbb{Z}_{>l_0}$ there exists $\Phi \in \mathscr{F}_l^\flat$ such that*

$$L(1/2, \pi_\Phi)\, L(1/2, \pi_\Phi \times \eta_D) > 0, \quad y_S(\Phi) \in U.$$

To prove these results, we invoke the following deep results on automorphic representations of $\mathbf{GSp}_2(\mathbb{A})$: Suppose $\Phi \in S_l(\mathbf{Sp}_2(\mathbb{Z}))$ is a joint Hecke eigenform which is *not* a Saito-Kurokawa lift from cusp forms on $\mathbf{SL}_2(\mathbb{Z})$. Then,

- (**The Ramanujan property of** Φ, conjectured by Kurokawa and proved by Weissauer [25]) The automorphic represenation π_Φ of $\mathbf{GSp}_2(\mathbb{A})$ is tempered, i.e., the Satake parameter $y_p(\Phi)$ lies in $[Y_p^0]$ for all $p < \infty$.
- (**The existence of transfer to** $\mathbf{GL}_4$ due to Pitale-Saha-Schmidt [17]) There exists an irreducible cuspidal automorphic representation Π of $\mathbf{GL}_4(\mathbb{A})$ of symplectic type such that $L(s,\Pi) = L(s,\pi_\Phi)$. As a consequence of this, invoking a result by Lapid, they deduce the non-negativity $L\left(\frac{1}{2}, \pi_\Phi\right) \geqslant 0$, which is what we need.
- (**Refined form of Boechrere's conjecture** due to Liu [12], furthur computed by Dickson-Pitale-Saha-Schmidt [5], and proved by Furusawa and Morimoto [7], [8]) For any fundamental discriminant $D < 0$ and for any character χ of $\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$,

$$\frac{|R(\Phi, D, \chi^{-1})|^2}{\|\Phi\|^2} = \frac{2^{4l-4}\pi^{2l+1}}{(2l-2)!} w_D^2 |D|^{l-1} \frac{L(1/2, \pi_\Phi \times \mathcal{AI}(\chi))}{L(1, \pi_\Phi, \mathrm{Ad})}.$$

2.1 Related works

- Kowalski-Saha-Tsimerman [10] obtained (among other things) an asymptotic formula of $\sum_{\Phi\in\mathscr{F}_l} L(s,\pi_\Phi)\, \omega^{\Phi}_{l,-4,1}$ with s being in the convergent range of the Euler product. Note that $\mathbb{Q}(\sqrt{-4}) = \mathbb{Q}(\mathrm{i})$ has class number 1 so that $\chi = 1$. Their tool is Kitaoka's formula (a Siegel modular analogue of Petersson's formula) and Sugano's formula of spherical Bessel functions over $\mathbb{Q}_p$.
- Blomer [3] proved the formula

$$\sum_{\Phi\in\mathscr{F}_l} L(1/2, \pi_\Phi)\, \omega^{\Phi}_{l,-4,1} = 2L(1,\eta_{-4})(\log l - \log(4\pi^2)) + L'(1,\eta_{-4}) + O(l^{-1}),$$

for average of central L-values. This result is consistent with our result for $D = -4$. More strikingly, an asymptotic formula of the second moment

$$\sum_{\Phi \in \mathscr{F}_l} |L(1/2, \pi_\Phi)|^2 \, \omega_{l,-4,1}^{\Phi} \quad (l \to \infty)$$

is elaborated.

- Waibel [24], employing the method by [3], proved a second moment formula for the central spinor L-values of Siegel cusp forms with fixed even weight and varying square free levels of Siegel parabolic type.

3 Main results II (for forms on type IV symmetric domain)

For details of this section, we refer to [23]. Given a $\mathbb{Z}$-module $\mathscr{M}$ and a commutative ring R, we use the notation $\mathscr{M}_R$ to denote the R-module $\mathscr{M} \otimes_{\mathbb{Z}} R$.

3.1 Notation and preliminaries

Let $m \in \mathbb{Z}_{\geqslant 3}$ be an *odd* integer. Let $\mathscr{L} \cong \mathbb{Z}^{m+2}$ be a lattice of signature $(2-, m+)$ (=free $\mathbb{Z}$-module endowed with a quadratic form $Q : \mathscr{L} \to \mathbb{Z}$ whose scalar extension to $\mathscr{L}_{\mathbb{Q}}$ is non-degenerate) satisfying

(A) $\mathscr{L}$ is maximal even-integral, i.e., $Q(\mathscr{L}) \subset 2\mathbb{Z}$ and $\mathscr{L}$ is maximal among all $\mathbb{Z}$-lattices in $\mathscr{L}_{\mathbb{Q}}$ with this property.

(B) $\mathscr{L}$ admits the orthogonal splitting

$$\mathscr{L} = \langle \varepsilon_1, \varepsilon_1' \rangle_{\mathbb{Z}} \oplus \mathscr{L}_1, \quad \mathscr{L}_1 = \langle \varepsilon_0, \varepsilon_0' \rangle_{\mathbb{Z}} \oplus \mathscr{L}_0$$

with $\langle \varepsilon_j, \varepsilon_j' \rangle_{\mathbb{Z}}$ hyperbolic planes. Thus, $\mathscr{L}_0$ is positive definite and maximal even-integral.

Let $(X, Y) := \frac{1}{2}(Q(X+Y) - Q(X) - Q(Y))$ $(X, Y \in \mathscr{L}_{\mathbb{Q}})$ be the associated bi-linear form; then $(\mathscr{L}, \mathscr{L}) \subset \mathbb{Z}$ due to (A). Let $O := O_{\mathscr{L}}$ be the orthogonal group (scheme over $\mathbb{Z}$) defined by $\mathscr{L}$. Set $\mathscr{D} := \mathscr{L}_{1,\mathbb{R}} + \mathsf{i}\Omega^-$ a complex domain in $\mathscr{L}_{1,\mathbb{C}} \cong \mathbb{C}^m$, where $\Omega^- := \{Y \in \Omega \mid (\xi, \varepsilon_0 - \varepsilon_0') < 0\}$ is the connected component containing the point $\mathfrak{z}_0/\mathsf{i} := \varepsilon_0 - \varepsilon_0'$ of the cone $\Omega := \{Y \in \mathscr{L}_{1,\mathbb{R}} \mid Q[Y] < 0\}$. The Lie group $G := O(\mathbb{R})^0 \cong \mathrm{SO}_0(2, m)$ acts on $\mathscr{D}$ holomorphically in the way described as follows. For $(\mathfrak{z}, g) \in \mathscr{D} \times G$, define $g\langle \mathfrak{z} \rangle \in \mathscr{D}$ and $J(g, \mathfrak{z}) \in \mathbb{C}^*$ by the relation

$$g\, P(\mathfrak{z}) = J(g, \mathfrak{z})\, P(g\langle \mathfrak{z} \rangle),$$

where

$$P(\mathfrak{z}) = (-2^{-1}Q[\mathfrak{z}])\, \varepsilon_1 + \mathfrak{z} + \varepsilon_1' \in \mathscr{L}_{\mathbb{C}} \cong \mathbb{C}^{m+2}.$$

Then $G \times \mathscr{D} \ni (g, z) \longmapsto g\langle \mathfrak{z} \rangle \in \mathscr{D}$ is the action of G on $\mathscr{D}$ such that $\mathfrak{z} \mapsto g\langle \mathfrak{z} \rangle$ is a holomorphic automorphism of $\mathscr{D}$. Actually, this action is extended to an action of the disconnected group $O(\mathbb{R})$ (with 4-connected component) on $\mathscr{L}_{1,\mathbb{R}} + \mathsf{i}\Omega$; let $O(\mathbb{Q})^+$ denote the element of $\gamma \in O(\mathbb{Q})$ which preserves the connected component $\mathscr{D}$ of $\mathscr{L}_{1,\mathbb{R}} + \mathsf{i}\Omega$. The function $J(g, \mathfrak{z})$ satisfies the automorphy condition $J(gg', \mathfrak{z}) = J(g, g'\langle \mathfrak{z} \rangle)\, J(g', \mathfrak{z})$ for all $g, g' \in G$ and $\mathfrak{z} \in \mathscr{D}$. The complex manifold $\mathscr{D}$ is endowed with the G-invariant Kähler 2-form $\omega_{\mathscr{D}}(\mathfrak{z}) := 2^{-1}\mathsf{i}\partial\bar{\partial}\, Q[\mathrm{Im}(\mathfrak{z})]$, which yields the Bergmann metric of $\mathscr{D}$. Let $\mathbf{K}_\infty := \mathrm{Stab}_G(\mathfrak{z}_0)$; then, $\mathbf{K}_\infty \cong \mathrm{SO}(2) \times \mathrm{SO}(m)$ is a maximal compact subgroup of G and we have a G-isomorphism

$$G/\mathbf{K}_\infty \cong \mathscr{D}, \quad g\mathbf{K}_\infty \mapsto g\langle \mathfrak{z}_0 \rangle.$$

For a prime p, let $\mathbf{K}_p^*$ denote the kernel of the natural group homomorphism $O(\mathbb{Z}_p) \longrightarrow \mathrm{Aut}(\mathscr{L}_{\mathbb{Z}_p}^\vee/\mathscr{L}_{\mathbb{Z}_p})$, where $\mathscr{L}^\vee := \{X \in \mathscr{L}_\mathbb{Q} \mid (\mathscr{L}, X) \subset \mathbb{Z}\}$ is the dual lattice of $\mathscr{L}$; $O(\mathbb{Z}_p) = \{g \in O(\mathbb{Q}_p) \mid g(\mathscr{L}_{\mathbb{Z}_p}) = \mathscr{L}_{\mathbb{Z}_p}\}$ is a maximal compact subgroup of $O(\mathbb{Q}_p)$ and $\mathbf{K}_p^*$ is an open subgroup $O(\mathbb{Z}_p)$. Set $\mathbf{K}_\mathbf{f}^* := \prod_p \mathbf{K}_p^*$. Let $l \in \mathbb{Z}_{>0}$. A function $F : \mathscr{D} \times O(\mathbb{A}_\mathbf{f}) \to \mathbb{C}$ is called a *holomorphic cusp form of weight* l if it satisfies the conditions:

(i) $F(\gamma\langle\mathfrak{z}\rangle, \gamma g_\mathbf{f} k) = J(\gamma, \mathfrak{z})^l F(g)$ for all $\gamma \in O(\mathbb{Q})^+$, $\mathfrak{z} \in \mathscr{D}$ and $g_\mathbf{f} \in O(\mathbb{A}_\mathbf{f})$, $k \in \mathbf{K}_\mathbf{f}^*$.

(ii) For any $g_\mathbf{f} \in O(\mathbb{A}_\mathbf{f})$, the function $\mathfrak{z} \to F(\mathfrak{z}, g_\mathbf{f})$ on $\mathscr{D}$ is holomorphic.

(iii) $|Q[\mathrm{Im}(\mathfrak{z})]|^{l/2} F(\mathfrak{z}, g_\mathbf{f})$ is bounded on $\mathscr{D} \times O(\mathbb{A}_\mathbf{f})$.

Let $\mathfrak{S}_l$ denote the space of all the holomorphic cusp forms of weight l; then $\mathfrak{S}_l$ is finite dimensional, and $\dim_\mathbb{C} \mathfrak{S}_l \asymp l^m$ $(l \to +\infty)$ by the Hirzebruch-Mumford proportionality principle. We endow $\mathfrak{S}_l$ with the inner product:

$$\langle F|F_1\rangle := \int_{O(\mathbb{Q})^+\backslash(\mathscr{D}\times O(\mathbb{A}_\mathbf{f}))} F(\mathfrak{z}, g_\mathbf{f})\overline{F_1(\mathfrak{z}, g_\mathbf{f})}\, \mathrm{d}\mu_\mathscr{D}(\mathfrak{z})\, \mathrm{d}g_\mathbf{f},$$

where $\mathrm{d}_\mathscr{D}\mu(\mathfrak{z})$ is the Kaehler volume element on $\mathscr{D}$ and $\mathrm{d}g_\mathbf{f}$ is a (unique) Haar measure such that $\mathrm{vol}(\mathbf{K}_\mathbf{f}^*) = 1$. Moreover, the space $\mathfrak{S}_l$ has a natural action of the Hecke algebra $\mathscr{H}_p^+$ for all p, where $\mathscr{H}_p^+$ is defined to be the center of the Hecke algebra of the pair $(O(\mathbb{Q}_p), \mathbf{K}_p^*)$. For $g_\mathbf{f} \in O(\mathbb{A}_\mathbf{f})$, there exists a $\mathbb{Z}$-lattice $\mathscr{L}_1(g_\mathbf{f}) \subset \mathscr{L}_{1,\mathbb{Q}}$ such that for any $F \in \mathfrak{S}_l$, the function $\mathfrak{z} \mapsto F(\mathfrak{z}, g_\mathbf{f})$ is given by the Fourier expansion

$$F(\mathfrak{z}, g_\mathbf{f}) = \sum_{\eta\in\mathscr{L}_1(g_\mathbf{f})\cap(-\Omega^-)} a_F(g_\mathbf{f};\, \eta) \exp(2\pi\mathrm{i}(\eta, \mathfrak{z})), \quad \mathfrak{z} \in \mathscr{D},$$

where $a_F(g_\mathbf{f};\, \eta) \in \mathbb{C}$ will be refereed to as the Fourier coefficients.

From now on, we *fix* $\xi \in \mathscr{L}_{1,\mathbb{Q}}$ such that

(a) (signature condition) $\xi \in \mathrm{i}\,\mathscr{D}$, or explicitly $Q[\xi] < 0$ and $(\xi, \mathfrak{z}_0/\mathrm{i}) < 0$.

(b) (primitivity) ξ is a primitive vector of the lattice $\mathscr{L}_1^\vee$.

(c) (maximality) $\mathscr{L}_1^\xi := \mathscr{L}_1 \cap \xi^\perp$ is a maximal even-lattice in the quadratic space $\xi^\perp$.

Now define $\mathbb{Q}$-algebraic subgroups of O as

$$O_1 := \mathrm{Stab}_O(\varepsilon_1, \varepsilon_1'), \quad O^\xi := \mathrm{Stab}_O(\xi), \quad O_1^\xi := O^\xi \cap O_1.$$

By the signature conditions, we have $O_1(\mathbb{R}) \cong O(1, m-1)$, $O^\xi(\mathbb{R}) \cong O(1, m)$ and $O_1^\xi(\mathbb{R}) \cong O(m-1)$. In particular, $O_1^\xi(\mathbb{R})$ is compact. Let $\mathbf{K}_1^{\xi,*} := \prod_p \mathbf{K}_{1,p}^{\xi_1*}$ with $\mathbf{K}_{1,p}^{\xi*} := \{u \in O_1^\xi(\mathbb{Z}_p) \mid u(X) - X \in \mathscr{L}_{1,\mathbb{Z}_p}^\xi\ (\forall X \in (\mathscr{L}_{1,\mathbb{Z}_p}^\xi)^\vee)\}$, and

$$f : O_1^\xi(\mathbb{Q})\backslash O_1^\xi(\mathbb{A}_\mathbf{f})/\mathbf{K}_1^{\xi*} \longrightarrow \mathbb{C}$$

be a joint eigenfunction of $(\mathscr{H}_{1,p}^\xi)^+$ for all p, where $(\mathscr{H}_{1,p}^\xi)+$ denote the center of the Hecke algebra of $(O_1^\xi(\mathbb{Q}_p), \mathbf{K}_{1,p}^{\xi*})$. For $F \in \mathfrak{S}_l$ with the Fourier coefficients $a_F(g_\mathbf{f}; \eta)$, define

$$a_F^f(\xi) := {\mu_\xi}^{-1} \sum_{j=1}^h \frac{f(u_j)}{e_\xi(j)}\, a_F(u_j;\, \xi),$$

where $u_j \in O_1^\xi(\mathbb{A}_\mathbf{f})$ $(1 \leqslant j \leqslant h)$ are such that

$$O_1^\xi(\mathbb{Q})\backslash O_1^\xi(\mathbb{A}_\mathbf{f})/\mathbf{K}_1^{\xi *} = \{\bar{u}_1, \dots, \bar{u}_h\} \quad \text{and}$$
$$e_\xi(j) = \#[O_1^\xi(\mathbb{Q}) \cap u_j\, \mathbf{K}_1^{\xi *}\, u_j^{-1}], \quad \mu_\xi = \sum_{j=1}^{h} e_\xi(j)^{-1}.$$

Next we introduce an Eisenstein series on $O^\xi(\mathbb{A})$. Let P^ξ be the maximal parabolic subgroup of O^ξ stabilizing the vector ε_1 up to constant. Then

$$P^\xi(\mathbb{A}) = \left\{ \left[\begin{smallmatrix} t & * & * \\ & h_0 & * \\ & & t^{-1} \end{smallmatrix}\right] \in O^\xi(\mathbb{A}) \mid t \in \mathbb{A}^\times,\, h_0 \in O_1^\xi(\mathbb{A}) \right\}.$$

Let $\mathbf{K}_p^{\xi *} := \mathbf{K}_p^* \cap O^\xi(\mathbb{Q}_p)$ for $p < \infty$, and $\mathbf{K}_\infty^\xi$ be a maximal compact subgroup of $O^\xi(\mathbb{R})$ stabilizing ξ up to constants; set $\mathbf{K}^{\xi *} := \prod_p \mathbf{K}_p^{\xi *}\, \mathbf{K}_\infty^\xi$. By means of the Iwasawa decomposition $O^\xi(\mathbb{A}) = P^\xi(\mathbb{A})\, \mathbf{K}^\xi$, we define a function $f^{(s)}$ on $O^\xi(\mathbb{A})$ as $f^{(s)}(h) := f(h_0)\, |t|_\mathbb{A}^{s+\frac{m-1}{2}}$ for $h = \left[\begin{smallmatrix} t & * & * \\ & h_0 & * \\ & & t^{-1} \end{smallmatrix}\right] \in P^\xi(\mathbb{A})$ and $k \in \mathbf{K}^{\xi *}$. Then the Eisenstein series relevant to our purpose is

$$E_{P^\xi}^{O^\xi}(f, s; h) := \sum_{\gamma \in P^\xi(\mathbb{Q})\backslash O^\xi(\mathbb{Q})} f^{(s)}(\gamma h), \quad h \in O^\xi(\mathbb{A}),$$

which is convergent on $\mathrm{Re}(s) > (m-1)/2$. By Murase-Sugano [13], the Euler product $L(f, s) = \prod_p L_p(f, s)\, (\mathrm{Re}(s) > (m-1)/2)$ is defined in such a way that the local p-factor $L_p(f, s)$ when $(\mathscr{L}_1^\xi)_{\mathbb{Z}_p}^\vee = (\mathscr{L}_1^\xi)_{\mathbb{Z}_p}$ coincides with the common definition á la Langlands; then it is proved that $L(f, s)$ has a meromorphic continuation to $\mathbb{C}$ in such a way that the completed L-function $\Lambda(f, s) := \Gamma_{\mathscr{L}_1^\xi}(s) \times L(f, s)$ with the gamma-factor being

$$\Gamma_{\mathscr{L}_1^\xi}(s) = \prod_{j=1}^{(m-1)/2} \Gamma_\mathbb{C}(s - j + (m-1)/2)\, \{\#((\mathscr{L}_1^\xi)^\vee/\mathscr{L}_1^\xi)\}^{s/2}$$

satisfies the functional equation $\Lambda(f, 1-s) = \Lambda(f, s)$ and admits possible poles only at $s = \frac{m-1}{2} - j\ (j \in [0, m-2])$; in particular, $\Lambda(f, s)$ has a possible simple pole at $s = 1$ when m is odd. From this result, they deduced the meromorphic continuation and the functional equation $\widehat{E_{P^\xi}^{O^\xi}}(f, -s; h) = \widehat{E_{P^\xi}^{O^\xi}}(f, s; h)$ for the normalized Eisenstein series

$$\widehat{E_{P^\xi}^{O^\xi}}(f, s; h) := \Lambda(f, -s)\, E_{P^\xi}^{O^\xi}(f, s; h). \tag{3.1}$$

3.1.1 Integral representation of L-function

Let $F \in \mathfrak{S}_l$ be a joint eigenfunction of the Hecke algebras $\mathscr{H}_p^+$ for all $p < \infty$. Let $L(F, s) = \prod_{p<\infty} L_p(F, s)$ be the Euler product defined by Murase-Sugano ([13]). Set

$$\Gamma_{\mathscr{L}, l}(s) := \Gamma_\mathbb{C}(s - m/2 + l) \prod_{j=1}^{(m-1)/2} \Gamma_\mathbb{C}(s + m/2 - j)\, \{2^{-1}\#(\mathscr{L}^\vee/\mathscr{L})\}^{s/2}$$

and $\Lambda(F, s) := \Gamma_{\mathscr{L}, l}(s)\, L(F, s)$, the completed L-function of F. Let F^O denote the function of $O(\mathbb{A})$ defined by $F^O(g_\infty g_\mathbf{f}) = J(g_\infty, \mathfrak{z}_0)^{-l} F(g_\infty\langle\mathfrak{z}_0\rangle, g_\mathbf{f})$ for $g_\infty \in O(\mathbb{R})$ and $g_\mathbf{f} \in O(\mathbb{A}_\mathbf{f})$.

The following identity is partly due to Andrianov ([1], [2]) and Sugano ([19], [20]) and is stated in this form in [15]:

$$\int_{O^\xi(\mathbb{Q})\backslash O^\xi(\mathbb{A})} \widehat{E_{P^\xi}^{O^\xi}}(f, s-1/2; h)\, F^O(h\, b_\infty^\xi)\, \mathrm{d}h = C_l^\xi\, a_F^f(\xi)\, \Lambda(F,s) \quad (\mathrm{Re}(s) \gg 0),$$

where $b_\infty^\xi \in O_1(\mathbb{R})$ is an element such that $b_\infty^\xi(\frac{\varepsilon_0'-\varepsilon_0}{\sqrt{2}}) = |Q(\xi)|^{-1/2}\xi$, and C_l^ξ is a positive constant which can be explicitly described once the normalization of Haar measure on $O^\xi(\mathbb{A})$ is fixed. Suppose $a_F^f(\xi) \neq 0$ for some ξ and f; then $\Lambda(F,s)$ has a meromorphic continuation to $\mathbb{C}$ satisfying the functional equation $\Lambda(F,1-s) = \Lambda(F,s)$ with possible poles only at $s = m/2 - j$ $(0 \leqslant j \leqslant m-1)$; in particular, $L(F,s)$ is regular at $s = 1/2$. When m is odd, the center of the functional equation $s = 1/2$ is a (unique) critical point of the L-function $L(F,s)$.

3.2 Statement of the main result

Let the notation and the assumptions be as before; in particular ξ satisfies three conditions (a), (b) and (c). Let $\mathscr{U}$ be an irreducible $O_1^\xi(\mathbb{A}_\mathbf{f})$-subrepresentation of $L^2(O_1^\xi(\mathbb{Q})\backslash O_1^\xi(\mathbb{A}_\mathbf{f}))$ such that the space of $\mathbf{K}_1^{\xi *}$-fixed vectors $\mathscr{U}(\mathbf{K}_1^{\xi *})$ in $\mathscr{U}$ is not zero.

Lemma 3.1. *Suppose* $2\xi \in \mathscr{L}_1$.
(1) *For each prime p, let* $\mathfrak{r}_p^\xi$ *be the reflection of* $\mathscr{L}_{1,\mathbb{Q}_p}$ *with respect to* ξ. *Then,*

$$(\mathfrak{r}_p^\xi)_{p<\infty} \in h_\mathbf{f}^\xi\, \mathbf{K}_1^* \text{ for some } h_\mathbf{f}^\xi \in O_1^\xi(\mathbb{A}_\mathbf{f}).$$

(2) *There exists an involutive operator* $\tau_\mathbf{f}^\xi$ *on* $\mathscr{U}(\mathbf{K}_1^{\xi *})$ *such that*

$$\tau_\mathbf{f}^\xi(f)(h) = f(h\, h_\mathbf{f}^\xi), \quad f \in \mathscr{U}(\mathbf{K}_1^{\xi *})$$

for any $h_\mathbf{f}^\xi \in O_1^\xi(\mathbb{A}_\mathbf{f})$ *as in* (1).
(3) *The involution* $\tau_\mathbf{f}^\xi$ *commutes with all the Hecke operators from* $(\mathscr{H}_{1,p}^\xi)^+$ $(p < \infty)$. *There exists an orthonormal basis* $\mathscr{B}(\mathscr{U}; \mathbf{K}_1^{\xi *})$ *of* $\mathscr{U}(\mathbf{K}_1^{\xi *})$ *which diagonalizes the action of* $\langle (\mathscr{H}_{i,p}^\xi)^+ \ (p<\infty), \tau_\mathbf{f}^\xi \rangle$.

Set $\mathscr{B}_\mathscr{U}^{(\pm 1)} := \{f \in \mathscr{B}(\mathscr{U}; \mathbf{K}_1^{\xi *}) \mid \tau_\mathbf{f}^\xi(f) = \pm f\}$. For $F \in \mathfrak{S}_l$ and $f \in \mathscr{U}(\mathbf{K}_1^{\xi *})$, we define[2]

$$A_F^f(\xi) := \frac{(4\pi\sqrt{-2Q[\xi]})^{-l+\frac{m}{2}}\, \Gamma(2l - \frac{m-1}{2})^{1/2}\, a_F^f(\xi)}{\|f\|\, \|F\|},$$

where

$$\|f\|^2 = \mu_\xi^{-1} \sum_{j=1}^h \frac{|f(u_j)|^2}{e_\xi(j)}, \quad \|F\|^2 = \int_{O(\mathbb{Q})^+\backslash(\mathscr{D}\times O(\mathbb{A}_\mathbf{f}))} |F(\mathfrak{z}, g_\mathbf{f})|^2\, \mathrm{d}\mu_\mathscr{D}(\mathfrak{z})\, \mathrm{d}g_\mathbf{f}.$$

Theorem 3.2. *Suppose* ξ *satisfies* $2\xi \in \mathscr{L}_1$ *as well as conditions* (a), (b) and (c). Let $\varepsilon \in \{\pm 1\}$ be such that $\#(\mathscr{B}_\mathscr{U}^\varepsilon) \neq \varnothing$. Then, there exists $C > 1$ such that, as $l \to \infty$ with $(-1)^l = \varepsilon$,

$$\frac{\widetilde{\Gamma}(l)}{l^m} \sum_{F\in\mathscr{F}_l} \frac{1}{\#(\mathscr{B}_\mathscr{U}^\varepsilon)} \sum_{f\in\mathscr{B}_\mathscr{U}^\varepsilon} L(F,1/2)\, |A_F^f(\xi)|^2 = B_\mathscr{L}(\xi)\, L(\mathscr{U},s)|_{s=1}^* + O(C^{-l}),$$

[2]The quantity $A_F^f(\xi)$ should be viewed as an analogue of $A_\phi(1)$ considered in §1.

where $L(\mathscr{U},s)|_{s=1}^{*}$ denotes the leading Laurent coefficient of $L(\mathscr{U},s)$ (:=$L(f,s)$ for any $f\in\mathscr{B}(\mathscr{U};\mathbf{K}^{*}_{\mathscr{L}_1^{\xi}})$) at $s=1$, where $\mathscr{F}_l$ is any orthonormal basis of $\mathfrak{S}_l$ consisting of Hecke eigen forms, and

$$\widetilde{\Gamma}(l):=\frac{l^m\,\Gamma(l-\frac{m}{2})\,\Gamma(l-m+1)}{\Gamma(l-\frac{m-1}{4})\,\Gamma(l-\frac{m-3}{4})},\quad B_{\mathscr{L}}(\xi):=16\,(2^{-1}\mathfrak{d}(\mathscr{L}))^{-\frac{1}{2}}\left(\tfrac{\pi}{4}\right)^{-\frac{m-1}{2}}.$$

Remark. We have $\widetilde{\Gamma}(l)=1+O(l^{-1})$ as $l\to\infty$. Note that $\#(\mathscr{F}_l)=\dim_{\mathbb{C}}(\mathfrak{S}_l)\asymp l^m$ and $m=\dim_{\mathbb{C}}\mathscr{D}$.

Let S be a finite set of prime numbers such that $p\in S$ is relatively prime to $\#(\mathscr{L}_1^{\vee}/\mathscr{L}_1)$ and $Q(\xi)$. For $p\in S$, choose a maximal set of isotropic vectors $(e_j)_{j=1}^{r_p}$ and $(e'_j)_{j=1}^{r_p}$ in $\mathscr{L}_{\mathbb{Z}_p}$ satisfying $(e_i,e'_j)=\delta_{ij}$ and $\mathscr{L}_{\mathbb{Z}_p}=\bigoplus_{j=1}^{r_p}(\mathbb{Z}_pe_j+\mathbb{Z}_pe'_j)\oplus\mathscr{M}$ (Witt decomposition) with $\mathscr{M}:=\{X\in\mathscr{L}_{\mathbb{Z}_p}\mid (X,e_j)=(X,e'_j)=0\,(\forall j\in[1,r_p]\}$. Let B_p be the Borel subgroup of $O(\mathbb{Q}_p)$ stabilizing the isotropic flag $\{F_j:=\langle e_1,\dots,e_j\rangle_{\mathbb{Q}_p}\mid j\in[1,r_p]\}$. Then, $O(\mathbb{Q}_p)=B_p\mathbf{K}_p$ (Iwasawa decomposition) holds. For $g\in O(\mathbb{Q}_p)$, a coset $b(g)\in B_p/B_p\cap\mathbf{K}_p$ is well-defined by the relation $g\in b(g)\mathbf{K}_p$. Let T_p be the maximal $\mathbb{Q}_p$-split torus of $O(\mathbb{Q}_p)$ such that there exist $\mathbb{Q}_p$-rational characters $\chi_j:T_p\to\mathbb{Q}_p^{\times}$ satisfying $t(e_j)=\chi_j(t)e_j$, $t(e'_j)=\chi_j(t)^{-1}e'_j$ for all $j\in[1,r_p]$ and $t(X)=X$ for all $X\in\mathscr{M}$. Since T_p is a Levi subgroup of B_p, each χ_j is viewed as a character of B_p by the natural surjection $B_p\to T_p$. Set $\mathfrak{X}_p:=(\mathbb{C}/2\pi(\log p)^{-1}\mathbb{Z})^{r_p}$; by identifying $\mathfrak{X}_p$ with the space of continuous characters of T_p trivial on $T_p\cap\mathbf{K}_p$, we have a natural action of the Weyl group W_p of $(T_p,O(\mathbb{Q}_p))$ on $\mathfrak{X}_p$. For $\nu=(\nu_j)_{j=1}^{r_p}\in\mathfrak{X}_p$, let $\omega_\nu:O(\mathbb{Q}_p)\to\mathbb{C}$ be the zonal spherical function of Satake parameter ν, which is defined by

$$\omega_\nu(g):=\int_{\mathbf{K}_p}\prod_{j=1}^{r_p}|\chi_j(b(g))|_p^{\nu_j+\rho_j}\,\mathrm{d}k,\quad g\in O(\mathbb{Q}_p).$$

By [18], the map $\nu\mapsto\omega_\nu$ yields a bijection from $\mathfrak{X}_p/W_p$ onto the set of zonal spherical functions on $O(\mathbb{Q}_p)$. Let $\mathfrak{X}_p^{0+}\subset\mathfrak{X}_p$ denote the locus of zonal spherical functions of positive type. For $\nu\in\mathfrak{X}_p^{0+}$, let $\pi^{O(\mathbb{Q}_p)}(\nu)$ denote the smooth spherical representation generated by the right-translations of the function ω_ν on $O(\mathbb{Q}_p)$; it is known that $\pi^{O(\mathbb{Q}_p)}(\nu)$ is irreducible and unitarizable ([4]). Let $\mathscr{H}_p$ denote the Hecke algebra for $(O(\mathbb{Q}_p),\mathbf{K}_p)$, which is the same as $\mathscr{H}_p^{+}$ due to $\mathscr{L}_{\mathbb{Z}_p}^{\vee}=\mathscr{L}_{\mathbb{Z}_p}$. For $\phi\in\mathscr{H}_p$, its spherical Fourier transform $\widehat{\phi}:\mathfrak{X}_p\to\mathbb{C}$ is defined by

$$\widehat{\phi}(\nu):=\int_{G(\mathbb{Q}_p)}\phi(g)\,\omega_{-\nu}(g)\,\mathrm{d}g,\quad g\in O(\mathbb{Q}_p),$$

where $\mathrm{d}g$ is the unique Haar measure on $O(\mathbb{Q}_p)$ such that $\mathrm{vol}(\mathbf{K}_p)=1$. Then it is known that there exists a Radon measure μ_p^{Pl} (Plancherel measure) on $[\mathfrak{X}_p^{0+}]:=\mathfrak{X}^{0+}/W_p$ such that

$$\phi(g)=\int_{[\mathfrak{X}_p^{0+}]}\widehat{\phi}(\nu)\,\omega_\nu(g)\,\mathrm{d}\mu_p^{\mathrm{Pl}}(\nu),\quad \phi\in\mathscr{H}_p.$$

Since $\#(\mathscr{L}_1^{\vee}/\mathscr{L}_1)=|Q(\xi)|^{-1}\#((\mathscr{L}_1^{\xi})^{\vee}/\mathscr{L}_1^{\xi})$, we have $(\mathscr{L}_1^{\xi})_{\mathbb{Z}_p}^{\vee}=(\mathscr{L}_1^{\xi})_{\mathbb{Z}_p}$ for $p\in S$. Thus, in the same way, we have the space $[\mathfrak{X}_p^{0+}(\xi)]$ of Satake parameters for zonal spherical functions on $O_1^{\xi}(\mathbb{Q}_p)$ of positive type, the spherical representation $\pi^{O_1^{\xi}(\mathbb{Q}_p)}(z)$ of $O_1^{\xi}(\mathbb{Q}_p)$ for $z\in[\mathfrak{X}_p^{0+}(\xi)]$. Let $\mathscr{U}\cong\bigotimes_p\mathscr{U}_p$ be a restricted tensor decomposition of $\mathscr{U}$ to irreducible smooth representations $\mathscr{U}_p$ of $O_1^{\xi}(\mathbb{Q}_p)$. Since $\mathscr{U}\subset L^2(O_1^{\xi}(\mathbb{Q})\backslash O_1^{\xi}(\mathbb{A}_{\mathbf{f}}))$ and $\mathscr{U}(\mathbf{K}_1^{\xi*})\neq\{0\}$, each $\mathscr{U}_p$ for $p\in S$ is unitarizable and spherical. As such, $\mathscr{U}_p$ for $p\in S$ is isomorphic to $\pi^{O_1^{\xi}(\mathbb{Q}_p)}(z_p)$ with $z_p\in[\mathfrak{X}_p^{0+}(\xi)]$ being the Satake parameter of $\mathscr{U}_p$. We say that $\mathscr{U}$ is tempered at $p\in S$, if z_p is purely imaginary. Set $[\mathfrak{X}_S^{0+}]:=\prod_{p\in S}[\mathfrak{X}_p^{0+}]$ and $[\mathfrak{X}_S^{0+}(\xi)]:=\prod_{p\in S}[\mathfrak{X}_S^{0+}(\xi)]$.

Theorem 3.3. *Suppose $2\xi \in \mathscr{L}_1$. Let $\varepsilon \in \{\pm 1\}$ be such that $\mathscr{B}^{\varepsilon}_{\mathscr{U}} \neq \varnothing$. Let S be a finite set of prime numbers such that $p \in S$ is relatively prime to $\#(\mathscr{L}^{\vee}/\mathscr{L})$ and $Q(\xi)$. Suppose $\mathscr{U}_p$ is tempered at all $p \in S$. Let $\phi_S = \otimes_{p\in S}\phi_p$ be an element of $\bigotimes_{p\in S}\mathscr{H}_p$.*

- $\widehat{\phi_S} := \prod_{p\in S}\widehat{\phi_p} \in C_{\rm c}([\mathfrak{X}_S^{0+}])$: *the spherical Fourier transformation of $\phi_S = \otimes_{p\in S}\phi_p$.*
- $\nu_S(F) := \{\nu_p(F)\}_{p\in S} \in [\mathfrak{X}_S^{0+}]$: *the Satake parameter of F at S.*
- $a(\mathscr{U}) := -\mathrm{ord}_{s=1}L(\mathscr{U}, s) \in \{0, 1\}$.

Then, as $l \to \infty$ with $(-1)^l = \varepsilon$,

$$\frac{1}{(\log l)^{a(\mathscr{U})}}\frac{\widetilde{\Gamma}(l)}{l^m}\sum_{F\in\mathscr{F}_l}\frac{1}{\#(\mathscr{B}^{\varepsilon}_{\mathscr{U}})}\sum_{f\in\mathscr{B}^{\varepsilon}_{\mathscr{U}}} L(F,1/2)\,|A_F^f(\xi)|^2\widehat{\phi_S}(\nu_S(F))$$

$$\longrightarrow B_{\mathscr{L}}(\xi)\,L(\mathscr{U},s)|^*_{s=1}\,\Lambda^{\mathscr{U}_S}(\widehat{\phi_S}),$$

where, $\Lambda^{\mathscr{U}}$ is a linear functional on $C_{\rm c}([\mathfrak{X}_S^{0+}])$ such that the value $\Lambda^{\mathscr{U}}(\alpha)$ at $\alpha = \otimes_{p\in S}\alpha_p$ is

$$\prod_{p\in S}\frac{\prod_{j=1}^{(m+1)/2}\zeta_p(2j)}{L(1,\pi_p^H(z_p);\mathrm{Ad})\,L(1,\pi_p^H(z_p))}\int_{[\mathfrak{X}_p^{0+}]}\alpha_p(\nu)\,\frac{L\left(\frac{1}{2},\pi_p^H(z_p)\boxtimes\pi_p^G(\nu)\right)L\left(\frac{1}{2},\pi_p^G(\nu)\right)}{L(1,\pi_p^G(\nu);\mathrm{Ad})}\,\mathrm{d}\mu_p^{\rm Pl}(\nu),$$

where we set $H = O_1^{\xi}(\mathbb{Q}_p)$ and $G = O(\mathbb{Q}_p)$ and $z_p \in [\mathfrak{X}_p^{0+}(\xi)]$ is the Satake parameter of $\mathscr{U}_p$ at $p \in S$.

4 Overview of proofs

Our method is based on a computation of a Fourier integral of a deliverately designed Poincaré series. Contrary to [3] and [24], and in a similar spirit to [11], neither Petersson-Kitaoka's formula nor the approximate functional equation is used. The most novel part in the definition of our Poincaré series is the usage of the archimedean *Shintani function* $\Phi_l^{\xi}(s) : O(\mathbb{R}) \to \mathbb{C}$, which is a smooth function on $O(\mathbb{R})$ defined by the formula

$$\Phi_l^{\xi}(s,g_\infty) := (-1)^l 2^{-(s+\frac{m-1}{2})}A(g_\infty)^{-l}\left\{\mathrm{i}\,\mathrm{sgn}\left(\mathrm{Im}\tfrac{B(g_\infty)}{A(g_\infty)}\right)\tfrac{B(g_\infty)}{A(g_\infty)}\right\}^{-(s+\frac{m-1}{2})},\quad g_\infty \in O(\mathbb{R}),$$

where

$$A(g_\infty) := |Q(\xi)|^{-1/2}(\xi, g(v_0^{\mathbb{C}})),\quad B(g_\infty) := (\varepsilon_1, g(v_0^{\mathbb{C}}))$$

with $v_0^{\mathbb{C}} := \frac{\varepsilon_0-\varepsilon_0'}{\sqrt{2}} + i\frac{-\varepsilon_1+\varepsilon_1'}{\sqrt{2}} \in \mathscr{L}_{\mathbb{C}}$ (see [21, §4], [22, §4.1]). It turns out that $\Phi = \Phi_l^{\xi}(s)$ is a unique C^{∞}-function on $O(\mathbb{R})$ that satisfies the conditions:

- $\Phi(g_\infty k) = J(k,\mathfrak{z}_0)^{-l}\Phi(g_\infty),\ k \in \mathbf{K}_\infty$,
- $J(g_\infty,\mathfrak{z}_0)^l\Phi(g_\infty)$ on $O(\mathbb{R})^0/\mathbf{K}_\infty \cong \mathscr{D}$ is holomorphic,
- $\Phi(hg_\infty) = |t|^{s+\frac{m-1}{2}}\,\Phi(g_\infty),\ h = \begin{bmatrix} t & * & * \\ & l & * \\ & & t^{-1}\end{bmatrix} \in P^{\xi}(\mathbb{R})$,
- $\Phi(b_\infty^{\xi}) = 1$.

We remark that the unramified Shintani funcions over $\mathfrak{p}$-adic fields ([14]) played an important role in the formulation of the refined Gan-Gross-Prasad conjecture (or the Ichino-Ikeda conjecture) originally due to [9] (see also [12]). For a Hecke function $\phi_S \in \bigotimes_{p\in S}\mathscr{H}_p$, let ϕ denote the function on $O(\mathbb{A}_{\mathbf{f}})$ defined as $\phi(g) = \prod_{p\in S}\phi_p(g_p)\prod_{p\notin S}\mathbb{1}_{\mathbf{K}_p^*}(g_p)$ for $g = (g_p)_{p<\infty} \in O(\mathbb{A}_{\mathbf{f}})$. Then set

$$\Phi_{\mathbf{f}}(\phi_S; g_{\mathbf{f}}) := \int_{O^\xi(\mathbb{A}_{\mathbf{f}})} f^{(s)}(h)\phi(h^{-1}g_{\mathbf{f}})\,\mathrm{d}h, \quad g_{\mathbf{f}} \in O(\mathbb{A}_{\mathbf{f}}),$$

where $f^{(s)}$ is the function used to define the Eisenstein series $E_{P^\xi}^{O^\xi}(f,s)$ associated to the Hecke eigenfunction f on $O_1^\xi(\mathbb{Q})\backslash O_1^\xi(\mathbb{A}_{\mathbf{f}})$. Define a smooth function $\Phi_l^\xi(\phi_S|s)$ on $O(\mathbb{A})$ by

$$\Phi_l^{f,\xi}(\phi_S|s; g_\infty g_{\mathbf{f}}) := \Phi_l^\xi(s; g_\infty)\,\Phi_{\mathbf{f}}(\phi_S; g_{\mathbf{f}}), \quad g_\infty g_{\mathbf{f}} \in O(\mathbb{R})O(\mathbb{A}_{\mathbf{f}}).$$

Choose an entire function $\beta(s)$ on $\mathbb{C}$ such that for any compact set $I \subset \mathbb{R}$ and for any $N > 0$ the estimation $e^{\pi|t|}\beta(\sigma + \mathrm{i}t) \ll_{I,N} (1 + |t|)^{-N}$ holds for $\sigma \in I$ and $t \in \mathbb{R}$, and set

$$\widehat{\Phi}_l^{f,\xi}(\phi_S|\beta, g) := \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \beta(s)D_*(s)\Lambda(f,-s)\,\Phi_l^{f,\xi}(\phi_S|s; g)\,\mathrm{d}s,$$

where $D_*(s) := \prod_{j\in[0,m-1]-\{\frac{m-1}{2}\}}(s - \frac{m-1}{2} + j)$, which is introduced to kill the possible poles of $\Lambda(f,s)$, the normalizing factor of the Eisenstein series (*cf.* (3.1)). Now, our adelic Poincaré series is defined by the infinite sum

$$\widehat{\mathbb{F}}_l^{f,\xi}(\phi_S|\beta; g) := \sum_{\gamma\in P^\xi(\mathbb{Q})\backslash O(\mathbb{Q})} \widehat{\Phi}_l^{f,\xi}(\phi_S|\beta, g), \quad g \in O(\mathbb{A}),$$

which is shown to be absolutely and normally convergent on $O(\mathbb{A})$ yielding a cusp form in $\mathfrak{S}_l$ for $l \gg 1$. Moreover, its spectral expansion in terms of an orthonormal basis $\mathscr{F}_l$ of $\mathfrak{S}_l$ is given as

$$\begin{aligned}&\widehat{\mathbb{F}}_l^{f,\xi}(\phi_S|\beta; g)\\ &= \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \beta(s)\left\{-2\pi^{\frac{m}{2}}\Gamma\left(\tfrac{m}{2}\right)^{-1} C_l^\xi\, B_l^\xi(s)\sum_{F\in\mathscr{F}_l} D_*(s)\overline{L\left(F, \bar{s} + \tfrac12\right)\, a_F^{\bar{f}}(\xi)}\,\lambda_F(\phi)\,F(g)\right\}\mathrm{d}s,\end{aligned}\tag{4.1}$$

where $B_l^\xi(s)$ is an entire function studied in [21, §4 (Proposition 30)]. We deduce a trace-formula-like identity by computing the integral (= Fourier-Bessel integral)

$$\int_{O_1^\xi(\mathbb{Q})\backslash O_1^\xi(\mathbb{A}_{\mathbf{f}})} \bar{f}(h_0)\,\mathrm{d}h \int_{\mathscr{L}_{1,\mathbb{A}_{\mathbf{f}}}} \widehat{\mathbb{F}}_l^{f,\xi}\left(\phi_S|\beta; \begin{bmatrix} 1 & -{}^tQX & -2^{-1}Q(X) \\ & l & X \\ & & 1\end{bmatrix}\begin{bmatrix} r & 0 & 0 \\ & h_0 & 0 \\ & & r^{-1}\end{bmatrix} b_\infty^\xi\right)\psi((\xi, X))^{-1}\mathrm{d}X$$

in two ways. We use the spectral expansion in (4.1) to relate this integral to the weighted average of the L-functions in Theorems 3.2 and 3.3. We invoke Liu's computation ([12]) of local period of zonal spherical functions to compute the main term in the geometric side (see [22, §5.2]). To deduce Theorems 2.1 and 2.2, we specialize the asymptotic formulas in Theorems 3.2 and 3.3 to the setting

$$\mathscr{L} := \left\{Y = \begin{bmatrix} X & -x'w \\ x''w & {}^tX\end{bmatrix} \mid X \in \mathbb{Z}^3,\ x', x'' \in \mathbb{Z}\right\} \cong \mathbb{Z}^5, \quad Q(Y) := \tfrac12\det(Y^2), \quad Y \in \mathscr{L}$$

with $w = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right]$, and transcribe the formula in the language of Siegel modular forms through the exceptional isomorphism $\rho : \mathbf{PGSp}_2 \to \mathrm{SO}(Q)$ defined by $\rho(g)Y = gYg^{-1}$.

References

[1] A.N. Andrianov, *Dirichlet series with Euler products in the theory of Siegel modular forms of genus* 2, Trudy Mat. Inst. Steklov. **112** (1971), 73–94.

[2] A.N. Andrianov, *Euler products corresponding to Siegel's modular forms of genus* 2, Uspekhi Mat. Nauk **29** no. 3 (1974), 43–110.

[3] V. Blomer, *Spectral summation formulae for* GSp(4) *and moments of spinor L-functions*, J. Eur. Math. Soc. **21** (2019), no. 6, 1751–1774.

[4] P. Cartier, *Representations of* $\mathfrak{p}$*-adic groups: a survey*, In: Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), part 1, 111–155, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[5] M. Dickson, A. Pitale, A. Saha and R. Schmidt, *Explicit refinements of Böchere's conjecture for Siegel modular forms of square-free level*, preprint, arXiv:1512.07204v6.

[6] S.A. Evdokimov, *Characterization of the Maass space of Siegel modular cusp forms of genus* 2, Math. Sb. (N.S.) **112(154)** (1980), no. 1(5), 133–142, 144.

[7] M. Furusawa and K. Morimoto, *Refined global Gross-Prasad conjecture on special Bessel periods and Boecherer's conjecture*, J. Eur. Math. Soc. **23** (2021) no. 4, 1295–1331.

[8] M. Furusawa and K. Morimoto, *On the Gross-Prasad conjecture with its refinement for* $(\mathrm{SO}(5), \mathrm{SO}(3))$ *and the generalized Böcherer conjecture*, preprint, arXiv:2205.09503.

[9] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, Geom. Funct. Anal. **19** (2010), no. 5, 1378–1425.

[10] E. Kowalski, A. Saha and J. Tsimerman, *Local spectral equidistribution for Siegel modular forms and applications*, Compos. Math. **148**, (2012), no. 2, 335–384.

[11] W. Kohnen, *Nonvanishing of Hecke L-functions associated to cusp forms inside the critical strip*, J. Number Theory **67** (1997), no. 2, 182–189.

[12] Y. Liu, *Refined global Gan-Gross-Prasad conjecture for Bessel periods*, J. reine angew. Math. **717** (2016), 133–194.

[13] A. Murase and T. Sugano, *On standard L-functions attached to automorphic forms on definite orthogonal groups*, Nagoya Math. J. **152** (1998), 57–96.

[14] A. Murase, T. Sugano and S. Kato, *Whittaker Shintani functions for orthogonal groups*, Tohoku Math. J. **55** (2003), no. 1, 1–64.

[15] A. Murase and T. Sugano, *Holomorphic automorphic forms and their L-functions on type IV domains*, Surikaisekikenkyusho Kokyuroku No. 1342, (2003), 93–106.

[16] T. Oda, *On the poles of Andrianov L-functions*, Math. Ann. **256** (1981), no. 3, 323–340.

[17] A. Pitale, A. Saha and R. Schmidt, *Transfer of Siegel cusp forms of degree* 2, Mem. Amer. Math. Soc. **232** (2014), no. 1090.

[18] I. Satake, *Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields*, Publ. Math. IHES. **18** (1963), 5–69.

[19] T. Sugano, *On holomorphic cusp forms on quaternion unitary groups of degree* 2, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1985), no. 3, 521–568.

[20] T. Sugano, *On Dirichlet series attached to holomorphic cusp forms on* $SO(2,q)$, In: Automorphic forms and number theory (Sendai, 1983), 333–362, Adv. Stud. Pure Math., 7, North-Holland, Amsterdam, 1985.

[21] M. Tsuzuki, *Spectral average of central values of automorphic L-functions for holomorphic cusp forms on* $SO_0(m,2)$*, I*, J. Number Theory **132** (2012), no. 11, 2407–2454.

[22] M. Tsuzuki, *Spectral average of central values of automorphic L-functions for holomorphic cusp forms on* $SO_0(m,2)$*, II*, preprint, arXiv:1906.01172v2.

[23] M. Tsuzuki, *A weighted equidistribution theorem for Siegel modular forms of degree* 2, preprint, arXiv:2001.02859v1.

[24] F. Waibel, *Moments of spinor L-functions and symplectic Kloosterman sums*, Q. J. Math. **70** (2019), no. 4, 1411–1436.

[25] R. Weissauer, Endoscopy for GSp(4) and the cohomology of Siegel threefolds, Lecture Notes in Mathematics, 1968, Springer-Verlag, Berlin, 2009.