Local intertwining relation for metaplectic groups

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1 Introduction

These notes are based on my speech in the 13th Fukuoka Number Theory Symposium. See [6] for the detail of the notes.

Representations of p-adic groups have been important objects of study, and we want to classify these representations. The Langlands classification says that irreducible representations of p-adic groups can be classified in terms of tempered ones. Thus the problem is reduced to the classification of tempered ones.

Let F be a p-adic field (i.e., a finite extension of $\mathbb{Q}_p$, for some prime number p) with the (absolute) Galois group Γ_F and the (absolute) Weil group W_F, and G a connected reductive algebraic group defined over F. We write WD_F for the Weil-Deligne group $W_F \times \mathrm{SL}(2, \mathbb{C})$. The local Langlands correspondence (LLC) proposes a classification of tempered irreducible admissible representations of $G(F)$ in terms of tempered admissible L-parameters of G. Let $\hat{G}$ be the connected complex Langlands dual group of G. We write $\Pi_{\mathrm{temp}}(G)$ for the set of equivalence classes of tempered irreducible smooth admissible representations of $G(F)$, and $\Phi_{\mathrm{temp}}(G)$ for the set of equivalence classes of tempered admissible L-parameters $WD_F \to \hat{G} \rtimes W_F$.

The basic form of LLC is the following:

Conjecture 1.1. (1) *There exists a canonical map*

$$LL : \Pi_{\mathrm{temp}}(G) \longrightarrow \Phi_{\mathrm{temp}}(G)$$

with some important properties.
(2) *For each $\phi \in \Phi_{\mathrm{temp}}(G)$, the fiber $\Pi_\phi = \Pi_\phi(G) = LL^{-1}(\phi)$ is a finite set.*

There are further expected properties.

When we treat a non-quasi-split groups, the local Langlands correspondence of Vogan type is more exquisite. Let G^* be a quasi-split connected reductive algebraic group over a p-adic field F. This treats pure inner twists of G^* at the same time. For each $\phi \in \Phi_{\mathrm{temp}}(G)$, we let $S_\phi = S_\phi(G)$ denote the centralizer $\mathrm{Cent}(\phi(WD_F), \hat{G})$ of $\phi(WD_F)$ in $\hat{G}$, and $\pi_0(S_\phi)$ denotes its component group. Then the local Langlands correspondence of Vogan type proposes the following:

Conjecture 1.2. (1) *There exists a canonical map*

$$LLV : \bigsqcup_{(\xi,z)} \Pi_{\mathrm{temp}}(G) \longrightarrow \Phi_{\mathrm{temp}}(G^*),$$

as (ξ, z) runs over the isomorphism classes of pure inner twists of G^*, i.e., $\xi : G^* \to G$ is an inner twist and $z \in Z^1(\Gamma_F, G^*)$ is a 1-cocycle such that $\xi^{-1} \circ \sigma \circ \xi \circ \sigma^{-1} = \mathrm{Ad}(z(\sigma))$ for all $\sigma \in \Gamma_F$. This map satisfies some important properties.
(2) *For each $\phi \in \Phi_{\mathrm{temp}}(G^*)$, the fiber $\Pi_\phi = LLV^{-1}(\phi)$ is a finite set.*

(3) *For each* $\phi \in \Phi_{\mathrm{temp}}(G^*)$*, there exists a bijective map*

$$\iota : \Pi_\phi \longrightarrow \mathrm{Irr}(\pi_0(S_\phi)),$$

where $\mathrm{Irr}(\pi_0(S_\phi))$ *denotes the set of equivalence classes of irreducible representations of the finite group* $\pi_0(S_\phi)$*. This bijection* ι *satisfies the endoscopic character relations and other nice properties. Moreover, once we fix a Whittaker datum of* G^**, then the map* ι *is uniquely determined.*

These conjectures are proved for some classical groups. For the general linear groups $\mathrm{GL}(N)$, it has been proved by Harris-Taylor and Henniart. Arthur [1] established LLC for quasi-split $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$ and $\mathrm{Sp}(2n)$. Moreover, Mœglin-Renard [9] gives a classification of irreducible tempered representations of non-quasi-split odd special orthogonal groups over p-adic fields, hence LLC of Vogan type for $\mathrm{SO}(2n+1)$. Mok [10] and Kaletha-Mínguez-Shin-White [7, Chapter 2] proved for inner forms of unitary groups, hence LLC of Vogan type for $\mathrm{U}(n)$.

Although in general the map LL or LLV may not be bijective (i.e., each packet may not be a singleton), there is a formula that describes how the bijection ι classifies the elements in a same packet, in terms of intertwining operators. This can distinguish the elements of some packets Π_ϕ more precisely, with the eigenvalues of intertwining operators. We call this formula the local intertwining relation. In [1], Arthur proved the local intertwining relation for quasi-split special orthogonal and symplectic groups ([1, Theorem 2.4.1]). Mok [10] and Kaletha-Mínguez-Shin-White [7, Chapter 2] proved for inner forms of unitary groups.

Our main result is that we have formulated and proved a local intertwining relation for the metaplectic group $\mathrm{Mp}(2n)$, under the assumption that the local intertwining relation for non-quasi-split $\mathrm{SO}(2n+1)$ holds.

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2 LLC & LIR for $\mathrm{SO}(2n+1)$

Before LLC & LIR for $\mathrm{SO}(2n+1)$, we shall begin with a brief review of the orthogonal groups and their L-parameters. Let V be a $(2n+1)$-dimensional vector space over F equipped with a non-degenerate quadratic form $q = q_V$ of discriminant 1 (i.e., determinant $(-1)^n$). If $n \geq 1$, there are precisely two such quadratic spaces V up to isomorphism. One of them, to be denoted by V^+, has maximal isotropic subspaces of dimension n, whereas the other has maximal isotropic subspaces of dimension $n-1$ and is denoted by V^-. As such, we call the former the split quadratic space and the latter the non-split one. We shall write

$$\epsilon(V) = \begin{cases} +1, & V = V^+; \\ -1, & V = V^-. \end{cases}$$

If $n = 0$, there is only one such V. In this case we put $V^+ = V$, and $\epsilon(V) = +1$. Let

$$\mathrm{O}(V) = \{\, h \in \mathrm{GL}(V) \mid q_V(hv) = q_V(v) \quad \text{for all } v \in V \,\}$$

be the associated orthogonal group. Then observe that $\mathrm{O}(V) = \mathrm{SO}(V) \times \{\pm 1\}$, where $\mathrm{SO}(V) = \mathrm{O}(V) \cap \mathrm{SL}(V)$ is the special orthogonal group. The group $\mathrm{SO}(V)$ is split (resp. non-quasi-split) if V is the split (resp. non-split) quadratic space. If $n \geq 1$, up to isomorphism, there are precisely two pure inner twists of $\mathrm{SO}(V^+)$, namely $\mathrm{SO}(V^+)$ and $\mathrm{SO}(V^-)$.

Let $\underline{k} = (k_1, \dots, k_m)$ be a sequence of positive integers such that $k_1 + \cdots k_m \leq r$. Put $n_0 = n - (k_1 + \cdots k_m)$. Let $Q_{\underline{k}}$ be a parabolic subgroup of $\mathrm{SO}(V)$ with a Levi subgroup $L_{\underline{k}} \cong \mathrm{GL}(k_1) \times \mathrm{GL}(k_2) \times \cdots \times \mathrm{GL}(k_m) \times \mathrm{SO}(V_{n_0})$, where V_{n_0} is a $(2n_0+1)$-dimensional quadratic space with $\epsilon(V_{n_0}) = \epsilon(V)$. We shall write $U_{\underline{k}}$ for the unipotent radical of $Q_{\underline{k}}$.

Next, we recall the notion of L-parameters of $\mathrm{GL}(k)$ and $\mathrm{SO}(2n+1)$. See [2] for detail.

Let D be a finite dimensional vector space over $\mathbb{C}$. We say that a homomorphism $\phi : WD_F \to \mathrm{GL}(D)$ is a representation of $WD_F = W_F \times \mathrm{SL}(2, \mathbb{C})$ if

- $\phi(\mathrm{Frob}_F)$ is semi-simple, where $\mathrm{Frob}_F \in W_F$ is a geometric Frobenius;
- the restriction of ϕ to $\mathrm{SL}(2, \mathbb{C})$ is algebraic;
- the restriction of ϕ to W_F is smooth.

We call ϕ tempered if the image of W_F is bounded. We say that ϕ is symplectic (resp. orthogonal) if there exists a non-degenerate bilinear form $B : D \times D \to \mathbb{C}$ such that

$$\begin{cases} B(\phi(w)x, \phi(w)y) = B(x, y), \\ B(y, x) = \delta B(x, y), \end{cases} \tag{1}$$

with $\delta = -1$ (resp. $\delta = +1$), for any $x, y \in D$ and $w \in WD_F$. In this case, ϕ is self-dual, i.e., ϕ is equivalent to its contragredient $\phi^\vee$.

Let $\phi : WD_F \to \mathrm{GL}(D)$ be a tempered symplectic representation. Take a non-degenerate bilinear form B that satisfies (1) with $\delta = -1$. Then, as in [2, §4], we can write

$$\phi = \bigoplus_{i \in I_\phi^+} \ell_i \phi_i \ \oplus \ \bigoplus_{i \in I_\phi^-} \ell_i \phi_i \ \oplus \ \bigoplus_{j \in J_\phi} \ell_j (\phi_j \oplus \phi_j^\vee),$$

where ℓ_i are positive integers, and $I_\phi^\pm$, J_ϕ are indexing sets for mutually inequivalent irreducible representations ϕ_i of WD_F such that

- ϕ_i is symplectic for $i \in I_\phi^+$;
- ϕ_i is orthogonal and ℓ_i is even for $i \in I_\phi^-$;
- ϕ_j is not self-dual for $j \in J_\phi$.

Let S_ϕ be the centralizer of the image $\mathrm{Im}(\phi)$ in $\mathrm{Sp}(D, B)$. Then by [2, §4], we have

$$S_\phi \cong \prod_{i \in I_\phi^+} \mathrm{O}(\ell_i, \mathbb{C}) \times \prod_{i \in I_\phi^-} \mathrm{Sp}(\ell_i, \mathbb{C}) \times \prod_{j \in J_\phi} \mathrm{GL}(\ell_j, \mathbb{C}),$$

and the component group $\pi_0(S_\phi)$ can be identified with a free $\mathbb{Z}/2\mathbb{Z}$-module

$$\pi_0(S_\phi) = \bigoplus_{i \in I_\phi^+} (\mathbb{Z}/2\mathbb{Z}) a_i$$

of rank $\# I_\phi^+$, where $\{a_i\}_{i \in I_\phi^+}$ is a basis with a_i associated to ϕ_i.

Let $\Phi_{\mathrm{temp}}(\mathrm{GL}(k))$ be the set of equivalence classes of L-parameters of $\mathrm{GL}(k)$. It can be identified with the set of equivalence classes of tempered representations $\phi : WD_F \to \mathrm{GL}(k, \mathbb{C})$ of dimension k. Now let $\Phi_{\mathrm{temp}}(\mathrm{SO}(2n+1))$ be the set of equivalence classes of L-parameters of $\mathrm{SO}(2n+1)$. Then by [2, §11, §8], we can identify $\Phi_{\mathrm{temp}}(\mathrm{SO}(2n+1))$ with the set of equivalence classes of tempered symplectic representations $\phi : WD_F \to \mathrm{Sp}(2n, \mathbb{C})$ of dimension $2n$.

The local Langlands correspondence for $\mathrm{SO}(2n+1)$ (established by Arthur, Mœglin-Renard), is the following:

Theorem 2.1 (LLC for $\mathrm{SO}(2n+1)$)**.** *There exists a canonical map*

$$\Pi_{\mathrm{temp}}(\mathrm{SO}(V^+)) \sqcup \Pi_{\mathrm{temp}}(\mathrm{SO}(V^-)) \longrightarrow \Phi_{\mathrm{temp}}(\mathrm{SO}(2n+1)),$$

such that

- *packets Π_ϕ (the preimage of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{SO}(2n+1))$) are finite sets;*
- *for each $\phi \in \Phi_{\mathrm{temp}}(\mathrm{SO}(2n+1))$, there is a bijective map*

$$\iota : \Pi_\phi \longrightarrow \mathrm{Irr}(\pi_0(S_\phi));$$

- *some other properties.*

As one of the "some other properties", LLC has the following compatibility with the parabolic inductions of representations.

Let $\underline{k} = (k_1, \ldots, k_m)$ such that $k_1 + \cdots + k_m \le r$, and put $n_0 = n - (k_1 + \cdots + k_m)$. For $(G, Q, L) = (\mathrm{SO}(V), Q_{\underline{k}}, L_{\underline{k}})$, it is known that $\widehat{G} = \mathrm{Sp}(2n, \mathbb{C})$, $\widehat{L} = \mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_m, \mathbb{C}) \times \mathrm{Sp}(2n_0, \mathbb{C})$, and there exists a standard embedding $\widehat{L} \hookrightarrow \widehat{G}$ as a Levi subgroup of a standard parabolic subgroup $\widehat{Q}$ of $\widehat{G}$.

Let ϕ be a tempered L-parameter for G with the image $\mathrm{im}(\phi)$ in $\widehat{L}$. This is of the form

$$\phi = \phi_1 \oplus \cdots \oplus \phi_m \oplus \phi_0 \oplus \phi_m^\vee \oplus \cdots \oplus \phi_1^\vee \tag{2}$$

where $\phi_i \in \Phi_{\mathrm{temp}}(\mathrm{GL}(k_i))$ for $i = 1, \ldots, m$, and $\phi_0 \in \Phi_{\mathrm{temp}}(\mathrm{SO}(2n_0+1))$. Then $\phi_L = \phi_1 \oplus \cdots \oplus \phi_m \oplus \phi_0$ is the corresponding L-parameter for L. Then we have

$$\Pi_\phi = \left\{ \sigma \;\middle|\; \sigma \subset \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0), \quad \sigma_0 \in \Pi_{\phi_0} \right\},$$

where τ_i is the representation of $\mathrm{GL}(k_i, F)$ which corresponds to ϕ_i, $i = 1, \ldots, m$. Moreover for $\sigma_0 \in \Pi_{\phi_0}$, we have

$$\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0) = \bigoplus_{\substack{\sigma \in \Pi_\phi \\ \iota(\sigma)|_{\pi_0(S_{\phi_0})} = \iota(\sigma_0)}} \sigma.$$

Note that

$$\Pi_{\phi_L} = \{ \tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0 \mid \sigma_0 \in \Pi_{\phi_0} \}.$$

The local intertwining relation (LIR) is a relation which enable us to distinguish the irreducible components of one induced representation. Before that, we need to define some notion around L-parameters. Let $A_{\widehat{L}}$ be the maximal central split torus of $\widehat{L}$. Then $\widehat{L} = \mathrm{Cent}(A_{\widehat{L}}, \widehat{G})$, and one has $A_{\widehat{L}} \cong (\mathbb{C}^\times)^m$. Put

$$\begin{aligned}
\mathfrak{N}_\phi(L, G) &= \mathrm{Norm}(A_{\widehat{L}}, S_\phi) / \mathrm{Cent}(A_{\widehat{L}}, S_\phi^\circ), \\
\mathrm{W}_\phi(L, G) &= \mathrm{Norm}(A_{\widehat{L}}, S_\phi) / \mathrm{Cent}(A_{\widehat{L}}, S_\phi), \\
S_\phi^\natural(L, G) &= \mathrm{Norm}(A_{\widehat{L}}, S_\phi) / \mathrm{Norm}(A_{\widehat{L}}, S_\phi^\circ).
\end{aligned}$$

We have a natural surjection

$$\mathfrak{N}_\phi(L, G) \longrightarrow S_\phi^\natural(L, G), \tag{3}$$

natural inclusions

$$\mathrm{W}_\phi(L,G) \subset \mathrm{W}(\widehat{L},\widehat{G}),$$
$$S_\phi^\natural(L,G) \subset \pi_0(S_\phi),$$

and a natural short exact sequence

$$1 \longrightarrow \pi_0(S_{\phi_0}) \longrightarrow \mathfrak{N}_\phi(L,G) \longrightarrow \mathrm{W}_\phi(L,G) \longrightarrow 1.$$

By applying [1, p.104] or [7, p.103, after (2.4.1)] to $\mathrm{SO}(2n+1)$, the injection $\pi_0(S_{\phi_0}) \to \mathfrak{N}_\phi(L,G)$ admits a canonical splitting

$$\mathfrak{N}_\phi(L,G) = \pi_0(S_{\phi_0}) \times \mathrm{W}_\phi(L,G).$$

Now, let us return to LIR. For $w \in \mathrm{W}_\phi(L,\mathrm{SO}(V))$, let

$$R_Q(w, \tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0) \in \mathrm{End}_{\mathrm{SO}(V)}(\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0))$$

be the normalized self intertwining operator defined below. The local intertwining relation for $\mathrm{SO}(2n+1)$ is the following hypothesis, and it has already been proved in case $V = V^+$ by Arthur [1, §2.4].

Hypothesis 2.2. *Let $x_w \in S_\phi^\natural(L,\mathrm{SO}(V))$ be the image of $w \in \mathrm{W}_\phi(L,\mathrm{SO}(V))$ under the natural surjection* (3)*. Then, the restriction of $R_Q(w, \tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0)$ to $\sigma \subset \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0)$ is the scalar multiplication by $\iota(\sigma)(x_w)$.*

Even if V is not split, this hypothesis is expected to hold. As $w \in \mathrm{W}_\phi(L,\mathrm{SO}(V))$ runs, x_w runs over $S_\phi^\natural(L,\mathrm{SO}(V))$, and LIR enables us to distinguish the elements in a packet.

In the rest of this section, we give the definition of the normalized self intertwining operator $R_Q(w,\sigma_L)$, where σ_L denotes $\tau_1 \otimes \cdots \otimes \tau_m \otimes \sigma_0$. Let $\psi : F \to \mathbb{C}^1$ be a nontrivial additive character, $\mathbf{spl}_{\mathrm{SO}(V^+)}$ the standard F-splitting of $\mathrm{SO}(V^+)$ ([6]). Take a repersentation space $\mathscr{V}_{\sigma_0}$ of σ_0, and $\mathscr{V}_{\tau_i}$ of τ_i for $i = 1, \ldots, m$. For any $\underline{s} = (s_1, \ldots s_m) \in \mathbb{C}^m$, we realize the representation $\tau_{i,s_i} = \tau_i \otimes |\det|_F^{s_i}$ on $\mathscr{V}_{\tau_i}$ by setting $\tau_{i,s_i}(a)v = |\det a|_F^{s_i}\tau_i(a)v$ for $v \in \mathscr{V}_{\tau_i}$ and $a \in \mathrm{GL}(k_i, F)$. Moreover, we put $\sigma_{L,\underline{s}} = \tau_{1,s_1} \otimes \cdots \otimes \tau_{m,s_m} \otimes \sigma_0$.

We define an unnormalized intertwining operator

$$\mathcal{M}\left(\widetilde{w}_Q, \sigma_{L,\underline{s}}\right) : \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_{L,\underline{s}}) \to \mathrm{Ind}_Q^{\mathrm{SO}(V)}(w\sigma_{L,\underline{s}}),$$

by the meromorphic continuation of the integral

$$\mathcal{M}\left(\widetilde{w}_Q, \sigma_{L,\underline{s}}\right)\mathscr{F}_{\underline{s}}(h) = \int_{(wUw^{-1}\cap U)\backslash U} \mathscr{F}_{\underline{s}}((\widetilde{w}_Q)^{-1}uh)du,$$

where $U = U_{\underline{k}}$, $\widetilde{w}_Q \in \mathrm{SO}(V)$ is Langlands-Shelstad's representative ([8]) of w with repect to the splitting $\mathbf{spl}_{\mathrm{SO}(V^+)}$, and $w\sigma_{L,\underline{s}}$ is the representation of L on $\mathscr{V}_{\tau_1} \otimes \cdots \otimes \mathscr{V}_{\tau_m} \otimes \mathscr{V}_{\sigma_0}$ given by

$$w\sigma_{L,\underline{s}}(l) = \sigma_{L,\underline{s}}((\widetilde{w}_Q)^{-1}l\widetilde{w}_Q), \quad l \in L.$$

The integral above converges absolutely on some open set of $\mathbb{C}^m$ in $\underline{s}$, and has meromorphic continuation to $\underline{s} \in \mathbb{C}^m$. Thus the operator is well-defined for $\underline{s} \in \mathbb{C}^m$ except finite poles modulo $(2\pi i/\log q_F)\mathbb{Z}^m$. Then we shall normalize the operator $\mathcal{M}\left(\widetilde{w}_Q, \sigma_{L,\underline{s}}\right)$ to be holomorphic at $\underline{s} = 0$. Put $Q^w = (\widetilde{w}_Q)^{-1}Q\widetilde{w}_Q$, and let $\phi_1, \ldots, \phi_m, \phi_0$ be the L-parameters that corresponds to $\tau_1, \ldots, \tau_m, \sigma_0$ via LLC. Put

$$\phi = \phi_1 \oplus \cdots \oplus \phi_m \oplus \phi_0 \oplus \phi_m^\vee \oplus \cdots \oplus \phi_1^\vee,$$

then we have $\phi \in \Phi_{\mathrm{temp}}(\mathrm{SO}(2n+1))$ and $\mathrm{im}(\phi) \subset \widehat{L}$. The twist of ϕ by $\underline{s}$ is defined by

$$\phi_{\underline{s}} = (\phi_1 \otimes \| - \|^{s_1}) \oplus \cdots \oplus (\phi_m \otimes \| - \|^{s_m}) \oplus \phi_0 \oplus (\phi_m^\vee \otimes \| - \|^{-s_m}) \oplus \cdots \oplus (\phi_1^\vee \otimes \| - \|^{-s_1}),$$

where $\| - \|$ denotes the norm on W_F. We write $\rho_{Q^w|Q}$ for the adjoint representation of $\widehat{L}$ on the quotient

$$\widehat{\mathfrak{n}}_{Q^w} / \widehat{\mathfrak{n}}_{Q^w} \cap \widehat{\mathfrak{n}}_Q,$$

where $\widehat{\mathfrak{n}}_{Q^w}$ denotes the Lie algebra of the unipotent radical of $\widehat{Q^w}$. We define a normalized intertwining operator

$$\mathcal{R}_Q(w, \sigma_{L,\underline{s}}) = \epsilon(V)^{\dim y(w,\phi)} \gamma(0, \rho_{Q^w|Q}^\vee \circ \phi_{\underline{s}}, \psi) \mathcal{M}\left(\widetilde{w}_Q, \sigma_{L,\underline{s}}\right),$$

where a representation $y(w,\phi)$ is defined as follows. Let $y : \mathbb{Z}/2\mathbb{Z} \to \{0,1\}$ be a map such that $y(2\mathbb{Z}) = 0$ and $y(1+2\mathbb{Z}) = 1$. The canonical realization $\mathrm{W}(\widehat{L}, \mathrm{Sp}(2n,\mathbb{C})) \hookrightarrow \mathfrak{S}_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$ gives us an expression

$$w = \sigma_w \ltimes (d_i)_{i=1}^m$$

of w as an element of $\mathfrak{S}_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$. Now, for a representation $\phi_{\underline{s}}$ of WD_F, we define another representation $y(w, \phi_{\underline{s}})$ by

$$y(w, \phi_{\underline{s}}) = \bigoplus_{i=1}^m y(d_i)\phi_i \otimes \| - \|^{s_i}.$$

By [1, Proposition 2.3.1], we have that the operator $\mathcal{R}_Q(w, \sigma_{L,\underline{s}})$ is holomorphic at $\underline{s} = 0$, and the operator

$$\mathcal{R}_Q(w, \sigma_L) = \mathcal{R}_Q(w, \sigma_{L,0})$$

is therefore defined.

Now we can define the normalized self-intertwining operator. Let $w \in \mathrm{W}_\phi(L, \mathrm{SO}(V))$, which is equivalent to $w\sigma_L \cong \sigma_L$. We take the unique isomorphism

$$\mathcal{A}_w : \mathscr{V}_{\tau_1} \otimes \cdots \otimes \mathscr{V}_{\tau_m} \longrightarrow \mathscr{V}_{\tau_1} \otimes \cdots \otimes \mathscr{V}_{\tau_m}$$

such that:

- $(\mathcal{A}_w \otimes 1_{\mathscr{V}_{\sigma_0}}) \circ w\sigma_L(l) = \sigma_L(l) \circ (\mathcal{A}_w \otimes 1_{\mathscr{V}_{\sigma_0}})$ for any $l \in L$;
- $\Lambda \circ \mathcal{A}_w = \Lambda$, where $\Lambda : \mathscr{V}_{\tau_1} \otimes \cdots \otimes \mathscr{V}_{\tau_m} \to \mathbb{C}$ is the unique (up to a scalar) Whittaker functional with respect to the Whittaker datum $(B_{\underline{k}}^{\mathrm{GL}}, \psi_{\underline{k}})$. Here $B_{\underline{k}}^{\mathrm{GL}}$ is the Borel subgroup consisting of upper triangular matrices in $\mathrm{GL}(k_1, F) \times \cdots \times \mathrm{GL}(k_m, F)$, where we realize $\mathrm{GL}(k_1,F) \times \cdots \times \mathrm{GL}(k_m, F)$ in $\mathrm{GL}(k, F)$ as a group of block diagonal matrices, and $\psi_{\underline{k}}$ is the generic character of the unipotent radical $U_{\underline{k}}^{\mathrm{GL}}$ of $B_{\underline{k}}^{\mathrm{GL}}$ given by $\psi_{\underline{k}}(x) = \psi(x_{1,2} + \cdots + x_{k-1,k})$, for $x = (x_{i,j})_{i,j} \in U_{\underline{k}}^{\mathrm{GL}} \subset \mathrm{GL}(k,F)$.

Then the normalized self-intertwining operator

$$R_Q(w, \sigma_L) : \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L) \longrightarrow \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L),$$

is defined by

$$[R_Q(w, \sigma_L)\mathscr{F}](h) = \mathcal{A}_w \otimes 1_{\mathscr{V}_{\sigma_0}} \left(\mathcal{R}_Q(w, \sigma_L)\mathscr{F}(h)\right).$$

3 LLC for $\mathrm{Mp}(2n)$

We shall begin with a brief review of the metaplectic groups. Let $(W, \langle -, - \rangle_W)$ be a symplectic vector space of dimension $2n$ over F, with the associated symplectic group

$$\mathrm{Sp}(W) = \left\{ g \in \mathrm{GL}(W) \;\middle|\; \langle gw, gw' \rangle_W = \langle w, w' \rangle_W, \quad \text{for all } w, w' \in W \right\}.$$

The group $\mathrm{Sp}(W)$ has a unique nonlinear two-fold central extension $\mathrm{Mp}(W)$, which is called the metaplectic group:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1.$$

As a set, we may write

$$\mathrm{Mp}(W) = \mathrm{Sp}(W) \times \{\pm 1\}$$

with group law given by

$$(g, \epsilon) \cdot (g', \epsilon') = (gg', \epsilon\epsilon' c(g, g')),$$

where c is Ranga Rao's normalized cocycle, which is a 2-cocycle on $\mathrm{Sp}(W)$ valued in $\{\pm 1\}$. See [11] for detail. For any subset $A \subset \mathrm{Sp}(W)$, we write $\widetilde{A}$ for its preimage under the covering map $\mathrm{Mp}(W) \to \mathrm{Sp}(W)$. For any subset $B \subset \mathrm{Mp}(W)$, we write $\overline{B}$ for its image under the covering map $\mathrm{Mp}(W) \to \mathrm{Sp}(W)$.

Let $\underline{k} = (k_1, \ldots, k_m)$ be a sequence of positive integers such that $k_1 + \cdots + k_m \leq n$, and put $k_0 = 0$, $n_0 = n - (k_1 + \cdots + k_m)$. Let $\overline{P_{\underline{k}}}$ be a parabolic subgroup of $\mathrm{Sp}(W)$ with a Levi subgroup $\overline{M_{\underline{k}}} \cong \mathrm{GL}(k_1) \times \cdots \times \mathrm{GL}(k_m) \times \mathrm{Sp}(W_{n_0})$, where W_{n_0} is a $2n_0$-dimensional symplectic space. We shall write $N_{\underline{k}}$ for the unipotent radical of $\overline{P_{\underline{k}}}$. By a parabolic subgroup and of $\mathrm{Mp}(W)$ and its Levi subgroup, we mean the preimage of a parabolic subgroup of $\mathrm{Sp}(W)$ and its Levi subgroup. So, any parabolic subgroup of $\mathrm{Mp}(W)$ is conjugate to $P_{\underline{k}} = \widetilde{\overline{P_{\underline{k}}}}$, and its Levi subgroup is $M_{\underline{k}} = \widetilde{\overline{M_{\underline{k}}}} \cong \tilde{\mathrm{GL}}(k_1) \times_{\mu_2} \cdots \times_{\mu_2} \tilde{\mathrm{GL}}(k_m) \times_{\mu_2} \mathrm{Mp}(W_{n_0})$. It is known that the covering map splits over $N_{\underline{k}}$, and thus we have $P_{\underline{k}} = M_{\underline{k}} \ltimes N_{\underline{k}} \subset \mathrm{Mp}(W)$.

A representation of $\mathrm{Mp}(W)$ is said to be genuine if it does not factor through the covering map $\mathrm{Mp}(W) \to \mathrm{Sp}(W)$. We consider only the genuine representations of $\mathrm{Mp}(W)$. Let $\Pi_{\mathrm{temp}}(\mathrm{Mp}(2n))$ be the set of equivalenve classes of irreducible genuine tempered representations. In 2012, Gan and Savin established the local Shimura correspondence ([5]):

Theorem 3.1. *Fix a nontrivial additive character $\psi : F \to \mathbb{C}^1$. Then there is a bijection*

$$\Theta_\psi : \Pi_{\mathrm{temp}}(\mathrm{Mp}(2n)) \longleftrightarrow \Pi_{\mathrm{temp}}(\mathrm{SO}(V^+)) \sqcup \Pi_{\mathrm{temp}}(\mathrm{SO}(V^-)),$$

with natural properties.

This bijection is given by the local theta lift. By conbining LLC for $\mathrm{SO}(2n+1)$ with the local Shimura correspondence, one obtains LLC for $\mathrm{Mp}(2n)$ ([5, Corollary1.2.]):

Theorem 3.2 (LLC for $\mathrm{Mp}(2n)$)**.** *There exists a canonical map (depending on ψ)*

$$\Pi_{\mathrm{temp}}(\mathrm{Mp}(2n)) \longrightarrow \Phi_{\mathrm{temp}}(\mathrm{Mp}(2n)) := \Phi_{\mathrm{temp}}(\mathrm{SO}(2n+1)),$$

such that

- *packets $\Pi_{\phi,\psi}$ (the preimage of $\phi \in \Phi_{\mathrm{temp}}(\mathrm{Mp}(2n))$) are finite sets;*
- *for each $\phi \in \Phi_{\mathrm{temp}}(\mathrm{Mp}(2n))$, there is a bijective map (depending on ψ)*

$$\iota_\psi : \Pi_{\phi,\psi} \longrightarrow \mathrm{Irr}(\pi_0(S_\phi));$$

- *some other properties.*

As in the case of $\mathrm{SO}(2n+1)$, LLC for $\mathrm{Mp}(2n)$ also has the compatibility with the parabolic inductions. For $\underline{k} = (k_1, \ldots, k_m)$, put $n_0 = n - (k_1 + \cdots + k_m)$. For $(G, P, M) = (\mathrm{Mp}(W), P_{\underline{k}}, M_{\underline{k}})$, define $\widehat{G} = \widehat{\mathrm{SO}(V)} = \mathrm{Sp}(2n, \mathbb{C})$, $\widehat{M} = \widehat{L_{\underline{k}}} = \mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_m, \mathbb{C}) \times \mathrm{Sp}(2n_0, \mathbb{C})$, with a standard embedding $\widehat{M} \hookrightarrow \widehat{G}$ as a Levi subgroup of a standard parabolic subgroup $\widehat{P} = \widehat{Q_{\underline{k}}}$ of $\widehat{G}$.

Let ϕ be a tempered L-parameter for $\mathrm{Mp}(W)$ of the form (2). Then we have

$$\Pi_{\phi,\psi} = \left\{ \pi \;\middle|\; \pi \subset \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0), \quad \pi_0 \in \Pi_{\phi_0,\psi} \right\}.$$

Moreover for $\pi_0 \in \Pi_{\phi_0,\psi}$, we have

$$\mathrm{Ind}_P^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0) = \bigoplus_{\substack{\pi \in \Pi_{\phi,\psi} \\ \iota_\psi(\pi)|_{\pi_0(S_{\phi_0})} = \iota_\psi(\pi_0)}} \pi.$$

4 LIR for $\mathrm{Mp}(2n)$

In this section, we shall state the main theorem, that is LIR for $\mathrm{Mp}(2n)$, which is a similar relation to LIR for $\mathrm{SO}(2n+1)$. In order to do state it, we first define the normalized self intertwining operator $R_P(w, \pi_M)$. Let $\underline{k} = (k_1, \ldots, k_m)$ such that $k_1 + \cdots + k_m \leq r$, and put $n_0 = n - (k_1 + \cdots + k_m)$. Let π_0 be an irreducible genuine tempered representation of $\mathrm{Mp}(W_0)$ with an L-parameter ϕ_0. Also, take τ_i and ϕ_i be as above. Put $P = P_{\underline{k}}$, $M = M_{\underline{k}}$, and $\pi_M = \tilde{\tau}_1 \otimes \cdots \otimes \tilde{\tau}_m \otimes \pi_0$, where $\tilde{\tau}_i = \tau_i \otimes \chi_\psi$ is the irreducible genuine representation of $\tilde{\mathrm{GL}}(k_i)$ defined in [5, §2.4.]. Let $\psi : F \to \mathbb{C}^1$ be a nontrivial additive character, $\mathbf{spl}_{\mathrm{Sp}(W)}$ the standard F-splitting of $\mathrm{Sp}(W)$ ([6]).

We define $\mathscr{V}_{\pi_0}$, $\mathscr{V}_{\tau_i}$, and $\pi_{M,\underline{s}} = \tilde{\tau}_{1,s_1} \otimes \cdots \otimes \tilde{\tau}_{m,s_m} \otimes \pi_0$ as above.

We define an unnormalized intertwining operator

$$\mathcal{M}\left(\widetilde{w}_P, \pi_{M,\underline{s}}\right) : \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_{M,\underline{s}}) \to \mathrm{Ind}_P^{\mathrm{Mp}(W)}(w\pi_{M,\underline{s}}),$$

by the meromorphic continuations of the integral

$$\mathcal{M}\left(\widetilde{w}_P, \pi_{M,\underline{s}}\right) \mathscr{F}_{\underline{s}}(g) = \int_{(wNw^{-1} \cap N) \backslash N} \mathscr{F}_{\underline{s}}((\widetilde{w}_P)^{-1} ng) dn,$$

where $N = N_{\underline{k}}$, $\widetilde{w}_P \in \mathrm{Mp}(W)$ is Langlands-Shelstad's representative ([4, Definition 4.1.]) of w with repect to the splitting $\mathbf{spl}_{\mathrm{Sp}(W)}$.

We normalize the operator as follows. Put $\rho_{P^w|P} = \rho_{Q^w|Q}$, and

$$\mathcal{R}_P(w, \pi_{M,\underline{s}}, \psi) = \gamma_F(\psi)^{\dim y(w,\phi)} \gamma(\tfrac{1}{2}, y(w, \phi_{\underline{s}}), \psi)^{-1} \gamma(0, \rho^\vee_{P^w|P} \circ \phi_{\underline{s}}, \psi) \mathcal{M}\left(\widetilde{w}_P, \pi_{M,\underline{s}}\right),$$

where $\gamma_F(\psi) \in \mu_8(\mathbb{C})$ is the (unnormalized) Weil index of ψ. Then the operator $\mathcal{R}_P(w, \pi_{M,\underline{s}}, \psi)$ is holomorphic at $\underline{s} = 0$, and the operator

$$\mathcal{R}_P(w, \pi_M, \psi) = \mathcal{R}_P(w, \pi_{M,0}, \psi)$$

is therefore defined ([6, Lemma 7.2.]).

Now let $w \in \mathrm{W}_\phi(M, \mathrm{Mp}(W)) = \mathrm{W}_\phi(L_{\underline{k}}, \mathrm{SO}(V))$, and we define the normalized self-intertwining operator

$$R_P(w, \pi_M) : \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M) \longrightarrow \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M)$$

by

$$[R_P(w,\pi_M)\mathscr{F}](g) = \mathcal{A}_w \otimes 1_{\mathscr{V}_{\pi_0}} \left(\mathcal{R}_P(w,\pi_M,\psi)\mathscr{F}(g)\right).$$

Our main theorem is

Theorem 4.1. *Assume the local intertwining relation for* $\mathrm{SO}(2n+1)$ *(Hypothesis 2.2). Let* $x_w \in S_\phi^\natural(M_{\underline{k}}, \mathrm{Mp}(W))$ *be the image of* $w \in \mathrm{W}_\phi(M_{\underline{k}}, \mathrm{Mp}(W))$ *under the natural surjection (3). Then, the restriction of* $R_{P_{\underline{k}}}(w, \widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0)$ *to* $\pi \subset \mathrm{Ind}_{P_{\underline{k}}}^{\mathrm{Mp}(W)}(\widetilde{\tau_1} \otimes \cdots \otimes \widetilde{\tau_m} \otimes \pi_0)$ *is the scalar multiplication by* $\iota_\psi(\pi)(x_w)$*.*

Our main results are this theorem and the definition of the intertwining operator $R_P(w,\pi_M)$. Especially, the Weil index and the central value of a gamma factor in the normalizing factor could not be seen in the case of classical groups.

5 Outline of Proof

Since the LLC for $\mathrm{Mp}(2n)$ is defined by using the theta correspondence, it suffices for proving Theorem 4.1 to consider the relation between the theta correspondence and the intertwining operators. Therefore we can reduce the main theorem to the following proposition.

Proposition 5.1. *Let* ω *denotes the Weil representation of* $\mathrm{O}(V)\times\mathrm{Mp}(W)$ *relative to* ψ*. Let* $\underline{k}$ *be as above, and put* $L = L_{\underline{k}}$*,* $M = M_{\underline{k}}$*. Then for any* $\sigma_L \in \Pi_{\mathrm{temp}}(L)$ *and* $\pi_M \in \Pi_{\mathrm{temp}}(M)$ *such that* σ_L *and* π_M *are correspond via the local Shimura* Θ_ψ *relative to* ψ*, there exists a nonzero* $\mathrm{SO}(V) \times \mathrm{Mp}(W)$*-equivariant map*

$$\mathcal{T} : \omega \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L) \longrightarrow \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M)$$

such that
(a) for any irreducible constituent σ *of* $\mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L)$*, the restriction of* $\mathcal{T}$ *to* $\omega\otimes\sigma$ *is nonzero;*
(b) the diagram

$$\begin{array}{ccc} \omega \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L) & \xrightarrow{\mathcal{T}} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M) \\ {\scriptstyle 1_\omega\otimes R_Q(w,\sigma_L)}\downarrow & & \downarrow{\scriptstyle R_P(w,\pi_M)} \\ \omega \otimes \mathrm{Ind}_Q^{\mathrm{SO}(V)}(\sigma_L) & \xrightarrow{\mathcal{T}} & \mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M) \end{array}$$

commutes.

If $\underline{k}$ is an integer k, i.e., P and Q are maximal parabolic subgroups, then the proposition can be shown by using Gan-Ichino's mixed model of the Weil representation. See [3, §7.4] and [6, §9] for detail.

In the general case, we can prove Proposition 5.1 by induction in stages. Since the intertwining operators satisfy the multiplicativity in $w \in \mathrm{W}_\phi(M, \mathrm{Mp}(W)) = \mathrm{W}_\phi(L, \mathrm{SO}(V))$, we may assume that w is a simple reflection. Thus we divide into the two cases: the case of $w = (i, i+1) \ltimes 0 \in \mathrm{W}(\widehat{M}, \mathrm{Sp}(2n,\mathbb{C})) \subset \mathfrak{S}_m \ltimes (\mathbb{Z}/2\mathbb{Z})^m$ $(1 \le i \le m-1)$, and the case of $w = \mathrm{id} \ltimes (0,\ldots,0,1)$. In the first case, the intertwining operator $R_P(w,\pi_M)$ acts on $\mathrm{Ind}^{\mathrm{GL}(k)}(\tau_1 \otimes \cdots \otimes \tau_m)$ in an expression

$$\mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M) = \mathrm{Ind}^{\mathrm{Mp}(W)}(\widetilde{\mathrm{Ind}^{\mathrm{GL}(k)}(\tau_1 \otimes \cdots \otimes \tau_m)} \otimes \pi_0),$$

where Ind denote appropriate parabolic inductions, and the assertion follows from the mixed model and the theory for GL_k. Here, we have put $k = k_1 + \cdots + k_m$. In the second case, we have another expression

$$\mathrm{Ind}_P^{\mathrm{Mp}(W)}(\pi_M) = \mathrm{Ind}^{\mathrm{Mp}(W)}(\widetilde{\mathrm{Ind}^{\mathrm{GL}(k')}(\tau_1 \otimes \cdots \otimes \tau_{m-1})} \otimes \mathrm{Ind}^{\mathrm{Mp}(W')}(\widetilde{\tau_m} \otimes \pi_0)),$$

where $k' = k_1 + \cdots + k_{m-1}$, and W' is a $(n - k')$-dimensional symplectic subspace of W including W_0. Now the proposition can be proved by using the mixed model twice.

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