多変数 Bernoulli 多項式

渋川 元樹 (神戸大学)

概 要

Jack 多項式や補間 Jack 多項式等から Bernoulli 多項式の多変数類似を導入し, その基本的性質について述べる. また更にその多重化についても言及する.

1 Introduction

古典的な Bernoulli 数及び Bernoulli 多項式は, 母函数を用いて

$$\frac{u}{e^u-1}=\sum_{m=0}^{\infty}\frac{B_m}{m!}u^m,$$
$$\frac{u}{e^u-1}e^{zu}=\sum_{m=0}^{\infty}\frac{B_m(z)}{m!}u^m$$

によって定義される. この母函数から Bernoulli 多項式に関する次の諸公式が得られる [E]:

$$B_m(0)=B_m, \tag{1}$$
$$B_m(z+1)-B_m(z)=mz^{m-1}\quad(m\geq 0), \tag{2}$$
$$B'_m(z)=mB_{m-1}(z), \tag{3}$$
$$B_m(1-z)=(-1)^mB_m(z), \tag{4}$$
$$B_m(z)=\sum_{j=0}^{m}\binom{m}{j}B_jx^{m-j}, \tag{5}$$
$$z^m=\frac{1}{m+1}\sum_{n=0}^{m}\binom{m+1}{n}B_n(z)=\sum_{n=0}^{m}\frac{1}{m-n+1}\binom{m}{n}B_n(z), \tag{6}$$
$$\sum_{i=0}^{N-1}B_m\left(z+\frac{i}{N}\right)=N^{1-m}B_m(Nz), \tag{7}$$
$$B_m(z+1)=\sum_{i=0}^{m}\binom{m}{i}B_i\left(z\right). \tag{8}$$

Bernoulli 多項式に関しては様々な変奏が知られているが, 多変数類似は余り知られていないように思われる. ここでは Jack 多項式他を用いた多変数 Bernoulli 多項式を導入し, それが上述の (1)-(8) と類似の性質を満たすことを示す.

2 Prelimilaries

ここでは [Ka], [Ko], [L], [M], [S], [VK] から必要となる多変数類似について list する．まず r を正の整数, d を複素数として,

$$
\begin{aligned}
\mathcal{P} &:= \{\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r \mid m_1 \geq \cdots \geq m_r \geq 0\}, \\
\delta &:= (r-1, r-2, \ldots, 2, 1, 0) \in \mathcal{P}, \\
e_{r,k}(\mathbf{z}) &:= \sum_{1 \leq i_1 < \cdots < i_k \leq r} z_{i_1} \cdots z_{i_k} \quad (k = 1, \ldots, r), \quad e_{r,0}(\mathbf{z}) := 1, \quad |\mathbf{z}| := e_{r,1}(\mathbf{z}), \\
m_{\mathbf{m}}(\mathbf{z}) &:= \sum_{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathfrak{S}_r.\mathbf{m}} z_1^{\lambda_1} \cdots z_r^{\lambda_r}, \quad \mathfrak{S}_r.\mathbf{m} := \left\{(m_{\sigma(1)}, \ldots, m_{\sigma(r)}) \in \mathbb{Z}_{\geq 0}^r \mid \sigma \in \mathfrak{S}_r\right\}, \\
E_k(\mathbf{z}) &:= \sum_{j=1}^{r} z_j^k \partial_{z_j}, \quad D_k(\mathbf{z}) := \sum_{j=1}^{r} z_j^k \partial_{z_j}^2 + d \sum_{1 \leq j \neq l \leq r} \frac{z_j^k}{z_j - z_l} \partial_{z_j} \quad (k \in \mathbb{Z}_{\geq 0})
\end{aligned}
$$

とする．任意の分割 $\mathbf{m} = (m_1, \ldots, m_r) \in \mathcal{P}$ と複素数の r 組 $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$ について, **Jack 多項式 (Jack polynomials)** $P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)$ を以下の 2 条件を満たす $|\mathbf{m}|$ 次斉次多項式として定義する:

$$
\begin{aligned}
&(1)\ D_2(\mathbf{z}) P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) \sum_{j=1}^{r} m_j \left(m_j - 1 - d(r-j)\right), \\
&(2)\ P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) = m_{\mathbf{m}}(\mathbf{z}) + \sum_{\mathbf{k} < \mathbf{m}} c_{\mathbf{mk}} m_{\mathbf{k}}(\mathbf{z}).
\end{aligned}
$$

また**補間 Jack 多項式 (interpolation Jack polynomials)** $P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{z}; \frac{d}{2}\right)$ を次の 2 条件で定める:

$$
\begin{aligned}
&(1)^{\mathrm{ip}}\ P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right) = 0, \quad \text{unless } \mathbf{k} \subset \mathbf{m} \in \mathcal{P} \\
&(2)^{\mathrm{ip}}\ P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{z}; \frac{d}{2}\right) = P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right) + (\text{lower terms}).
\end{aligned}
$$

更に便宜上, 次のようにおく:

$$
\begin{aligned}
\Phi_{\mathbf{m}}^{(d)}(\mathbf{z}) &:= \frac{P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)}{P_{\mathbf{m}}\left(\mathbf{1}; \frac{d}{2}\right)} \quad \text{(normalized Jack polynomials)}, \\
\Psi_{\mathbf{m}}^{(d)}(\mathbf{z}) &:= \frac{P_{\mathbf{m}}\left(\mathbf{z}; \frac{d}{2}\right)}{P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right)} = \frac{P_{\mathbf{m}}\left(\mathbf{1}; \frac{d}{2}\right)}{P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{m} + \frac{d}{2}\delta; \frac{d}{2}\right)} \Phi_{\mathbf{m}}^{(d)}(\mathbf{z}), \\
\binom{\mathbf{z}}{\mathbf{k}}^{(d)} &:= \frac{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{z} + \frac{d}{2}\delta; \frac{d}{2}\right)}{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{k} + \frac{d}{2}\delta; \frac{d}{2}\right)} \quad \text{(generalized (or Jack) binomial coefficients)}, \\
{}_0\mathcal{F}_0{}^{(d)}\left(; \mathbf{z}, \mathbf{u}\right) &:= \sum_{\mathbf{m} \in \mathcal{P}} \Psi_{\mathbf{m}}^{(d)}(\mathbf{z}) \Phi_{\mathbf{m}}^{(d)}(\mathbf{u}) = \sum_{\mathbf{m} \in \mathcal{P}} \Phi_{\mathbf{m}}^{(d)}(\mathbf{z}) \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).
\end{aligned}
$$

主結果の導出に必要となる Jack 多項式及び補間 Jack 多項式の性質を挙げる.

Special values [M] (10.20), [Ko] (4.8) より

$$
P_{\mathbf{m}}\left(\mathbf{1}; \frac{d}{2}\right) = \prod_{(i,j) \in \mathbf{m}} \frac{j - 1 + \frac{d}{2}(r - i + 1)}{m_i - j + \frac{d}{2}(m'_j - i + 1)} = \prod_{1 \leq i < j \leq r} \frac{\left(\frac{d}{2}(j - i + 1)\right)_{m_i - m_j}}{\left(\frac{d}{2}(j - i)\right)_{m_i - m_j}}.
$$

また [Ko] の (7.4), (7.5) より

$$P^{\mathrm{ip}}_{\mathbf{m}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)=\prod_{(i,j)\in\mathbf{m}}\left(m_i-j+1+\frac{d}{2}(m'_j-i)\right)$$
$$=\prod_{j=1}^{r}\left(\frac{d}{2}(r-j)+1\right)_{m_j}\prod_{1\le i<j\le r}\frac{\left(\frac{d}{2}(j-i-1)+1\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i)+1\right)_{m_i-m_j}}.$$

Pieri type formulas for Jack polynomials [L] の Section 14 より, 任意の分割 $\mathbf{m}$ について

$$e^{|\mathbf{z}|}\Psi^{(d)}_{\mathbf{m}}(\mathbf{z})=\sum_{\mathbf{n}\in\mathcal{P}}\binom{\mathbf{n}}{\mathbf{m}}^{(d)}\Psi^{(d)}_{\mathbf{n}}(\mathbf{z}).$$

これを

$$e^{|\mathbf{z}|}=\sum_{N\ge 0}\frac{1}{N!}|\mathbf{z}|^N$$

と展開して, 次数ごとに比較すると

$$\frac{|\mathbf{z}|^N}{N!}\Psi^{(d)}_{\mathbf{m}}(\mathbf{z})=\sum_{|\mathbf{n}|-|\mathbf{m}|=N}\binom{\mathbf{n}}{\mathbf{m}}^{(d)}\Psi^{(d)}_{\mathbf{n}}(\mathbf{z}). \tag{9}$$

また [L] の Section 14 より

$$\binom{\mathbf{m}^i}{\mathbf{m}}^{(d)}=\left(m_i+1+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m}^i).$$

ただし $\epsilon_i:=(0,\dots,0,\overset{\overset{i}{\vee}}{1},0,\dots,0)\in\mathbb{Z}^r$, $\mathbf{m}^i:=\mathbf{m}+\epsilon_i$,

$$h^{(d)}_{\pm,i}(\mathbf{m}):=\prod_{1\le k\ne i\le r}\frac{m_i-m_k-\frac{d}{2}(i-k)\pm\frac{d}{2}}{m_i-m_k-\frac{d}{2}(i-k)}.$$

ここで $\mathbf{m}^i\notin\mathcal{P}$ ならば $h^{(d)}_{-,i}(\mathbf{m}^i)=0$ であることに注意しておく. よって特に (9) で $N=1$ とすると,

$$|\mathbf{z}|\Psi^{(d)}_{\mathbf{m}}(\mathbf{z})=\sum_{i=1}^{r}\binom{\mathbf{m}^i}{\mathbf{m}}^{(d)}\Psi^{(d)}_{\mathbf{m}^i}(\mathbf{z})=\sum_{i=1}^{r}\Psi^{(d)}_{\mathbf{m}^i}(\mathbf{z})\left(m_i+1+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m}^i). \tag{10}$$

Properties of ${}_0\mathcal{F}_0{}^{(d)}$ ここでもやはり [L] の Section 14 より

$$E_0(\mathbf{z}){}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)={}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)|\mathbf{u}|. \tag{11}$$

これから特に

$${}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{1}+\mathbf{z},\mathbf{u}\right)=e^{E_0(\mathbf{z})}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)={}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)e^{|\mathbf{u}|}. \tag{12}$$

Other formulas [S] Prop. 2.3 or [Ka] (17) より

$$|\mathbf{z}|^N = N! \sum_{|\mathbf{m}|=N} \Psi_{\mathbf{m}}^{(d)}(\mathbf{z}). \tag{13}$$

まとめると, 次のようになる:

$$P_m\left(1;\frac{d}{2}\right)=1 \implies P_{\mathbf{m}}\left(\mathbf{1};\frac{d}{2}\right)=\prod_{1\le i<j\le r}\frac{\left(\frac{d}{2}(j-i+1)\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i)\right)_{m_i-m_j}},$$

$$P_m^{\mathrm{ip}}\left(m;\frac{d}{2}\right)=m! \implies P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)=\prod_{j=1}^{r}\left(\frac{d}{2}(r-j)+1\right)_{m_j}$$
$$\times\prod_{1\le i<j\le r}\frac{\left(\frac{d}{2}(j-i-1)+1\right)_{m_i-m_j}}{\left(\frac{d}{2}(j-i)+1\right)_{m_i-m_j}},$$

$$\Phi_m^{(d)}(z):=z^m \implies \Phi_{\mathbf{m}}^{(d)}(\mathbf{z}):=\frac{P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right)}{P_{\mathbf{m}}\left(\mathbf{1};\frac{d}{2}\right)},$$

$$\Psi_m^{(d)}(z):=\frac{z^m}{m!} \implies \Psi_{\mathbf{m}}^{(d)}(\mathbf{z}):=\frac{P_{\mathbf{m}}\left(\mathbf{1};\frac{d}{2}\right)\Phi_{\mathbf{m}}^{(d)}(\mathbf{z})}{P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)}=\frac{P_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right)}{P_{\mathbf{m}}^{\mathrm{ip}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)},$$

$$\binom{m}{k}:=\frac{P_k^{\mathrm{ip}}\left(m;\frac{d}{2}\right)}{P_k^{\mathrm{ip}}\left(k;\frac{d}{2}\right)} \implies \binom{\mathbf{m}}{\mathbf{k}}^{(d)}:=\frac{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{m}+\frac{d}{2}\delta;\frac{d}{2}\right)}{P_{\mathbf{k}}^{\mathrm{ip}}\left(\mathbf{k}+\frac{d}{2}\delta;\frac{d}{2}\right)},$$

$$e^{zw}=\sum_{m=0}^{\infty}\frac{1}{m!}z^m w^m \implies {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{w}\right):=\sum_{\mathbf{m}\in\mathcal{P}}\Psi_{\mathbf{k}}^{(d)}(\mathbf{z})\Phi_{\mathbf{k}}^{(d)}(\mathbf{w}),$$

$$e^z\frac{z^m}{m!}=\sum_{n=0}^{\infty}\binom{n}{m}\frac{z^n}{n!} \implies e^{|\mathbf{z}|}\Psi_{\mathbf{m}}^{(d)}(\mathbf{z})=\sum_{\mathbf{n}\in\mathcal{P}}\binom{\mathbf{n}}{\mathbf{m}}^{(d)}\Psi_{\mathbf{n}}^{(d)}(\mathbf{z}),$$

$$\frac{z^N}{N!}\frac{z^m}{m!}=\binom{N+m}{m}\frac{z^{N+m}}{(N+m)!} \implies \frac{|\mathbf{z}|^N}{N!}\Psi_{\mathbf{m}}^{(d)}(\mathbf{z})=\sum_{|\mathbf{n}|-|\mathbf{m}|=N}\binom{\mathbf{n}}{\mathbf{m}}^{(d)}\Psi_{\mathbf{n}}^{(d)}(\mathbf{z}),$$

$$z\frac{z^m}{m!}=\frac{z^{m+1}}{(m+1)!}(m+1) \implies |\mathbf{z}|\Psi_{\mathbf{m}}^{(d)}(\mathbf{z})=\sum_{i=1}^{r}\Psi_{\mathbf{m}^i}^{(d)}(\mathbf{z})\left(m_i+1+\frac{d}{2}(r-i)\right)h_{-,i}^{(d)}(\mathbf{m}^i),$$

$$\partial_z e^{zw}=e^{zw}w \implies E_0(\mathbf{z})_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{w}\right)={}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{w}\right)|\mathbf{w}|,$$

$$e^{(1+z)w}=e^w e^{zw} \implies {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{1}+\mathbf{z},\mathbf{w}\right)=e^{|\mathbf{w}|}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{w}\right),$$

$$z^N=N!\frac{z^N}{N!} \implies |\mathbf{z}|^N=N!\sum_{|\mathbf{m}|=N}\Psi_{\mathbf{m}}^{(d)}(\mathbf{z}).$$

3 Multivariate Bernoulli polynomials

以上の set up を踏まえ, 多変数 Bernoulli 多項式 $B_{\mathbf{m}}^{(d)}(\mathbf{z})$, あるいは $B_{\mathbf{m}}\left(\mathbf{z};\frac{d}{2}\right)$ を次の母函数で定める:

$$\frac{u}{e^u-1}e^{zu}=\sum_{m=0}^{\infty}B_m(z)\Psi_m(u) \implies \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)=\sum_{\mathbf{m}\in\mathcal{P}}B_{\mathbf{m}}^{(d)}(\mathbf{z})\,\Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).$$

Remark 1. 最初は Bernoulli 多項式の Jacobi-Trudi 行列式から定まる多変数化を念頭に

$$\prod_{j=1}^{r} \frac{u_j}{e^{u_j}-1} {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)$$

を母函数にして考えた. しかし, これは一変数に類する良い性質を見つけることができなかったので, 上述の type を母函数にすることにした. 以下, 記すように現段階ではこちらの style の多変数化の方が色々と良い多変数類似になっているように思われる.

Theorem 2. (1) $\mathbf{z}=\mathbf{0}$ の特殊値:

$$B_{\mathbf{m}}^{(d)}(\mathbf{0}) = B_{|\mathbf{m}|}.$$

(2) 差分関係式:

$$B_{\mathbf{m}}^{(d)}(\mathbf{z}+\mathbf{1}) - B_{\mathbf{m}}^{(d)}(\mathbf{z}) = \sum_{i=1}^{r} \Phi_{\mathbf{m}_i}^{(d)}(\mathbf{z}) \left(m_i + \frac{d}{2}(r-i)\right) h_{-,i}^{(d)}(\mathbf{m}).$$

(3) 微分関係式:

$$E_0(\mathbf{z}) B_{\mathbf{m}}^{(d)}(\mathbf{z}) = \sum_{i=1}^{r} B_{\mathbf{m}_i}^{(d)}(\mathbf{z}) \left(m_i + \frac{d}{2}(r-i)\right) h_{-,i}^{(d)}(\mathbf{m}).$$

(4) 対称性:

$$B_{\mathbf{m}}^{(d)}(\mathbf{1}-\mathbf{z}) = (-1)^{|\mathbf{m}|} B_{\mathbf{m}}^{(d)}(\mathbf{z})$$

(5) 展開公式:

$$B_{\mathbf{m}}^{(d)}(\mathbf{z}) = \sum_{N=0}^{|\mathbf{m}|} B_N \sum_{\mathbf{n}\subset\mathbf{m},|\mathbf{m}|-|\mathbf{n}|=N} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} \Phi_{\mathbf{n}}^{(d)}(\mathbf{z}).$$

(6) 逆表示:

$$\Phi_{\mathbf{m}}^{(d)}(\mathbf{z}) = \sum_{\mathbf{n}\subset\mathbf{m}} \frac{1}{|\mathbf{m}|-|\mathbf{n}|+1} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} B_{\mathbf{n}}^{(d)}(\mathbf{z}).$$

(7) N 倍角公式:

$$\sum_{i=0}^{N-1} B_{\mathbf{m}}^{(d)}\left(\mathbf{z}+\frac{i}{N}\right) = N^{1-|\mathbf{m}|} B_{\mathbf{m}}^{(d)}(N\mathbf{z})$$

(8) 二項公式:

$$B_{\mathbf{m}}^{(d)}(\mathbf{z}+\mathbf{1}) = \sum_{\mathbf{n}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} B_{\mathbf{n}}^{(d)}(\mathbf{z})$$

Proof. (1) 多変数 Bernoulli 多項式の母函数より

$$\sum_{\mathbf{m}\in\mathcal{P}} B_{\mathbf{m}}^{(d)}(\mathbf{0})\, \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) = \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1} = \sum_{N=0}^{\infty} \frac{B_N}{N!} |\mathbf{u}|^N.$$

ここで (13) より

$$\sum_{\mathbf{m}\in\mathcal{P}} B_{\mathbf{m}}^{(d)}(\mathbf{0})\, \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) = \sum_{N=0}^{\infty} B_N \sum_{|\mathbf{m}|=N} \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) = \sum_{\mathbf{m}\in\mathcal{P}} B_{|\mathbf{m}|} \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).$$

(2) (12) と (10) より

$$
\begin{aligned}
&\sum_{\mathbf{m}\in\mathcal{P}} \left(B^{(d)}_{\mathbf{m}}(\mathbf{z}+\mathbf{1}) - B^{(d)}_{\mathbf{m}}(\mathbf{z})\right)\Psi^{(d)}_{\mathbf{m}}(\mathbf{z}) \\
&\quad = \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}\left({}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z}+\mathbf{1},\mathbf{u}\right) - {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\right) \\
&\quad = \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}\left(e^{|\mathbf{u}|}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) - {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\right) \\
&\quad = |\mathbf{u}|{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \\
&\quad = \sum_{\mathbf{m}\in\mathcal{P}} \Phi^{(d)}_{\mathbf{m}}(\mathbf{z})|\mathbf{u}|\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) \\
&\quad = \sum_{\mathbf{m}\in\mathcal{P}} \Phi^{(d)}_{\mathbf{m}}(\mathbf{z})\sum_{i=1}^{r}\Psi^{(d)}_{\mathbf{m}^i}(\mathbf{u})\left(m_i+1+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m}^i) \\
&\quad = \sum_{\mathbf{m}\in\mathcal{P}}\sum_{i=1}^{r}\Phi^{(d)}_{\mathbf{m}_i}(\mathbf{z})\left(m_i+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}
$$

(3) (11) と (10) より

$$
\begin{aligned}
\sum_{\mathbf{m}\in\mathcal{P}} E_0(\mathbf{z})B^{(d)}_{\mathbf{m}}(\mathbf{z})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) &= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}E_0(\mathbf{z}){}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \\
&= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)|\mathbf{u}| \\
&= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{\mathbf{m}}(\mathbf{z})|\mathbf{u}|\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{\mathbf{m}}(\mathbf{z})\sum_{i=1}^{r}\Psi^{(d)}_{\mathbf{m}^i}(\mathbf{u})\left(m_i+1+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m}^i) \\
&= \sum_{\mathbf{m}\in\mathcal{P}}\sum_{i=1}^{r} B^{(d)}_{\mathbf{m}_i}(\mathbf{z})\left(m_i+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}
$$

(4) (12) より

$$
\begin{aligned}
\sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{\mathbf{m}}(\mathbf{1}-\mathbf{z})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) &= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{1}-\mathbf{z},\mathbf{u}\right) \\
&= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},-\mathbf{u}\right)e^{|\mathbf{u}|} \\
&= \frac{|-\mathbf{u}|}{e^{|-\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},-\mathbf{u}\right) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{\mathbf{m}}(\mathbf{z})\Psi^{(d)}_{\mathbf{m}}(-\mathbf{u}) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} (-1)^{|\mathbf{m}|}B^{(d)}_{\mathbf{m}}(\mathbf{z})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}
$$

(5) (9) より

$$
\sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{\mathbf{m}}(\mathbf{z})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) = \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)
$$

$$
\begin{aligned}
&= \sum_{N=0}^{\infty} \frac{B_N}{N!} |\mathbf{u}|^N \sum_{\mathbf{n}\in\mathcal{P}} \Phi_{\mathbf{n}}^{(d)}(\mathbf{z}) \Psi_{\mathbf{n}}^{(d)}(\mathbf{u}) \\
&= \sum_{N=0}^{\infty} B_N \sum_{\mathbf{n}\in\mathcal{P}} \Phi_{\mathbf{n}}^{(d)}(\mathbf{z}) \frac{|\mathbf{u}|^N}{N!} \Psi_{\mathbf{n}}^{(d)}(\mathbf{u}) \\
&= \sum_{N=0}^{\infty} B_N \sum_{\mathbf{n}\in\mathcal{P}} \Phi_{\mathbf{n}}^{(d)}(\mathbf{z}) \sum_{|\mathbf{m}|-|\mathbf{n}|=N} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} \sum_{N=0}^{|\mathbf{m}|} B_N \sum_{\mathbf{n}\subset\mathbf{m}, |\mathbf{m}|-|\mathbf{n}|=N} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} \Phi_{\mathbf{n}}^{(d)}(\mathbf{z}) \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).
\end{aligned}
$$

(6) (9) より

$$
\begin{aligned}
\sum_{\mathbf{m}\in\mathcal{P}} \Phi_{\mathbf{m}}^{(d)}(\mathbf{z}) \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) &= {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \\
&= \frac{e^{|\mathbf{u}|}-1}{|\mathbf{u}|} \sum_{\mathbf{n}\in\mathcal{P}} B_{\mathbf{n}}^{(d)}(\mathbf{z}) \Psi_{\mathbf{n}}^{(d)}(\mathbf{u}) \\
&= \sum_{N=0}^{\infty} \frac{1}{N+1} \frac{1}{N!} |\mathbf{u}|^N \sum_{\mathbf{n}\in\mathcal{P}} B_{\mathbf{n}}^{(d)}(\mathbf{z}) \Psi_{\mathbf{n}}^{(d)}(\mathbf{u}) \\
&= \sum_{N=0}^{\infty} \frac{1}{N+1} \sum_{\mathbf{n}\in\mathcal{P}} B_{\mathbf{n}}^{(d)}(\mathbf{z}) \frac{|\mathbf{u}|^N}{N!} \Psi_{\mathbf{n}}^{(d)}(\mathbf{u}) \\
&= \sum_{N=0}^{\infty} \frac{1}{N+1} \sum_{\mathbf{n}\in\mathcal{P}} B_{\mathbf{n}}^{(d)}(\mathbf{z}) \sum_{|\mathbf{m}|-|\mathbf{n}|=N} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} \sum_{\mathbf{n}\subset\mathbf{m}} \frac{1}{|\mathbf{m}|-|\mathbf{n}|+1} B_{\mathbf{n}}^{(d)}(\mathbf{z}) \binom{\mathbf{m}}{\mathbf{n}}^{(d)} \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).
\end{aligned}
$$

(7) (12) と等比級数の和の公式より

$$
\begin{aligned}
\sum_{\mathbf{m}\in\mathcal{P}} \sum_{i=0}^{N-1} B_{\mathbf{m}}^{(d)}\left(\mathbf{z}+\frac{i}{N}\mathbf{1}\right) \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) &= \sum_{i=0}^{N-1} \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1} {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z}+\frac{i}{N}\mathbf{1},\mathbf{u}\right) \\
&= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1} {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \sum_{i=0}^{N-1} e^{\frac{i}{N}|\mathbf{u}|} \\
&= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1} {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \frac{e^{|\mathbf{u}|}-1}{e^{\frac{|\mathbf{u}|}{N}}-1} \\
&= N \frac{\frac{|\mathbf{u}|}{N}}{e^{\frac{|\mathbf{u}|}{N}}-1} {}_0\mathcal{F}_0{}^{(d)}\left(;N\mathbf{z},\frac{\mathbf{u}}{N}\right) \\
&= N \sum_{\mathbf{m}\in\mathcal{P}} B_{\mathbf{m}}^{(d)}(N\mathbf{z}) \Psi_{\mathbf{m}}^{(d)}\left(\frac{\mathbf{u}}{N}\right) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} N^{1-|\mathbf{m}|} B_{\mathbf{m}}^{(d)}(N\mathbf{z}) \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}).
\end{aligned}
$$

(8) (12) より

$$
\sum_{\mathbf{m}\in\mathcal{P}} B_{\mathbf{m}}^{(d)}(\mathbf{z}+\mathbf{1}) \Psi_{\mathbf{m}}^{(d)}(\mathbf{u}) = \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1} {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z}+\mathbf{1},\mathbf{u}\right)
$$

$$
\begin{aligned}
&= \frac{|\mathbf{u}|}{e^{|\mathbf{u}|}-1} e^{|\mathbf{u}|} {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \\
&= \sum_{\mathbf{n}\in\mathcal{P}} B^{(d)}_{\mathbf{n}}(\mathbf{z})\, e^{|\mathbf{u}|} \Psi^{(d)}_{\mathbf{n}}(\mathbf{u}) \\
&= \sum_{\mathbf{n}\in\mathcal{P}} B^{(d)}_{\mathbf{n}}(\mathbf{z}) \sum_{\mathbf{m}\in\mathcal{P}} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} \sum_{\mathbf{n}\subset\mathbf{m}} \binom{\mathbf{m}}{\mathbf{n}}^{(d)} B^{(d)}_{\mathbf{n}}(\mathbf{z})\, \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}
$$

□

4 A multiple analogue of the multivariate Bernoulli polynomials

n 個の複素パラメータ

$$\boldsymbol{\omega} := (\omega_1, \ldots, \omega_n) \in \mathbb{C}^n$$

に対し, 今回の多変数 Bernoulli 多項式の更に多重化を

$${}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \prod_{i=1}^{n} \frac{|\mathbf{u}|}{e^{\omega_i |\mathbf{u}|}-1} = \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega})\, \Psi^{(d)}_{\mathbf{m}}(\mathbf{u})$$

により定めると, これは Barnes 型の Bernoulli 多項式

$$e^{zu} \prod_{j=1}^{n} \frac{u}{e^{\omega_j u}-1} = \sum_{m\geq 0} B_{n,m}(z \mid \boldsymbol{\omega})\, \Psi_m(u)$$

の多変数化になっている. 実際,

$$
\begin{aligned}
\widehat{\boldsymbol{\omega}}(j) &:= (\omega_1, \cdots, \omega_{j-1}, \omega_{j+1}, \cdots, \omega_r) \in \mathbb{C}^{r-1} \\
&= (\omega_1, \cdots, \widehat{\omega}_j, \cdots, \omega_r), \\
\boldsymbol{\omega}^-[j] &:= (\omega_1, \cdots, -\omega_j, \cdots, \omega_r) \in \mathbb{C}^r
\end{aligned}
$$

とすると $B_{n,m}(z \mid \boldsymbol{\omega})$ の満たす諸公式 [N] (12)–(17)

$$
\begin{aligned}
B_{n,m}(cz \mid c\boldsymbol{\omega}) &= c^{m-n} B_{n,m}(z \mid \boldsymbol{\omega}) \quad (c \in \mathbb{C}^*), \\
B_{n,m}(|\boldsymbol{\omega}| - z \mid \boldsymbol{\omega}) &= (-1)^m B_{n,m}(z \mid \boldsymbol{\omega}), \\
B_{n,m}(z+\omega_j \mid \boldsymbol{\omega}) - B_{n,m}(z \mid \boldsymbol{\omega}) &= m B_{n-1,m-1}(z \mid \widehat{\boldsymbol{\omega}}(j)), \\
B_{n,m}(z \mid \boldsymbol{\omega}^-[j]) &= -B_{n,m}(z+\omega_j \mid \boldsymbol{\omega}), \\
B_{n,m}(z \mid \boldsymbol{\omega}) + B_{n,m}(z \mid \boldsymbol{\omega}^-[j]) &= -m B_{n-1,m-1}(z \mid \widehat{\boldsymbol{\omega}}(j)), \\
\frac{d}{dz} B_{n,m}(z \mid \boldsymbol{\omega}) &= m B_{n,m-1}(z \mid \boldsymbol{\omega})
\end{aligned}
$$

の多変数類似が成立することがわかる.

Theorem 3. (1)
$$B^{(d)}_{n,\mathbf{m}}(c\mathbf{z}\mid c\boldsymbol{\omega})=c^{|\mathbf{m}|-n}B^{(d)}_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega})\quad (c\in\mathbb{C}^*).$$

(2)
$$B^{(d)}_{n,\mathbf{m}}(|\boldsymbol{\omega}|\mathbf{1}-\mathbf{z}\mid\boldsymbol{\omega})=(-1)^{|\mathbf{m}|}B^{(d)}_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega}).$$

(3)
$$B^{(d)}_{n,\mathbf{m}}(\mathbf{z}+\omega_j\mathbf{1}\mid\boldsymbol{\omega})-B^{(d)}_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega})=\sum_{i=1}^{r}B^{(d)}_{n-1,\mathbf{m}_i}(z\mid\widehat{\boldsymbol{\omega}}(j))\left(m_i+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m}). \quad (14)$$

(4)
$$B^{(d)}_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega}^-[j])=-B^{(d)}_{n,\mathbf{m}}(\mathbf{z}+\omega_j\mathbf{1}\mid\boldsymbol{\omega}). \quad (15)$$

(5)
$$B^{(d)}_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega})+B^{(d)}_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega}^-[j])=-\sum_{i=1}^{r}B^{(d)}_{n-1,\mathbf{m}_i}(z\mid\widehat{\boldsymbol{\omega}}(j))\left(m_i+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m}).$$

(6)
$$E_0(\mathbf{z})B^{(d)}_{n,\mathbf{m}}(\mathbf{z}\mid\boldsymbol{\omega})=\sum_{i=1}^{r}B^{(d)}_{n,\mathbf{m}_i}(z\mid\boldsymbol{\omega})\left(m_i+\frac{d}{2}(r-i)\right)h^{(d)}_{-,i}(\mathbf{m}).$$

Proof. (1) 多重多変数 Bernoulli 多項式の母函数と Jack 多項式の斉次性より

$$\begin{aligned}\sum_{\mathbf{m}\in\mathcal{P}}B^{(d)}_{n,\mathbf{m}}(c\mathbf{z}\mid c\boldsymbol{\omega})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u})&={}_0\mathcal{F}_0{}^{(d)}\left(;c\mathbf{z},\mathbf{u}\right)\prod_{i=1}^{n}\frac{|\mathbf{u}|}{e^{c\omega_i|\mathbf{u}|}-1}\\&=c^{-n}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},c\mathbf{u}\right)\prod_{i=1}^{n}\frac{|c\mathbf{u}|}{e^{\omega_i|c\mathbf{u}|}-1}\\&=c^{-n}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},c\mathbf{u}\right)\prod_{i=1}^{n}\frac{|c\mathbf{u}|}{e^{\omega_i|c\mathbf{u}|}-1}\\&=c^{-n}\sum_{\mathbf{m}\in\mathcal{P}}B^{(d)}_{n,\mathbf{m}}\left(\mathbf{z}\mid\boldsymbol{\omega}\right)\Psi^{(d)}_{\mathbf{m}}(c\mathbf{u})\\&=\sum_{\mathbf{m}\in\mathcal{P}}c^{|\mathbf{m}|-n}B^{(d)}_{n,\mathbf{m}}\left(\mathbf{z}\mid\boldsymbol{\omega}\right)\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).\end{aligned}$$

(2) (12) より

$$\begin{aligned}\sum_{\mathbf{m}\in\mathcal{P}}B^{(d)}_{n,\mathbf{m}}(|\boldsymbol{\omega}|\mathbf{1}-\mathbf{z}\mid\boldsymbol{\omega})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u})&={}_0\mathcal{F}_0{}^{(d)}\left(;|\boldsymbol{\omega}|\mathbf{1}-\mathbf{z},\mathbf{u}\right)\prod_{i=1}^{n}\frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1}\\&=e^{|\boldsymbol{\omega}||\mathbf{u}|}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},-\mathbf{u}\right)\prod_{i=1}^{n}\frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1}\\&={}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},-\mathbf{u}\right)\prod_{i=1}^{n}\frac{|\mathbf{u}|e^{\omega_i|\mathbf{u}|}}{e^{\omega_i|\mathbf{u}|}-1}\\&={}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},-\mathbf{u}\right)\prod_{i=1}^{n}\frac{|-\mathbf{u}|}{e^{-\omega_i|\mathbf{u}|}-1}\end{aligned}$$

$$= \sum_{\mathbf{m}\in\mathcal{P}} (-1)^{|\mathbf{m}|} B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}) \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).$$

(3) (12) と (10) より

$$\begin{aligned}
&\sum_{\mathbf{m}\in\mathcal{P}} (B^{(d)}_{n,\mathbf{m}}(\mathbf{z}+\omega_j\mathbf{1} \mid \boldsymbol{\omega}) - B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}))\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) \\
&= \left({}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z}+\omega_j\mathbf{1},\mathbf{u}\right) - {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right)\right) \prod_{i=1}^{n} \frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1} \\
&= (e^{\omega_j|\mathbf{u}|}-1){}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \prod_{i=1}^{n} \frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1} \\
&= |\mathbf{u}|{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \prod_{1\le i\ne j\le n} \frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1} \\
&= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{n-1,\mathbf{m}}(z \mid \widehat{\boldsymbol{\omega}}(j))|\mathbf{u}|\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{n-1,\mathbf{m}}(z \mid \widehat{\boldsymbol{\omega}}(j)) \sum_{i=1}^{r} \Psi^{(d)}_{\mathbf{m}^i}(\mathbf{u}) \left(m_i+1+\frac{d}{2}(r-i)\right) h^{(d)}_{-,i}(\mathbf{m}^i) \\
&= \sum_{\mathbf{m}\in\mathcal{P}} \sum_{i=1}^{r} B^{(d)}_{n-1,\mathbf{m}_i}(z \mid \widehat{\boldsymbol{\omega}}(j)) \left(m_i+\frac{d}{2}(r-i)\right) h^{(d)}_{-,i}(\mathbf{m})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}$$

(4) (12) より

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathcal{P}} B_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}^-[j])\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) &= {}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \frac{|\mathbf{u}|}{e^{-\omega_j|\mathbf{u}|}-1} \prod_{1\le i\ne j\le n} \frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1} \\
&= -e^{\omega_j|\mathbf{u}|}{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \frac{|\mathbf{u}|}{e^{\omega_j|\mathbf{u}|}-1} \prod_{1\le i\ne j\le n} \frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1} \\
&= -{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z}+\omega_j\mathbf{1},\mathbf{u}\right) \prod_{i=1}^{n} \frac{|\mathbf{u}|}{e^{\omega_i|\mathbf{u}|}-1} \\
&= \sum_{\mathbf{m}\in\mathcal{P}} -B_{n,\mathbf{m}}(\mathbf{z}+\omega_j\mathbf{1} \mid \boldsymbol{\omega})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).
\end{aligned}$$

(5) (15) と (14) より

$$\begin{aligned}
B_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}) + B_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}^-[j]) &= B_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}) - B_{n,\mathbf{m}}(\mathbf{z}+\omega_j\mathbf{1} \mid \boldsymbol{\omega}) \\
&= -\sum_{i=1}^{r} B^{(d)}_{n-1,\mathbf{m}_i}(z \mid \widehat{\boldsymbol{\omega}}(j)) \left(m_i+\frac{d}{2}(r-i)\right) h^{(d)}_{-,i}(\mathbf{m}).
\end{aligned}$$

(6) (11) と (10) より

$$\begin{aligned}
\sum_{\mathbf{m}\in\mathcal{P}} E_0(\mathbf{z}) B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega})\Psi^{(d)}_{\mathbf{m}}(\mathbf{u}) &= E_0(\mathbf{z}){}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \prod_{i=1}^{n} \frac{|\mathbf{u}|}{e^{c\omega_i|\mathbf{u}|}-1} \\
&= |\mathbf{u}|{}_0\mathcal{F}_0{}^{(d)}\left(;\mathbf{z},\mathbf{u}\right) \prod_{i=1}^{n} \frac{|\mathbf{u}|}{e^{c\omega_i|\mathbf{u}|}-1} \\
&= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega})|\mathbf{u}|\Psi^{(d)}_{\mathbf{m}}(\mathbf{u})
\end{aligned}$$

$$= \sum_{\mathbf{m}\in\mathcal{P}} B^{(d)}_{n,\mathbf{m}}(\mathbf{z} \mid \boldsymbol{\omega}) \sum_{i=1}^{r} \Psi^{(d)}_{\mathbf{m}^i}(\mathbf{u}) \left(m_i + 1 + \frac{d}{2}(r-i) \right) h^{(d)}_{-,i}(\mathbf{m}^i)$$
$$= \sum_{i=1}^{r} B^{(d)}_{n,\mathbf{m}_i}(z \mid \boldsymbol{\omega}) \left(m_i + \frac{d}{2}(r-i) \right) h^{(d)}_{-,i}(\mathbf{m}) \Psi^{(d)}_{\mathbf{m}}(\mathbf{u}).$$

□

5 Concluding remarks

以上のように, 我々の多変数 Bernoulli 多項式 $B^{(d)}_{\mathbf{m}}(\mathbf{z})$ は一変数の諸公式の類似が数多く成立する良い多変数類似であることがわかる. あとはこれに付随する何かしらの zeta 函数の類似物 (負の特殊値, ただし負の整数ではなく負の partitions?, がこの多変数 Bernoulli 多項式になるような zeta) も構成できれば面白いが, それが何であるかは今のところ見当がつかない.

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