On the exponential Diophantine equation $a^x + b^y = c^z$

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Abstract

Let m be a positive integer, and let p be a prime with $p \equiv 1 \pmod 4$. Then we show that the exponential Diophantine equation $(3pm^2-1)^x + (p(p-3)m^2+1)^y = (pm)^z$ and $(12m^2+1)^x + (13m^2-1)^y = (5m)^z$ have only the positive integer solution $(x, y, z) = (1, 1, 2)$ under some conditions, respectively. As a corollary, we derive that the exponential Diophantine equation $(15m^2-1)^x + (10m^2+1)^y = (5m)^z$ has only the positive integer solution $(x, y, z) = (1, 1, 2)$. The proof is based on elementary methods and Baker's method.

1 Introduction

Let a, b, c be fixed relatively prime positive integers greater than one. The exponential Diophantine equation

$$a^x + b^y = c^z \tag{1.1}$$

in positive integers x, y, z has been actively studied by a number of authors. This field has a rich history. Using elementary methods such as congruences, the quadratic reciprocity law and the arithmetic of quadratic (or cubic) fields, we can completely solve most of equation (1.1) for small values of a, b, c. (cf. Nagell[N], Hadano[H] and Uchiyama[U].) It is known that the number of solutions (x, y, z) is finite, and all solutions can be effectively determined by means of Baker's method for linear forms in logarithms.

In 1956, Sierpiński[S1] showed that the equation $3^x + 4^y = 5^z$ has only the positive integer solution $(x, y, z) = (2, 2, 2)$. Jeśmanowicz[J] proved that the only positive integer solution of each of the equations

$$5^x + 12^y = 13^z, \quad 7^x + 24^y = 25^z, \quad 9^x + 40^y = 41^z, \quad 11^x + 60^y = 61^z$$

is given by $(x, y, z) = (2, 2, 2)$, and proposed the following conjecture concerning primitive Pythagorean triples:

Conjecture 1.1 (Jeśmanowicz' Conjecture)**.** *Let m, n be relatively prime positive integers with $m \not\equiv n \pmod 2$ and $m > n$. Then the exponential Diophantine equation*

$$(m^2 - n^2)^x + (2mn)^y = (m^2 + n^2)^z$$

has only the positive integer solution $(x, y, z) = (2, 2, 2)$.

This is one of famous unsolved problems in the field of exponential Diophantine equations. Conjecture 1.1 has been verified to be true in many special cases:

- $n = 1$ (Lu[Lu], 1959), $m - n = 1$ (Demjanenko[D], 1965)
- $m^2 - n^2 \equiv \pm 1 \pmod{2mn}$, $m^2 + n^2 \equiv 1 \pmod{2mn}$ (Miyazaki[M2], 2013)
- $n = 2$ (Teari[T3], 2014), $n \equiv 2 \pmod 4$ and $n < 100$ (Miyazaki-Terai[MT2], 2015)

As an analogue of Jeśmanowicz' conjecture, the author proposed the following conjecture:

Conjecture 1.2 (Generalized Jeśmanowicz' conjecture: Terai's Conjecture)**.** *Let a, b, c, p, q, r be fixed positive integers satisfying $a^p + b^q = c^r$ with $p, q, r \geq 2$ and $\gcd(a, b) = 1$. Then the exponential Diophantine equation*

$$a^x + b^y = c^z$$

has only the positive integer solution $(x, y, z) = (p, q, r)$ except for three cases (taking $a < b$), where the equation has only the following solutions, respectively:

$$\begin{aligned} (a,b,c) &= (2, 2^k - 1, 2^k + 1), \quad & (x,y,z) &= (1,1,1),\ (k+2,2,2), \\ (a,b,c) &= (2,7,3), \quad & (x,y,z) &= (1,1,2),\ (5,2,4); \\ (a,b,c) &= (1,2,3), \quad & (x,y,z) &= (m,1,1),\ (n,3,2); \end{aligned}$$

where m, n are arbitrary and k is a positive integer with $k \geq 2$.

Conjecture 1.2 has been proved to be true in many cases. This conjecture, however, are still unsolved. (cf. [C], [M1], [T1].)

In the previous paper Terai[T2], the first author showed that if m is a positive integer with $1 \leq m \leq 20$ or $m \not\equiv 3 \pmod 6$, then the Diophantine equation

$$(4m^2 + 1)^x + (5m^2 - 1)^y = (3m)^z \tag{1.2}$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$. The proof is based on elementary methods and Baker's method. Su-Li[SL] proved that if $m \geq 90$ and $m \equiv 3 \pmod 6$, then equation (1.2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$ by means of the result of Bilu-Hanrot-Voutier [BHV] concerning the existence of primitive prime divisors in Lucas-numbers. Recently, Bertók has completely solved the remaining cases that $20 < m < 90$ and $m \equiv 3 \pmod 6$ via the help of exponential congruences. (cf. Bertók-Hajdu [BH].) In [MT1], we also showed that the Diophantine equations

$$(m^2 + 1)^x + (cm^2 - 1)^y = (am)^z \quad \text{with} \quad 1 + c = a^2,$$

have only the positive integer solution $(x, y, z) = (1, 1, 2)$ under some conditions, respectively. Moreover, Fu-Yang[FY] has recently shown that if $p + q = r^2$, $r|m$, $m > 36r^3 \log r$, then the Diophantine equation

$$(pm^2 + 1)^x + (qm^2 - 1)^y = (rm)^z,$$

have only the positive integer solution $(x, y, z) = (1, 1, 2)$.

In this paper, we consider the exponential Diophantine equations

$$(3pm^2 - 1)^x + (p(p-3)m^2 + 1)^y = (pm)^z \tag{1.3}$$

$$(12m^2 + 1)^x + (13m^2 - 1)^y = (5m)^z \tag{1.4}$$

with m positive integer and p prime . Our main results are the following:

Theorem 1.1. *Let m be a positive integer with $m \not\equiv 0 \pmod 3$. Let p be a prime with $p \equiv 1 \pmod 4$. Moreover, suppose that if $m \equiv 1 \pmod 4$, then $p < 3784$. Then equation* (1.3) *has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

Theorem 1.2. *Let m be a positive integer with $m \not\equiv 17, 33 \pmod{40}$. Then equation* (1.4) *has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

In particular, for $p=5$, we can completely solve equation (1.3) without any assumption on m. The proof is based on applying a result on linear forms in p-adic logarithms due to Bugeaud [Bu] to equation (1.3) with $m \equiv 0 \pmod 3$.

Corollary 1.1. *Then the exponential Diophantine equation*

$$(15m^2-1)^x + (10m^2+1)^y = (5m)^z \tag{1.5}$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

2 Preliminaries

In order to obtain an upper bound for a solution y of Pillai's equation $c^z - b^y = a$ under some conditions, we need a result on lower bounds for linear forms in the logarithms of two algebraic numbers. We will introduce here some notations. Let α_1 and α_2 be real algebraic numbers with $|\alpha_1| \geq 1$ and $|\alpha_2| \geq 1$. We consider the linear form

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where b_1 and b_2 are positive integers. As usual, the *logarithmic height* of an algebraic number α of degree n is defined as

$$h(\alpha) = \frac{1}{n}\left(\log|a_0| + \sum_{j=1}^{n} \log\max\left\{1, |\alpha^{(j)}|\right\}\right),$$

where a_0 is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$) and $(\alpha^{(j)})_{1\leq j\leq n}$ are the conjugates of α. Let A_1 and A_2 be real numbers greater than 1 with

$$\log A_i \geq \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\},$$

for $i \in \{1, 2\}$, where D is the degree of the number field $\mathbb{Q}(\alpha_1, \ \alpha_2)$ over $\mathbb{Q}$. Define

$$b' = \frac{b_1}{D\log A_2} + \frac{b_2}{D\log A_1}.$$

We choose to use a result due to Laurent [[L], Corollary 2] with $m = 10$ and $C_2 = 25.2$.

Proposition 1 (Laurent[L]). *Let Λ be given as above, with $\alpha_1 > 1$ and $\alpha_2 > 1$. Suppose that α_1 and α_2 are multiplicatively independent. Then*

$$\log|\Lambda| \geq -25.2\, D^4 \left(\max\left\{\log b' + 0.38, \frac{10}{D}\right\}\right)^2 \log A_1 \log A_2.$$

Next, we shall quote a result on linear forms in p-adic logarithms due to Bugeaud [Bu]. Here we consider the case where $y_1 = y_2 = 1$ in the notation from [Bu, p.375].

Let p be an odd prime. Let a_1 and a_2 be non-zero integers prime to p. Let g be the least positive integer such that

$$\mathrm{ord}_p(a_1^g - 1) \geq 1, \quad \mathrm{ord}_p(a_2^g - 1) \geq 1,$$

where we denote the p-adic valuation by $\mathrm{ord}_p(\,\cdot\,)$. Assume that there exists a real number E such that

$$1/(p-1) < E \leq \mathrm{ord}_p(a_1^g - 1).$$

We consider the integer

$$\Lambda = a_1^{b_1} - a_2^{b_2},$$

where b_1 and b_2 are positive integers. We let A_1 and A_2 be real numbers greater than 1 with

$$\log A_i \geq \max\{\log |a_i|,\, E \log p\} \quad (i = 1, 2),$$

and we put $b' = b_1/\log A_2 + b_2/\log A_1$.

Proposition 2 (Bugeaud [Bu]). *With the above notation, if a_1 and a_2 are multiplicatively independent, then we have the upper estimate*

$$\mathrm{ord}_p(\Lambda) \leq \frac{36.1g}{E^3(\log p)^4}\big(\max\{\log b' + \log(E\log p) + 0.4,\, 6E\log p,\, 5\}\big)^2 \log A_1 \log A_2.$$

3 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1.

Let (x, y, z) be a solution of (1.3). Taking (1.3) modulo p implies that $(-1)^x + 1 \equiv 0 \pmod{p}$. Hence x is odd.

3.1 the case where m is even

Using a congruence method, we ca easily show that if m is even, then equation (1.3) has only the positive integer solution $(x, y, z) = (1, 1, 2)$.

Lemma 3.1. *If m is even, then equation* (1.3) *has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

Proof. If $z \leq 2$, then $(x, y, z) = (1, 1, 2)$ from (1.3). Hence we may suppose that $z \geq 3$. Taking (1.3) modulo m^3 implies that

$$-1 + 3pm^2x + 1 + p(p-3)m^2y \equiv 0 \pmod{m^3},$$

so

$$3px + p(p-3)y \equiv 0 \pmod{m},$$

which is impossible, since x is odd and m is even. We therefore conclude that if m is even, then equation (1.3) has only the positive integer solution $(x, y, z) = (1, 1, 2)$. □

3.2 the case where m is odd with $m \not\equiv 0 \pmod{3}$

Lemma 3.2. *If m is odd with $m \not\equiv 0 \pmod{3}$, then $x = 1$.*

Proof. Suppose that $x \geq 2$. We show that this will lead to a contradiction. The proof is devided into two cases: Case 1: $m \equiv 1 \pmod{4}$, Case 2: $m \equiv 3 \pmod{4}$.
Case 1: $m \equiv 1 \pmod{4}$. Then, taking (1.3) modulo 4 implies that $3^y \equiv 1 \pmod{4}$, so y is even.

On the other hand, taking (1.3) modulo 3, together with our assumption $m \not\equiv 0 \pmod{3}$, implies that

$$(-1)^x + (-1)^y \equiv (pm)^z \not\equiv 0 \pmod{3}, \tag{3.1}$$

which contradicts the fact that x is odd and y is even. Hence we obtain $x = 1$.

Case 2: $m \equiv 3 \pmod 4$. Then $\left(\dfrac{3pm^2-1}{p(p-3)m^2+1}\right) = 1$ and $\left(\dfrac{pm}{p(p-3)m^2+1}\right) = -1$, where $\left(\dfrac{*}{*}\right)$ denotes the Jacobi symbol. Indeed,

$$\left(\frac{3pm^2-1}{p(p-3)m^2+1}\right) = \left(\frac{3pm^2+p(p-3)m^2}{p(p-3)m^2+1}\right) = \left(\frac{p^2m^2}{p(p-3)m^2+1}\right) = 1$$

and

$$\begin{aligned}\left(\frac{pm}{p(p-3)m^2+1}\right) &= \left(\frac{p}{p(p-3)m^2+1}\right)\left(\frac{m}{p(p-3)m^2+1}\right) \\ &= -\left(\frac{p(p-3)m^2+1}{p}\right)\left(\frac{p(p-3)m^2+1}{m}\right) \\ &= -1,\end{aligned}$$

since $m \equiv 3 \pmod 4$ and $p \equiv 1 \pmod 4$. In view of these, z is even from (1.3).

Taking (1.3) modulo 4 implies that $3^y \equiv (pm)^z \equiv 3^z \equiv 1 \pmod 4$, since z is even. Hence y is even. Similarly, (3.1) also leads to a contradiction. We therefore obtain $x = 1$. □

3.3 Pillai's equation $c^z - b^y = a$

From Lemma 3.2, it follows that $x = 1$ in (1.3), provided that m is odd with $m \not\equiv 0 \pmod 3$. If $y \leq 2$, then we obtain $y = 1$ and $z = 2$ from (1.3). From now on, we may suppose that $y \geq 3$. Hence our theorem is reduced to solving Pillai's equation

$$c^z - b^y = a \tag{3.2}$$

with $y \geq 3$, where $a = 3pm^2 - 1$, $b = p(p-3)m^2 + 1$ and $c = pm$.

We now want to obtain a lower bound for y.

Lemma 3.3. $y > m^2 - 2$.

Proof. Since $y \geq 3$, equation (3.2) yields the following inequality:

$$(pm)^z \geq 3pm^2 - 1 + (p(p-3)m^2+1)^3 > (pm)^3.$$

Hence $z \geq 4$. Taking (3.2) modulo p^2m^4 implies that

$$3pm^2 - 1 + 1 + p(p-3)ym^2 \equiv 0 \pmod{p^2m^4},$$

so $3 + (p-3)y \equiv 0 \pmod{pm^2}$. Hence we have

$$y \geq \frac{1}{p-3}(pm^2-3) = \frac{p}{p-3}m^2 - \frac{3}{p-3} > m^2 - 2,$$

as desired. □

We next want to obtain an upper bound for y.

Lemma 3.4. $y < 2521 \log c$.

Proof. From (3.2), we now consider the following linear form in two logarithms:

$$\Lambda = z \log c - y \log b \quad (> 0).$$

Using the inequality $\log(1+t) < t$ for $t > 0$, we have

$$0 < \Lambda = \log(\frac{c^z}{b^y}) = \log(1 + \frac{a}{b^y}) < \frac{a}{b^y}. \tag{3.3}$$

Hence we obtain

$$\log \Lambda < \log a - y \log b. \tag{3.4}$$

On the other hand, we use Proposition 1 to obtain a lower bound for Λ. It follows from Proposition 1 that

$$\log \Lambda \ \geq \ -25.2 \ \left(\max\left\{\log b' + 0.38, 10\right\}\right)^2 \ (\log b)\ (\log c), \tag{3.5}$$

where $b' = \dfrac{y}{\log c} + \dfrac{z}{\log b}$.

We note that $b^{y+1} > c^z$. Indeed,

$$b^{y+1} - c^z = b(c^z - a) - c^z = (b-1)c^z - ab \geq p(p-3)m^2 \cdot p^2 m^2 - (3pm^2 - 1)(p(p-3)m^2 + 1) > 0.$$

Hence $b' < \dfrac{2y+1}{\log c}$.

Put $M = \dfrac{y}{\log c}$. Combining (3.4) and (3.5) leads to

$$y \log b < \log a + 25.2 \ \left(\max\left\{\log\left(2M + \frac{1}{\log c}\right) + 0.38,\ 10\right\}\right)^2 \ (\log b)\ (\log c),$$

so

$$M < 1 + 25.2 \ \left(\max\left\{\log\left(2M + 1\right) + 0.38,\ 10\right\}\right)^2,$$

since $\log c = \log(pm) \geq \log 5 > 1$. We therefore obtain $M < 2521$. This completes the proof of Lemma 3.4. $\square$

We are now in a position to prove Theorem 1.1. It follows from Lemmas 3.3, 3.4 that

$$m^2 - 2 < 2521 \log(pm). \tag{3.6}$$

We want to obtain an upper bound for p and then one for m. We first show that if $m \equiv 3 \pmod 4$, then $p < 3784$. Recall that z is even for the case $m \equiv 3 \pmod 4$, as seen in the proof of Lemma 3.2. Put $z = 2Z$ with Z positive integer. Now equation (3.2) can be written as

$$(c^2)^Z - b^y = c^2 - b.$$

Then $y \geq Z$. If $y = Z$, then we obtain $y = Z = 1$. If $y > Z$, then we consider a "gap" between the trivial solution $(y, Z) = (1, 1)$ and (possible) another solution (y, Z).

From $a + b = c^2$ and $a + b^y = c^{2Z}$, consider

$$\Lambda_0 = 2 \log c - \log b \ (> 0), \quad \Lambda = 2Z \log c - y \log b \ (> 0).$$

Then

$$y\Lambda_0 - \Lambda = 2(y - Z) \log c \geq 2 \log c,$$

so

$$y > \frac{2}{\Lambda_0}\log c.$$

By Lemma 3.4, we have $\dfrac{2}{\Lambda_0}\log c < 2512\log c$. Hence

$$\begin{aligned}\frac{2}{2521} < \Lambda_0 = \log\left(\frac{c^2}{b}\right) = \log\left(1+\frac{a}{b}\right) < \frac{a}{b}&\\ = \frac{3pm^2-1}{p(p-3)m^2+1} < \frac{3pm^2}{p(p-3)m^2} = \frac{3}{p-3}&.\end{aligned}$$

Consequently we obtain $p < 3784$. When $m \equiv 1 \pmod 4$,we could not prove that z is even in Lemma 3.2. We therefore suppose that if $m \equiv 1 \pmod 4$, then $p < 3784$. In any case, (3.6) yields $m \le 183$.

From (3.3), we have the inequality

$$\left|\frac{\log b}{\log c} - \frac{z}{y}\right| < \frac{a}{yb^y\log c},$$

which implies that $\left|\dfrac{\log b}{\log c} - \dfrac{z}{y}\right| < \dfrac{1}{2y^2}$, since $y \ge 3$. Thus $\dfrac{z}{y}$ is a convergent in the simple continued fraction expansion to $\dfrac{\log b}{\log c}$.

On the other hand, if $\dfrac{p_r}{q_r}$ is the r-th such convergent, then

$$\left|\frac{\log b}{\log c} - \frac{p_r}{q_r}\right| > \frac{1}{(a_{r+1}+2)q_r^2},$$

where a_{r+1} is the $(r+1)$-st partial quotient to $\dfrac{\log b}{\log c}$ (see e.g. Khinchin [K]). Put $\dfrac{z}{y} = \dfrac{p_r}{q_r}$. Note that $q_r \le y$. It follows, then, that

$$a_{r+1} > \frac{b^y\log c}{ay} - 2 \ge \frac{b^{q_r}\log c}{aq_r} - 2. \tag{3.7}$$

Finally, we checked by Magma [BC] that for each $p < 3784$ with $p \equiv 1 \pmod 4$, inequality (3.7) does not hold for any r with $q_r < 2521\log(pm)$ in the range $3 \le m \le 183$. This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

We can show Theorem 1.2 in the same way as in the proof of Theorem 1.1.

4.1 the case $m \not\equiv 0 \pmod 5$

We easily obtain the following two lemmas by elementary methods.

Lemma 4.1. *If m is even, then equation* (1.4) *has only the positive integer solution* $(x, y, z) = (1, 1, 2)$.

Lemma 4.2. *If m is odd and $m \not\equiv 17, 33 \pmod{40}$, then $y = 1$ and x is odd.*

By the congruence methods, we obtain a lower bound for x.

Lemma 4.3. $x \geq \dfrac{1}{12}(m^2 - 13)$.

By Baker's method, We also obtain an upper bound for x.

Lemma 4.4. $x < 2521 \log c$.

We are now in a position to prove Theorem 1.2. It follows from Lemmas 4.3, 4.4 that

$$\frac{1}{12}(m^2 - 13) < 2521 \log 5m.$$

Hence we obtain $m \leq 485$.

From (3.3), we have the inequality

$$\left| \frac{\log a}{\log c} - \frac{z}{x} \right| < \frac{b}{x a^x \log c},$$

which implies that $\left| \dfrac{\log a}{\log c} - \dfrac{z}{x} \right| < \dfrac{1}{2x^2}$, since $x \geq 3$. Thus $\dfrac{z}{x}$ is a convergent in the simple continued fraction expansion to $\dfrac{\log a}{\log c}$.

On the other hand, if $\dfrac{p_r}{q_r}$ is the r-th such convergent, then

$$\left| \frac{\log a}{\log c} - \frac{p_r}{q_r} \right| > \frac{1}{(a_{r+1} + 2) q_r^2},$$

where a_{r+1} is the $(r+1)$-st partial quotient to $\dfrac{\log a}{\log c}$ (see e.g. Khinchin [K]). Put $\dfrac{z}{x} = \dfrac{p_r}{q_r}$. Note that $q_r \leq x$. It follows, then, that

$$a_{r+1} > \frac{a^x \log c}{bx} - 2 \geq \frac{a^{q_r} \log c}{b q_r} - 2. \tag{4.1}$$

Finally, we checked by Magma [BC] that inequality (4.1) does not hold for any r with $q_r < 2521 \log(5m)$ in the range $3 \leq m \leq 485$.

4.2 the case $m \equiv 0 \pmod 5$

Let m be a positive integer with $m \equiv 0 \pmod 5$. Let (x, y, z) be a solution of (1.4). Taking (1.4) modulo $m (> 3)$, we see that y is odd. Here, we apply Proposition 2. For this we set $p := 5, a_1 := 12m^2 + 1, a_2 := 1 - 13m^2, b_1 := x, b_2 := y$, and

$$\Lambda := (12m^2 + 1)^x - (1 - 13m^2)^y.$$

Then we may take $g = 1, E = 2, A_1 = 12m^2 + 1, A_2 := 13m^2 - 1$. Hence we have

$$2z \leq \frac{36.1}{8(\log 5)^4} \left(\max\{\log b' + \log(2 \log 5) + 0.4,\ 12 \log 5\} \right)^2 \log(12m^2 + 1) \log(13m^2 - 1),$$

where $b' := \dfrac{x}{\log(13m^2 - 1)} + \dfrac{y}{\log(12m^2 + 1)}$. Suppose that $z \geq 4$. We will observe that this leads to a contradiction. Taking (1.4) modulo m^4, we find

$$12x + 13y \equiv 0 \pmod{m^2}.$$

In particular, we find $M := \max\{x, y\} \geq m^2/25$. Therefore, since $z \geq M$ and $b' \leq \frac{M}{\log m}$, we find

$$2M \leq \frac{36.1}{8(\log 5)^4}\left(\max\left\{\log\left(\frac{M}{\log m}\right) + \log(2\log 5) + 0.4,\ 12\log 5\right\}\right)^2 \times \log(12m^2+1)\log(13m^2-1). \tag{4.2}$$

If $m \geq 122009$, then

$$2M \leq \frac{36.1}{8(\log 5)^4}\left(\log\left(\frac{M}{\log m}\right) + \log(2\log 5) + 0.4\right)^2 \log(12m^2+1)\log(13m^2-1).$$

Since $m^2 \leq 25M$, the above inequality gives

$$2M \leq 0.7\,(\log M - \log(\log 122009) + 1.6)^2 \log(300M+1)\log(325M-1).$$

We therefore obtain $M \leq 3683$, which contradicts the fact that $M \geq m^2/25 \geq 595447844$.

If $m < 122009$, then inequality (4.2) gives

$$\frac{2}{25}m^2 \leq 251\log(12m^2+1)\log(13m^2-1).$$

This implies $m \leq 903$. Hence all x, y and z are also bounded. It is not hard to verify by Magma [BC] that there is no (m, x, y, z) under consideration satisfying (1.4). We conclude $z \leq 3$. In this case, one can easily show that $(x, y, z) = (1, 1, 2)$. This completes the proof of Theorem 1.2.

Remark. When $m \equiv \pm 2 \pmod 5$ and $m \equiv 1 \pmod 8$, i.e., $m \equiv 17, 33 \pmod{40}$, we could not prove Lemma 3.2. Hence the condition $m \not\equiv 17, 33 \pmod{40}$ is necessary to Theorem 1.2.

5 Proof of Corollary 1.1

Let (x, y, z) be a solution of (1.5). By Theorem 1.1, we may suppose that $m \equiv 0 \pmod 3$. Recall that x is odd. Here, we apply Proposition 2. For this, we set $p := 3, a_1 := 10m^2 + 1, a_2 := 1 - 15m^2, b_1 := y, b_2 := x$, and

$$\Lambda := (10m^2+1)^y - (1-15m^2)^x.$$

Then we may take $g = 1, E = 2, A_1 = 10m^2 + 1, A_2 := 15m^2 - 1$. Hence we have

$$z \leq \frac{36.1}{8(\log 3)^4}\left(\max\{\log b' + \log(2\log 3) + 0.4,\ 12\log 3\}\right)^2 \log(10m^2+1)\log(15m^2-1),$$

where $b' := \dfrac{y}{\log(15m^2-1)} + \dfrac{x}{\log(10m^2+1)}$. Suppose that $z \geq 4$. We will observe that this leads to a contradiction. Taking (1.5) modulo m^4, we find

$$15x + 10y \equiv 0 \pmod{m^2}.$$

In particular, we find $M := \max\{x, y\} \geq m^2/25$. Therefore, since $z \geq M$ and $b' \leq \frac{M}{\log m}$, we find

$$M \leq 3.1\left(\max\left\{\log\left(\frac{M}{\log m}\right) + \log(2\log 3) + 0.4,\ 12\log 3\right\}\right)^2 \log(10m^2+1)\log(15m^2-1). \tag{5.1}$$

If $m \geq 3450$, then

$$M \leq 3.1\left(\log\left(\frac{M}{\log m}\right) + \log(2\log 3) + 0.4\right)^2 \log(10m^2+1)\log(15m^2-1).$$

Since $m^2 \leq 25M$, the above inequality gives

$$M \leq 3.1\,(\log M - \log(\log 3450) + 1.19)^2 \log(250M+1)\log(375M-1).$$

We therefore obtain $M \leq 105186$, which contradicts the fact that $M \geq m^2/25 \geq 476100$.

If $m < 3450$, then inequality (5.1) gives

$$\frac{m^2}{25} \leq 539\log(10m^2+1)\log(15m^2-1).$$

This implies $m \leq 2062$. Hence all x, y and z are also bounded. It is not hard to verify by Magma [BC] that there is no (m, x, y, z) under consideration satisfying (1.5). We conclude $z \leq 3$. In this case, one can easily show that $(x, y, z) = (1, 1, 2)$. This completes the proof of Corollary 1.1.

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