Hypergeometric-type Bernoulli-Carlitz, Cauchy-Carlitz and Euler-Carlitz numbers

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1   Introduction

For $N \geq 1$, hypergeometric Bernoulli numbers $B_{N,n}$ ([8, 9, 11]) are defined by the generating function

$$\frac{1}{{}_1F_1(1; N+1; x)} = \frac{x^N/N!}{e^t - \sum_{n=0}^{N-1} x^n/n!} = \sum_{n=0}^{\infty} B_{N,n} \frac{x^n}{n!},$$

where

$$_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}} \frac{z^n}{n!}$$

is the confluent hypergeometric function with $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ $(n \geq 1)$ and $(x)^{(0)} = 1$. When $N = 1$, $B_n = B_{1,n}$ are classical Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

In addition, hypergeometric Cauchy numbers $c_{N,n}$ (see [13]) are defined by

$$\frac{1}{{}_2F_1(1, N; N+1; -x)} = \frac{(-1)^{N-1} x^N/N}{\log(1+t) - \sum_{n=1}^{N-1}(-1)^{N-1} x^n/n} = \sum_{n=0}^{\infty} c_{N,n} \frac{x^n}{n!},$$

where

$$_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}} \frac{z^n}{n!}$$

is the Gauss hypergeometric function. When $N = 1$, $c_n = c_{1,n}$ are classical Cauchy numbers defined by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}.$$

Hypergeometric Euler numbers $E_{N,n}$ ([18]) are defined by

$$\frac{1}{{}_1F_2(1; N+1, (2N+1)/2; t^2/4)} = \sum_{n=0}^{\infty} E_{N,n} \frac{t^n}{n!},$$

where $_1F_2(a; b, c; z)$ is the hypergeometric function defined by

$$_1F_2(a; b, c; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}(c)^{(n)}} \frac{z^n}{n!}.$$

When $N = 0$, then $E_n = E_{0,n}$ are classical Euler numbers.

On the other hand, L. Carlitz ([1]) introduced analogues of Bernoulli numbers for the rational function (finite) field $K = \mathbb{F}_r(T)$, which are called Bernoulli-Carlitz numbers now. Bernoulli-Carlitz numbers have been studied since then (e.g., see [2, 3, 5, 10, 19]). According to the notations by Goss [6], Bernoulli-Carlitz numbers are defined by

$$\frac{x}{e_C(x)} = \sum_{n=0}^{\infty} \frac{BC_n}{\Pi(n)} x^n.$$

Here, $e_C(x)$ are the Carlitz exponential defined by

$$e_C(x) = \sum_{i=0}^{\infty} \frac{x^{r^i}}{D_i},$$

where $D_i = [i][i-1]^r \cdots [1]^{r^{i-1}}$ $(i \geq 1)$ with $D_0 = 1$, and $[i] = T^{r^i} - T$. The Carlitz factorial $\Pi(i)$ is defined by

$$\Pi(i) = \prod_{j=0}^{m} D_j^{c_j}$$

for a non-negative integer i with r-ary expansion:

$$i = \sum_{j=0}^{m} c_j r^j \quad (0 \leq c_j < r).$$

As analogues of the classical Cauchy numbers c_n, Cauchy-Carlitz numbers CC_n ([12]) are introduced as

$$\frac{x}{\log_C(x)} = \sum_{n=0}^{\infty} \frac{CC_n}{\Pi(n)} x^n.$$

Here, $\log_C(x)$ is the Carlitz logarithm defined by

$$\log_C(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{r^i}}{L_i},$$

where $L_i = [i][i-1] \cdots [1]$ $(i \geq 1)$ with $L_0 = 1$.

In addition, we define the Carlitz hyperbolic cosine $\mathrm{Cosh}_C(x)$ by

$$\mathrm{Cosh}_C(x) = \sum_{i=0}^{\infty} \frac{x^{r^{2i}}}{D_{2i}}$$

and the Carlitz hyperbolic cosine $\mathrm{Sinh}_C(x)$ by

$$\mathrm{Sinh}_C(x) = \sum_{i=0}^{\infty} \frac{x^{r^{2i+1}}}{D_{2i+1}}.$$

In [12], Bernoulli-Carlitz numbers and Cauchy-Carlitz numbers are expressed explicitly by using the Stirling-Carlitz numbers of the second kind and of the first kind, respectively. These properties are the extensions that Bernoulli numbers and Cauchy numbers are expressed explicitly by using the Stirling numbers of the second kind and of the first kind, respectively.

2 Fundamental properties

For $N \geq 1$, define the *truncated Bernoulli-Carlitz numbers $BC_{N,n}$* and the *truncated Cauchy-Carlitz numbers $CC_{N,n}$* by

$$\frac{x^{r^N}/D_N}{e_C(x) - \sum_{i=0}^{N-1} x^{r^i}/D_i} = \sum_{n=0}^{\infty} \frac{BC_{N,n}}{\Pi(n)} x^n$$

and

$$\frac{(-1)^N x^{r^N}/L_N}{\log_C(x) - \sum_{i=0}^{N-1} (-1)^i x^{r^i}/L_i} = \sum_{n=0}^{\infty} \frac{CC_{N,n}}{\Pi(n)} x^n,$$

respectively. When $N = 0$, $BC_n = BC_{0,n}$ and $CC_n = CC_{0,n}$ are the original Bernoulli-Carlitz numbers and Cauchy-Carlitz numbers, respectively.

For $N \geq 1$, define the *truncated Euler-Carlitz numbers $EC_{N,n}$* by

$$\frac{x^{r^{2N}}/D_{2N}}{\mathrm{Cosh}_C(x) - \sum_{i=0}^{N-1} x^{r^{2i}}/D_{2i}} = \sum_{n=0}^{\infty} \frac{EC_{N,n}}{\Pi(n)} x^n.$$

When $N = 0$, $EC_n = EC_{0,n}$ are the *Euler-Carlitz numbers*, defined by

$$\frac{x}{\mathrm{Cosh}_C(x)} = \sum_{n=0}^{\infty} \frac{EC_n}{\Pi(n)} x^n.$$

Finally, define the *truncated Euler-Carlitz numbers of the second kind $\widehat{EC}_{N,n}$* by

$$\frac{x^{r^{2N+1}}/D_{2N+1}}{\mathrm{Sinh}_C(x) - \sum_{i=0}^{N-1} x^{r^{2i+1}}/D_{2i+1}} = \sum_{n=0}^{\infty} \frac{\widehat{EC}_{N,n}}{\Pi(n)} x^n.$$

When $N = 0$, $\widehat{EC}_n = \widehat{EC}_{0,n}$ are the *Euler-Carlitz numbers of the second kind*, defined by

$$\frac{x^r/D_1}{\mathrm{Sinh}_C(x)} = \sum_{n=0}^{\infty} \frac{\widehat{EC}_n}{\Pi(n)} x^n.$$

We obtain some explicit expressions of the hypergeometric Bernoulli-Carlitz numbers $BC_{N,n}$, the hypergeometric Cauchy numbers $CC_{N,n}$, the hypergeometric Bernoulli-Carlitz numbers $EC_{N,n}$ and the hypergeometric Bernoulli-Carlitz numbers $\widehat{EC}_{N,n}$, respectively.

Theorem 1. *For $n \geq 1$,*

$$BC_{N,n} = \Pi(n) \sum_{k=1}^{n} (-D_N)^k \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ r^{N+i_1}+\cdots+r^{N+i_k} = n+kr^N}} \frac{1}{D_{N+i_1} \cdots D_{N+i_k}}.$$

Remark. When $N = 0$, we have

$$BC_n = \Pi(n) \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ r^{i_1}+\cdots+r^{i_k} = n+k}} \frac{1}{D_{i_1} \cdots D_{i_k}}$$

which is Theorem 4.2 in [10].

We can express the hypergeometric Bernoulli-Carlitz numbers in terms of the binomial coefficients too.

Proposition 1. *For $n \geq 1$,*

$$BC_{N,n} = \Pi(n) \sum_{k=1}^{n} \binom{n+1}{k+1} (-D_N)^k \sum_{\substack{i_1,\dots,i_k \geq 0 \\ r^{N+i_1}+\cdots+r^{N+i_k}=n+kr^N}} \frac{1}{D_{N+i_1} \cdots D_{N+i_k}}.$$

Remark. When $N = 0$, we have

$$BC_n = \Pi(n) \sum_{k=1}^{n} \binom{n+1}{k+1} (-1)^k \sum_{\substack{i_1,\dots,i_k \geq 0 \\ r^{i_1}+\cdots+r^{i_k}=n+k}} \frac{1}{D_{i_1} \cdots D_{i_k}}$$

which is Proposition 4.4 in [10].

Next, we shall give an explicit formula for hypergeometric Cauchy-Carlitz numbers.

Theorem 2. *For $n \geq 1$,*

$$CC_{N,n} = \Pi(n) \sum_{k=1}^{n} (-L_N)^k \sum_{\substack{i_1,\dots,i_k \geq 1 \\ r^{N+i_1}+\cdots+r^{N+i_k}=n+kr^N}} \frac{(-1)^{i_1+\cdots+i_k}}{L_{N+i_1} \cdots L_{N+i_k}}.$$

Remark. When $N = 1$, we have

$$CC_n = \Pi(n) \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1,\dots,i_k \geq 1 \\ r^{i_1}+\cdots+r^{i_k}=n+k}} \frac{(-1)^{i_1+\cdots+i_k}}{L_{i_1} \cdots L_{i_k}}$$

which is Theorem 3 in [12].

We can express the hypergeometric Cauchy numbers in terms of the binomial coefficients too.

Proposition 2. *For $n \geq 1$,*

$$CC_{N,n} = \Pi(n) \sum_{k=1}^{n} \binom{n+1}{k+1} (-L_N)^k \sum_{\substack{i_1,\dots,i_k \geq 0 \\ r^{N+i_1}+\cdots+r^{N+i_k}=n+kr^N}} \frac{(-1)^{i_1+\cdots+i_k}}{L_{N+i_1} \cdots L_{N+i_k}}.$$

We can obtain explicit expressions of the hypergeometric Euler-Carlitz numbers $EC_{N,n}$.

Theorem 3. *For $n \geq 1$,*

$$EC_{N,n} = \Pi(n) \sum_{k=1}^{n} (-D_{2N})^k \sum_{\substack{i_1,\dots,i_k \geq 1 \\ r^{2N+2i_1}+\cdots+r^{2N+2i_k}=n+kr^{2N}}} \frac{1}{D_{2N+2i_1} \cdots D_{2N+2i_k}}.$$

We can get the form with binomial coefficients.

Proposition 3. *For* $n \geq 1$,

$$EC_{N,n} = \Pi(n) \sum_{k=1}^{n} \binom{n+1}{k+1} (-D_{2N})^k \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ r^{2N+2i_1}+\cdots+r^{2N+2i_k}=n+kr^{2N}}} \frac{1}{D_{2N+2i_1} \cdots D_{2N+2i_k}}.$$

When $N = 0$, we have the expression of the Euler-Carlitz numbers EC_n.

Corollary 1. *For* $n \geq 1$, *we have*

$$EC_n = \Pi(n) \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ r^{2i_1}+\cdots+r^{2i_k}=n+k}} \frac{1}{D_{2i_1} \cdots D_{2i_k}}$$

$$= \Pi(n) \sum_{k=1}^{n} (-1)^k \binom{n+1}{k+1} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ r^{2i_1}+\cdots+r^{2i_k}=n+k}} \frac{1}{D_{2i_1} \cdots D_{2i_k}}.$$

In a similar manner, we have an explicit expression of the hypergeometric Euler-Carlitz numbers of the second kind $\widehat{EC}_{N,n}$.

Theorem 4. *For* $n \geq 1$,

$$\widehat{EC}_{N,n} = \Pi(n) \sum_{k=1}^{n} (-D_{2N+1})^k \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ r^{2N+2i_1+1}+\cdots+r^{2N+2i_k+1}=n+kr^{2N+1}}} \frac{1}{D_{2N+2i_1+1} \cdots D_{2N+2i_k+1}}.$$

We can get the form with binomial coefficients. Note that each i takes 0 too in this case.

Proposition 4. *For* $n \geq 1$,

$$\widehat{EC}_{N,n} = \Pi(n) \sum_{k=1}^{n} \binom{n+1}{k+1} (-D_{2N+1})^k$$

$$\times \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ r^{2N+2i_1+1}+\cdots+r^{2N+2i_k+1}=n+kr^{2N+1}}} \frac{1}{D_{2N+2i_1+1} \cdots D_{2N+2i_k+1}}.$$

When $N = 0$, we have the expression of the Euler-Carlitz numbers of the second kind $\widehat{EC}_n$.

Corollary 2. *For* $n \geq 1$, *we have*

$$\widehat{EC}_n = \Pi(n) \sum_{k=1}^{n} (-D_1)^k \sum_{\substack{i_1,\ldots,i_k \geq 1 \\ r^{2i_1+1}+\cdots+r^{2i_k+1}=n+kr}} \frac{1}{D_{2i_1+1} \cdots D_{2i_k+1}}$$

$$= \Pi(n) \sum_{k=1}^{n} (-D_1)^k \binom{n+1}{k+1} \sum_{\substack{i_1,\ldots,i_k \geq 0 \\ r^{2i_1+1}+\cdots+r^{2i_k+1}=n+kr}} \frac{1}{D_{2i_1+1} \cdots D_{2i_k+1}}.$$

3 Incomplete Stirling-Carlitz numbers

In [12], as analogues of the Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ defined by

$$\frac{\left(-\log(1-t)\right)^k}{k!} = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{t^n}{n!},$$

the Stirling-Carlitz numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_C$ were introduced by

$$\frac{\left(\log_C(z)\right)^k}{\Pi(k)} = \sum_{n=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_C \frac{z^n}{\Pi(n)}. \tag{1}$$

As analogues of the Stirling numbers of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}$ defined by

$$\frac{(e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix} \frac{t^n}{n!},$$

the Stirling-Carlitz numbers of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}_C$ were introduced by

$$\frac{\left(e_C(z)\right)^k}{\Pi(k)} = \sum_{n=0}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix}_C \frac{z^n}{\Pi(n)}.$$

By the definition (1), we have

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_C = 0 \quad (n \geq 1), \quad \begin{bmatrix} n \\ m \end{bmatrix}_C = 0 \quad (n < m) \quad \text{and} \quad \begin{bmatrix} n \\ n \end{bmatrix}_C = 1 \quad (n \geq 0)$$

and

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix}_C = 0 \quad (n \geq 1), \quad \begin{Bmatrix} n \\ m \end{Bmatrix}_C = 0 \quad (n < m) \quad \text{and} \quad \begin{Bmatrix} n \\ n \end{Bmatrix}_C = 1 \quad (n \geq 0).$$

On the other hand, in [4, 14, 15, 16], so-called incomplete Stirling numbers of the fist kind and of the second kind were introduced as some generalizations of the classical Stirling numbers of the fist kind and of the second kind. One of the incomplete Stirling numbers is *restricted Stirling number*, and another is *associated Stirling number*. Associated Stirling numbers of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}_{\geq m}$ are given by

$$\frac{\left(e^x - E_{m-1}(x)\right)^k}{k!} = \sum_{n=0}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix}_{\geq m} \frac{x^n}{n!} \quad (m \geq 1),$$

where

$$E_m(x) = \sum_{n=0}^{m} \frac{x^n}{n!}.$$

When $m = 1$, $\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n \\ k \end{Bmatrix}_{\geq 1}$ is the classical Stirling numbers of the second kind. Restricted Stirling numbers of the second kind $\begin{Bmatrix} n \\ k \end{Bmatrix}_{\geq m}$ are given by

$$\frac{\left(E_m(x) - 1\right)^k}{k!} = \sum_{n=0}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix}_{\leq m} \frac{x^n}{n!} \quad (m \geq 1).$$

When $m \to \infty$, $\begin{Bmatrix} n \\ k \end{Bmatrix} = \begin{Bmatrix} n \\ k \end{Bmatrix}_{\leq \infty}$ is the classical Stirling numbers of the second kind.

Associated Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{\geq m}$ are given by

$$\frac{\left(-\log(1-x) + F_{m-1}(-x)\right)^k}{k!} = \sum_{n=0}^{\infty} \left[\begin{matrix} n \\ k \end{matrix}\right]_{\geq m} \frac{x^n}{n!} \quad (m \geq 1),$$

where

$$F_m(t) = \sum_{k=1}^{m} (-1)^{k+1} \frac{t^k}{k}.$$

When $m = 1$, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{\geq 1}$ is the classical Stirling numbers of the first kind. Restricted Stirling numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{\geq m}$ are given by

$$\frac{\left(-F_m(-x)\right)^k}{k!} = \sum_{n=0}^{\infty} \left[\begin{matrix} n \\ k \end{matrix}\right]_{\leq m} \frac{x^n}{n!} \quad (m \geq 1).$$

When $m \to \infty$, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{\leq \infty}$ is the classical Stirling numbers of the first kind.

Now, we introduce *associated Stirling-Carlitz numbers* and *restricted Stirling-Carlitz numbers*. The partial sum of the Carlitz exponential is denoted by

$$\mathcal{E}_m(x) = \sum_{i=0}^{m} \frac{x^{r^i}}{D_i}.$$

The associated Stirling-Carlitz numbers of the second kind $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_{C,\geq m}$ are defined by

$$\frac{\left(e_C(z) - \mathcal{E}_{m-1}(z)\right)^k}{\Pi(k)} = \sum_{n=0}^{\infty} \left\{\begin{matrix} n \\ k \end{matrix}\right\}_{C,\geq m} \frac{z^n}{\Pi(n)}. \tag{2}$$

The restricted Stirling-Carlitz numbers of the second kind $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_{C,\leq m}$ are defined by

$$\frac{\left(\mathcal{E}_m(z)\right)^k}{\Pi(k)} = \sum_{n=0}^{\infty} \left\{\begin{matrix} n \\ k \end{matrix}\right\}_{C,\leq m} \frac{z^n}{\Pi(n)}. \tag{3}$$

When $m = 0$ in (2) or $m \to \infty$ in (3), $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_C = \left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_{C,\geq 0} = \left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_{C,\leq \infty}$ is the original Stirling-Carlitz number of the second kind. The partial sum of the Carlitz logarithm is denoted by

$$\mathcal{F}_m(x) = \sum_{i=0}^{m} (-1)^i \frac{x^{r^i}}{L_i}.$$

The associated Stirling-Carlitz numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{C,\geq m}$ are defined by

$$\frac{\left(\log_C(z) - \mathcal{F}_{m-1}(z)\right)^k}{\Pi(k)} = \sum_{n=0}^{\infty} \left[\begin{matrix} n \\ k \end{matrix}\right]_{C,\geq m} \frac{z^n}{\Pi(n)}. \tag{4}$$

The restricted Stirling-Carlitz numbers of the first kind $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{C,\leq m}$ are defined by

$$\frac{\left(\mathcal{F}_m(z)\right)^k}{\Pi(k)} = \sum_{n=0}^{\infty} \left[\begin{matrix} n \\ k \end{matrix}\right]_{C,\leq m} \frac{z^n}{\Pi(n)}. \tag{5}$$

When $m = 0$ in (4) or $m \to \infty$ in (5), $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_C = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{C,\geq 0} = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{C,\leq \infty}$ is the original Stirling-Carlitz number of the first kind.

Due to associated Stirling-Carlitz numbers of the second kind in (2), we can obtain a more explicit expression of hypergeometric Bernoulli-Carlitz numbers, expressed in Theorem 1 or Proposition 1.

Theorem 5. *For $N \geq 1$ and $n \geq 1$, we have*

$$BC_{N,n} = \Pi(n) \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-D_N)^k \Pi(k)}{\Pi(n+kr^N)} \left\{ \begin{matrix} n+kr^N \\ k \end{matrix} \right\}_{C, \geq N}.$$

Bernoulli-Carlitz numbers can be expressed in term of the Stirling-Carlitz numbers of the second kind:

$$BC_n = \sum_{j=0}^{\infty} \frac{(-1)^j D_j}{L_j^2} \left\{ \begin{matrix} n \\ r^j - 1 \end{matrix} \right\}_C$$

([12, Theorem 2]). When $N = 0$, Theorem 5 is reduced to a different expression of Bernolli-Carlitz numbers in terms of the Stirling-Carlitz numbers of the second kind.

Corollary 3. *For $n \geq 1$, we have*

$$BC_n = \Pi(n) \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^k \Pi(k)}{\Pi(n+k)} \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\}_C.$$

Remark. This is an analogue of

$$B_n = \sum_{k=1}^{n} \frac{(-1)^k \binom{n+1}{k+1}}{\binom{n+k}{k}} \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\},$$

which is a simple formula appeared in [7, 20].

Similarly, due to associated Stirling-Carlitz numbers of the first kind in (4), we can obtain a more explicit expression of hypergeometric Cauchy-Carlitz numbers, expressed in Theorem 2 or Proposition 2.

Theorem 6. *For $N \geq 1$ and $n \geq 1$, we have*

$$CC_{N,n} = \Pi(n) \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^{Nk}(-L_N)^k \Pi(k)}{\Pi(n+kr^N)} \left[\begin{matrix} n+kr^N \\ k \end{matrix} \right]_{C, \geq N}.$$

Cauchy-Carlitz numbers can be expressed in term of the Stirling-Carlitz numbers of the first kind:

$$CC_n = \sum_{j=0}^{\infty} \frac{1}{L_j} \left[\begin{matrix} n \\ r^j - 1 \end{matrix} \right]_C$$

([12, Theorem 1]). When $N = 0$, Theorem 6 is reduced to a different expression of Cauchy-Carlitz numbers in terms of the Stirling-Carlitz numbers of the first kind.

Corollary 4. *For $n \geq 1$, we have*

$$CC_n = \Pi(n) \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^k \Pi(k)}{\Pi(n+k)} \left[\begin{matrix} n+k \\ k \end{matrix} \right]_C.$$

Remark. This is an analogue of

$$c_n = \sum_{k=1}^{n} \frac{(-1)^{n-k} \binom{n+1}{k+1}}{\binom{n+k}{k}} \left[\begin{matrix} n+k \\ k \end{matrix} \right],$$

which is Proposition 2 in [12].

4 Determinant expression of Bernoulli-Carlitz, Cauchy-Carlitz and Euler-Carlitz numbers

We give a determinant expression of truncated Cauchy-Carlitz numbers.

Theorem 7. *For integers $N \geq 0$ and $n \geq 1$,*

$$CC_{N,n} = \Pi(n) \begin{vmatrix} -a_1 & 1 & 0 & & & \\ a_2 & -a_1 & \ddots & & & \\ & & \ddots & \ddots & 0 & \\ & & & -a_1 & 1 \\ (-1)^n a_n & & & a_2 & -a_1 \end{vmatrix},$$

where

$$a_l = \frac{(-1)^{N+\log_r(l+r^N)} L_N \delta_l^*}{L_{\log_r(l+r^N)}} \quad (l \geq 1)$$

with

$$\delta_l^* = \begin{cases} 1 & \text{if } l = r^{N+i} - r^N \ (i = 0, 1, \dots); \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

We give a determinant expression of truncated Bernoulli-Carlitz numbers.

Theorem 8. *For integers $N \geq 0$ and $n \geq 1$,*

$$BC_{N,n} = \Pi(n) \begin{vmatrix} -d_1 & 1 & 0 & & & \\ d_2 & -d_1 & \ddots & & & \\ & & \ddots & \ddots & 0 & \\ & & & -d_1 & 1 \\ (-1)^n d_n & & & d_2 & -d_1 \end{vmatrix},$$

where

$$d_l = \frac{D_N \delta_l^*}{D_{\log_r(l+r^N)}} \quad (l \geq 1)$$

with δ_l^* as in (6).

The Euler-Carlitz numbers EC_n also have a similar determinant expression.

Theorem 9. *For $n \geq 0$, we have*

$$EC_n = \Pi(n) \begin{vmatrix} -d_1 & 1 & & & \\ d_2 & -d_1 & & & \\ \vdots & \vdots & \ddots & 1 & \\ (-1)^{n-1}d_{n-1} & (-1)^{n-2}d_{n-2} & \cdots & -d_1 & 1 \\ (-1)^n d_n & (-1)^{n-1}d_{n-1} & \cdots & d_2 & -d_1 \end{vmatrix},$$

where

$$d_l = \frac{\delta_l^*}{D_{\log_r(l+1)}} \quad (l \geq 1)$$

with

$$\delta_l^* = \begin{cases} 1 & \text{if } l = r^{2i} - 1 \ (i = 0, 1, \dots); \\ 0 & \text{otherwise.} \end{cases}$$

The Euler-Carlitz numbers of the second kind $\widehat{EC}_n$ also have a similar determinant expression.

Theorem 10. *For $n \geq 0$, we have*

$$\widehat{EC}_n = \Pi(n) \begin{vmatrix} -\widehat{d}_1 & 1 & & & \\ \widehat{d}_2 & -\widehat{d}_1 & & & \\ \vdots & \vdots & \ddots & 1 & \\ (-1)^{n-1}\widehat{d}_{n-1} & (-1)^{n-2}\widehat{d}_{n-2} & \cdots & -\widehat{d}_1 & 1 \\ (-1)^n\widehat{d}_n & (-1)^{n-1}\widehat{d}_{n-1} & \cdots & \widehat{d}_2 & -\widehat{d}_1 \end{vmatrix},$$

where

$$\widehat{d}_l = \frac{\widehat{\delta}_l^*}{D_{\log_r(l+1)}} \quad (l \geq 1)$$

with

$$\widehat{\delta}_l^* = \begin{cases} 1 & \text{if } l = r^{2i+1} - 1 \ (i = 0, 1, \dots); \\ 0 & \text{otherwise.} \end{cases}$$

References

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