Adelic Cartier divisors with base conditions and the Bonnesen–Diskant-type inequalities

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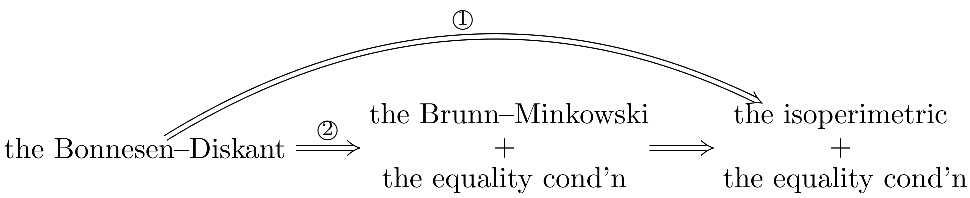
1 The background

1.1 Convex geometry (the classical case)

The classical Bonnesen–Diskant inequality in convex geometry fits into the following schema:



Let me explain it in detail.

The isoperimetric inequality. *A domain P enclosed by a Jordan curve of a given length $L > 0$ has area*

$$A \leq \frac{L^2}{4\pi},$$

and the equality holds iff P is a disk.

To show the inequality, it suffices to consider the case where P is convex; hence we enter into convex geometry. Let P, Q be convex bodies (viz. compact convex sets with positive volumes) in $\mathbb{R}^n$.

$$P + Q := \{p + q \,:\, p \in P,\ q \in Q\} \quad \text{(the Minkowski sum).}$$

The Brunn–Minkowski inequality. *Let P, Q be convex bodies in $\mathbb{R}^n$. Then*

$$\mathrm{Vol}_{\mathbb{R}^n}(P+Q)^{\frac{1}{n}} \geq \mathrm{Vol}_{\mathbb{R}^n}(P)^{\frac{1}{n}} + \mathrm{Vol}_{\mathbb{R}^n}(Q)^{\frac{1}{n}}$$

The equality holds iff $P = sQ + t$, $\exists s > 0$, $\exists t \in \mathbb{R}^n$ (viz. P and Q are homothetic).

For $i = 0, 1, 2, \ldots, n$, we define $V(P, i; Q, n-i)$ as those satisfying

$$\mathrm{Vol}_{\mathbb{R}^n}(P+Q) = \sum_{i=0}^{n} \binom{n}{i} V(P, i; Q, n-i);$$

viz.

$$n = 2 : V(P; Q) = \frac{1}{2}\left(\mathrm{Area}(P+Q) - \mathrm{Area}(P) - \mathrm{Area}(Q)\right),$$

$$n = 3 : V(P; Q, 2) = \frac{1}{6}(\mathrm{Vol}(P+Q+Q) - 2\,\mathrm{Vol}(P+Q) - \mathrm{Vol}(Q+Q) + \mathrm{Vol}(P) + 2\,\mathrm{Vol}(Q)).$$

Let (P, Q) be a pair of convex bodies in $\mathbb{R}^n$. We set $r(P, Q) := \max\{s > 0 : P \supset sQ + t, \exists t \in \mathbb{R}^n\}$ and $R(P, Q) := \min\{s > 0 : P \subset sQ + t, \exists t \in \mathbb{R}^n\}$. In particular, $r(P, Q) = R(P, Q)$ implies that P and Q are homothetic.

The Bonnesen–Diskant inequality. *Let P, Q be convex bodies in $\mathbb{R}^n$, and set $s_i := V(P, i; Q, n-i)$. Then*

$$\frac{s_{n-1}^{\frac{1}{n-1}} - \left(s_{n-1}^{\frac{n}{n-1}} - s_n s_0^{\frac{1}{n-1}}\right)^{\frac{1}{n}}}{s_0^{\frac{1}{n-1}}} \le r(P, Q) \le \frac{s_n}{s_{n-1}} \le \cdots \le \frac{s_1}{s_0}$$
$$\le R(P, Q) \le \frac{s_n^{\frac{1}{n-1}}}{s_1^{\frac{1}{n-1}} - \left(s_1^{\frac{n}{n-1}} - s_0 s_n^{\frac{1}{n-1}}\right)^{\frac{1}{n}}}.$$

① *The Bonnensen–Diskant inequality implies the isoperimetric inequality.* For simplicity, suppose $n = 2$. By the Bonnensen–Diskant inequality,

$$\frac{s_1 - \sqrt{s_1^2 - s_0 s_2}}{s_0} \le r \le R \le \frac{s_2}{s_1 - \sqrt{s_1^2 - s_0 s_2}} = \frac{s_1 + \sqrt{s_1^2 - s_0 s_2}}{s_0}$$
$$\therefore\ R - r \le \frac{2\sqrt{s_1^2 - s_0 s_2}}{s_0} \qquad \therefore\ \frac{s_0^2}{4}(R - r)^2 \le s_1^2 - s_0 s_2.$$

Moreover, if Q is a unit disk, then $s_0 = \mathrm{Area}(Q) = \pi$, $s_1 = V(P, Q) = \frac{1}{2}L$, and r (resp. R) is the usual inradius (resp. circumradius) of P. Hence

$$\frac{\pi^2}{4}(R - r)^2 \le \left(\frac{1}{2}L\right)^2 - \pi A \quad \therefore\ \pi^2(R - r)^2 \le L^2 - 4\pi A.$$

② *The Bonnesen–Diskant inequality implies the Brunn–Minkowski inequality.* For simplicity, suppose $n = 2$. We assume the middle part of the Bonnesen–Diskant inequality;

$$\frac{s_2}{s_1} \le \frac{s_1}{s_0} \qquad \therefore\ \mathrm{Area}(P)\,\mathrm{Area}(Q) \le V(P; Q)^2.$$

Then

$$\begin{aligned}\mathrm{Area}(P + Q) &= \mathrm{Area}(P) + 2V(P; Q) + \mathrm{Area}(Q)\\ &\ge \mathrm{Area}(P) + 2\,\mathrm{Area}(P)^{\frac{1}{2}}\,\mathrm{Area}(Q)^{\frac{1}{2}} + \mathrm{Area}(Q)\\ &= \left(\mathrm{Area}(P)^{\frac{1}{2}} + \mathrm{Area}(Q)^{\frac{1}{2}}\right)^2.\end{aligned}$$

Assume the Bonnesen–Diskant inequality

$$\frac{\mathrm{Area}(Q)^2}{4}(R - r)^2 \le V(P; Q)^2 - \mathrm{Area}(P)\,\mathrm{Area}(Q)$$

and the equality

$$\mathrm{Area}(P + Q) = \left(\mathrm{Area}(P)^{\frac{1}{2}} + \mathrm{Area}(Q)^{\frac{1}{2}}\right)^2.$$

Then

$$\mathrm{Area}(P) + 2V(P; Q) + \mathrm{Area}(Q) = \mathrm{Area}(P) + 2\,\mathrm{Area}(P)^{\frac{1}{2}}\,\mathrm{Area}(Q)^{\frac{1}{2}} + \mathrm{Area}(Q)$$
$$\therefore\ V(P; Q) = \sqrt{\mathrm{Area}(P)\,\mathrm{Area}(Q)} \quad \therefore\ R = r.$$

Hence P, Q are homothetic.

Ideas for the proof of the Bonnesen–Diskant inequality. For simplicity, suppose $n = 2$. Let P be the rectangle defined by $A = \mathrm{diag}(a_1, a_2)$ and let Q be the rectangle defined by $B = \mathrm{diag}(b_1, b_2)$. In this case, by easy calculations, we can easily verify

$$V(A; B) = \frac{1}{2}(a_1 b_2 + a_2 b_1), \quad r(A, B) = \min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}\right\}$$

and

$$V(A - tB; B) + t \det(B) \le V(A; B) \quad \text{(an Aleksandrov–Fenchel-type inequality).}$$

Since

$$\frac{d}{dt}\det(A - tB) = -2V(A - tB; B),$$

$$\det(A) = 2\int_{t=0}^{r(A,B)} V(A - tB; B)\, dt \le 2\int_{t=0}^{r(A,B)} (V(A; B) - t\det(B))\, dt.$$

By direct calculation of the RHS, we can conclude. □

1.2 The work of Boucksom–Favre–Jonsson

Teissier's problem. *Let X a complete variety over a field. Given nef and big line bundles P, Q on X, we set*

$$r(P, Q) := \sup\{s \in \mathbb{Q}_{>0} \,:\, P - sQ \text{ is effective}\},$$
$$s_i := \deg\left(P^{\cdot i} \cdot Q^{\cdot(\dim X - i)}\right) \quad (i = 0, 1, \ldots, \dim X).$$

Then can we give bounds for $r(P, Q)$ in terms of s_i?

Teissier's problem was solved by

- Boucksom–Favre–Jonsson '09 for smooth complete varieties over algebraically closed fields of characteristic zero,
- Cutkosky '15 for complete varieties over arbitrary fields,
- Fu–Xiao, ... for compact Kähler manifolds.

Let $\mathrm{Rat}(X)$ denote the field of rational functions on X. Given a class $\alpha \in N^1_{\mathbb{R}}(X)$ of an $\mathbb{R}$-Cartier divisor, we define

$$H^0(\alpha) := \left\{\phi \in \mathrm{Rat}(X)^\times \,:\, \alpha + (\phi) \ge 0\right\} \cup \{0\},$$

and set

$$\mathrm{vol}(\alpha) := \limsup_{m \to +\infty} \frac{(\dim X)!}{m^{\dim X}} \dim_k H^0(m\alpha).$$

We say that α is *big* if $\mathrm{vol}(\alpha) > 0$.

An answer to Teissier's problem by Boucksom–Favre–Jonsson: Let $\mathfrak{X}$ denote the Zariski–Riemann space of X. Let $N^p_{\mathbb{R}}(\mathfrak{X})$ (resp. $CN^p_{\mathbb{R}}(\mathfrak{X})$) denote the space of numerical Weil classes (resp. numerical Cartier classes) of codimension p.

- Given any big classes $\alpha_1, \ldots, \alpha_p \in CN^1_{\mathbb{R}}(\mathfrak{X})$, the following exists:
$$\langle \alpha_1 \cdots \alpha_p \rangle := \sup \{(\alpha_1 - \gamma_1) \cdots (\alpha_p - \gamma_p)\} \in N^p_{\mathbb{R}}(\mathfrak{X}),$$
where $\gamma_1, \ldots, \gamma_p \in CN^1_{\mathbb{R}}(\mathfrak{X})$ are psef classes such that $\alpha_1 - \gamma_1, \ldots, \alpha_p - \gamma_p$ are nef.
- Let $\alpha, \beta \in CN^1_{\mathbb{R}}(\mathfrak{X})$. If α is big, then
$$\frac{d}{dt} \mathrm{vol}(\alpha + t\beta)\Big|_{t=0} = (\dim X) \langle \alpha^{\dim X - 1} \rangle \cdot \beta.$$
- Let $\alpha, \beta \in CN^1_{\mathbb{R}}(\mathfrak{X})$ be nef and big Cartier classes, and let $n := \dim X$. Set
$$s_i := \deg(\alpha^i \cdot \beta^{n-i}), \quad r(\alpha, \beta) := \sup\{s > 0 \, : \, \alpha - s\beta \text{ is psef}\},$$
and $R(\alpha, \beta) := 1/r(\beta, \alpha)$. Then the same inequality as the Bonnesen–Diskant inequality holds true.

As a corollary,

The equality conditions for the Brunn–Minkowski inequality (BFJ)**.** *For nef and big Cartier classes $\alpha, \beta \in CN^1_{\mathbb{R}}(\mathfrak{X})$, TFAE.*

(1) $\mathrm{vol}(\alpha + \beta)^{\frac{1}{n}} = \mathrm{vol}(\alpha)^{\frac{1}{n}} + \mathrm{vol}(\beta)^{\frac{1}{n}}$.

(2) $s_i^2 = s_{i-1} s_{i+1}$ *for* $i = 1, \ldots, n-1$.

(3) $s_i^n = s_0^{n-i} s_n^i$ *for* $i = 0, \ldots, n$.

(4) $s_{n-1}^n = s_0 s_n^{n-1}$.

(5) $\dfrac{\alpha}{\mathrm{vol}(\alpha)^{\frac{1}{n}}} = \dfrac{\beta}{\mathrm{vol}(\beta)^{\frac{1}{n}}}$ *in* $CN^1_{\mathbb{R}}(\mathfrak{X})$.

1.3 The Arakerov setting (I [1])

Convex Geometry	Algebraic Geometry	Arakelov Geometry
convex bodies in $\mathbb{R}^n$	nef & big divisors	nef & big adelic divisors
n	$n = \dim X$	$n = \dim X + 1$
$\mathrm{vol}(P)$	$\mathrm{vol}(\alpha)$	$\widehat{\mathrm{vol}}(\overline{P})$
$s_i = V(P, i; Q, n-i)$	$s_i = \deg(\alpha^i \beta^{n-i})$	$s_i = \widehat{\deg}(\overline{P}^i \overline{Q}^{n-i})$
$r(P, Q)$	$r(\alpha, \beta)$	$r(\overline{P}, \overline{Q})$
$R(P, Q) = 1/r(Q, P)$	$R(\alpha, \beta) = 1/r(\beta, \alpha)$	$R(\overline{P}, \overline{Q}) = 1/r(\overline{Q}, \overline{P})$

Let K be a number field, and let M_K be the set of places of K. Let X be a normal projective K-variety, and let D be an $\mathbb{R}$-Cartier divisor on X. For each $v \in M_K$, X_v^{an} denotes the associated analytic space. Let $g_v : (X \setminus \mathrm{Supp}(D))_v^{\mathrm{an}} \to \mathbb{R}$ be a D-Green function on X_v^{an}: roughly,

"g_v = (a fundamental solution of a Poisson-type equation) + (a continuous function)".

Definition 1 (Adelic $\mathbb{R}$-Cartier divisors)**.** A couple $\left(D, \sum_{v\in M_K} g_v[v]\right)$ is said to be an *adelic* $\mathbb{R}$*-Cartier divisor* if

(1) (g_v) is invariant under the complex conjugation and

(2) $\exists$ an O_K-model $(\mathscr{X}, \mathscr{D})$ of (X, D) that defines g_v for all but finitely many v (so-called the "adelic condition").

We denote the $\mathbb{R}$-vector space of adelic $\mathbb{R}$-Cartier divisors on X by $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)$.

Remark 1. Let $r_v : X_v^{\mathrm{an}} \to \mathscr{X}_v$ denote the reduction map at v. The Green function defined by $(\mathscr{X}, \mathscr{D})$ is defined as

$$g_v^{(\mathscr{X},\mathscr{D})}(x) := -\log|f_x|^2(x),$$

where f_x is a local defining equation of $\mathscr{D}$ at $r_v(x)$.

Example 1. (1) For a $\phi \in \mathrm{Rat}(X)^\times$,

$$\widehat{(\phi)} := \left((\phi), \sum_{v\in M_K} -\log|\phi|_v^2[v]\right) \in \widehat{\mathrm{Div}}_{\mathbb{R}}(X).$$

(2) For a continuous function $f_v : X_v^{\mathrm{an}} \to \mathbb{R} \in C(X_v^{\mathrm{an}})$ (if $v|\infty$, we assume that f_v is invariant under the complex conjugation), one has $(0, f_v[v]) \in \widehat{\mathrm{Div}}_{\mathbb{R}}(X)$.

(3) If f_v is a constant function, $(0, f_v[v])$ corresponds to (a constant) $\times$ (the fiber over v).

We assign to each $\overline{D}$ a finite set

$$\widehat{\Gamma}^{\mathrm{ss}}(\overline{D}) := \left\{\phi \in \mathrm{Rat}(X)^\times \;:\; \overline{D} + \widehat{(\phi)} > 0\right\} \cup \{0\},$$

where $\overline{D} > 0$ means $D \geq 0$, $g_v \geq 0$ $(\forall v \in M_K)$, and $\inf_{x\in X_\infty^{\mathrm{an}}} g_\infty > 0$. We then define the arithmetic volume of $\overline{D}$ as

$$\widehat{\mathrm{vol}}(\overline{D}) := \limsup_{m\to+\infty} \frac{(\dim X+1)!}{m^{\dim X+1}} \log \sharp\widehat{\Gamma}^{\mathrm{ss}}(m\overline{D}).$$

Positivity of adelic $\mathbb{R}$-Cartier divisors: Given a $\overline{D} \in \widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ and an $x \in X(\overline{K})$, we set

$$h_{\overline{D}}(x) := \frac{1}{\deg(x)}\widehat{\deg}(\overline{D}|_x) \quad \text{(the height of } x \text{ with respect to } \overline{D}).$$

nef: $\overline{D}$ is *nef* if $h_{\overline{D}}(x) \geq 0$ $(\forall\, x \in X(\overline{K}))$ and g_v is semipositive $(\forall\, v \in M_K)$. If v: finite,

"g_v: semipositive $\Leftrightarrow$ g_v is uniformly approximable by nef models".

integrable: $\overline{D}$ is *integrable* if $\overline{D}$ can be written as the difference of two nef adelic $\mathbb{R}$-Cartier divisors. We denote the $\mathbb{R}$-vector space of integrable adelic $\mathbb{R}$-Cartier divisors on X by $\widehat{\mathrm{Int}}_{\mathbb{R}}(X)$.

big: $\overline{D}$ is *big* if $\widehat{\mathrm{vol}}(\overline{D}) > 0$.

psef: $\overline{D}$ is *psef* if $\overline{D} + \overline{A}$ is big for any big $\overline{A}$.

Arithmetic intersection numbers:

- One can extend the arithmetic intersection numbers to the map

$$\widehat{\mathrm{Int}}_{\mathbb{R}}(X)^{\times \dim X} \times \widehat{\mathrm{Div}}_{\mathbb{R}}(X) \to \mathbb{R},$$
$$(\overline{D}_1, \dots, \overline{D}_{\dim X+1}) \mapsto \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{\dim X+1}).$$

- An *approximation* of a big $\overline{D}$ is a couple $(\mu : X' \to X, \overline{M})$ having the following properties.
  - μ is a birational morphism of normal projective varieties.
  - $\overline{M}$ is a nef and big adelic $\mathbb{R}$-Cartier divisor on X' s.t. $\mu^*\overline{D} - \overline{M}$ is psef.

  We denote the set of approximations of $\overline{D}$ by $\widehat{\Theta}(\overline{D})$.

- Given big $\overline{D}_1, \dots, \overline{D}_p$ and nef and big $\overline{D}_{p+1}, \dots, \overline{D}_{\dim X+1}$, we set

$$\langle \overline{D}_1 \cdots \overline{D}_p \rangle \cdot \overline{D}_{p+1} \cdots \overline{D}_{\dim X+1} = \sup_{(\mu, \overline{M}_i) \in \widehat{\Theta}(\overline{D}_i)} \widehat{\deg}\left(\overline{M}_1 \cdots \overline{M}_p \cdot \mu^*\overline{D}_{p+1} \cdots \mu^*\overline{D}_{\dim X+1}\right).$$

The differentiability of the volume function ([1])**.** *Let $\overline{D}, \overline{E} \in \widehat{Div}_{\mathbb{R}}(X)$. If $\overline{D}$ is big, then*

$$\left.\frac{d}{dt}\widehat{\mathrm{vol}}(\overline{D} + t\overline{E})\right|_{t=0} = (\dim X + 1)\langle \overline{D}^{\dim X} \rangle \cdot \overline{E}.$$

The Bonnesen–Diskant bound for $r(\overline{P}, \overline{Q})$ ([1])**.** *Let $\overline{P}, \overline{Q} \in \widehat{Div}_{\mathbb{R}}(X)$ be nef and big adelic $\mathbb{R}$-Cartier divisors. Let $n := \dim X + 1$,*

$$s_i := \deg(\overline{P}^i \cdot \overline{Q}^{n-i}), \quad r(\overline{P}, \overline{Q}) := \sup\{s > 0 \,:\, \overline{P} - s\overline{Q} \textit{ is psef}\},$$

and $R(\overline{P}, \overline{Q}) := 1/r(\overline{Q}, \overline{P})$. Then the same inequality as the Bonnesen–Diskant inequality holds true.

As a corollary,

The equality conditions for the Brunn–Minkowski inequality ([1])**.** *For nef and big $\overline{P}, \overline{Q} \in \widehat{Div}_{\mathbb{R}}(X)$, TFAE.*

(1) $\widehat{\mathrm{vol}}(\overline{P} + \overline{Q})^{\frac{1}{n}} = \widehat{\mathrm{vol}}(\overline{P})^{\frac{1}{n}} + \widehat{\mathrm{vol}}(\overline{Q})^{\frac{1}{n}}$.

(2) $s_i^2 = s_{i-1}s_{i+1}$ *for* $i = 1, \dots, n-1$.

(3) $s_i^n = s_0^{n-i}s_n^i$ *for* $i = 0, \dots, n$.

(4) $s_{n-1}^n = s_0 s_n^{n-1}$.

(5) $\exists \phi_1, \dots, \phi_r \in \mathrm{Rat}(X)^\times$, $a_1, \dots, a_r \in \mathbb{R}$, *s.t.*

$$\frac{\overline{P}}{\widehat{\mathrm{vol}}(\overline{P})^{\frac{1}{n}}} = \frac{\overline{Q}}{\widehat{\mathrm{vol}}(\overline{Q})^{\frac{1}{n}}} + \sum_{i=1}^{r} a_i \widehat{(\phi_i)} \quad (\mathbb{R}\textit{-linearly equiv.})$$

The Bogomolov conjecture and the equidistribution: The equidistribution theorem of algebraic points with small heights was first exploited by Ullmo–Szpiro–Zhang to solve the Bogomolov conjecture.

Bogomolov conjecture for abelian varieties (solved by S. Zhang)**.** *Let A be an abelian variety over a number field, let h denote a Néron-Tate height, and let X be a subvariety of A. Let $\varepsilon > 0$. If $\{x \in X(\overline{\mathbb{Q}}) : h(x) \leq \varepsilon\}$ is Zariski dense, then X is a translation of an abelian subvariety by a torsion point.*

The differentiability of the arithmetic volume function along the directions of continuous functions implies the equidistribution theorem of algebraic points with small heights (Chambert-Loir, Chen, I).

The equidistribution theorem. *Let v be any place, let $\overline{D}$ be a big adelic $\mathbb{R}$-Cartier divisor, and let (x_n) be a sequence of algebraic points on X such that, given any subvariety Y, $x_n \notin Y(\overline{\mathbb{Q}})$ for every $n \gg 1$. If $h_{\overline{D}}(x_n)$ converges to*

$$\frac{\widehat{\mathrm{vol}}(\overline{D})}{(\dim X + 1)\mathrm{vol}(D)},$$

then for any $f \in C^0(X_v^{\mathrm{an}})$

$$\lim_{n\to\infty} \frac{1}{[K(x_n) : K]} \sum_{\substack{w \in M_{K(x_n)} \\ v|w}} [K(x_n)_w : K_v] f(x_n^w) = \frac{1}{\mathrm{vol}(D)} \langle \overline{D}^{\cdot \dim X} \rangle \cdot (0, 2f[v]),$$

where $K(x_n)$ denotes the field of definition for x_n.

2 The notion of "pairs" in Arakelov geometry (I [2, 3])

Motivation:

- Want to study the shapes of the arithmetic Okounkov bodies (explained later).
- Want to treat "open" arithmetic varieties (more generally, we should allow "singularities" along boudaries (eg. Faltings' height function on compact moduli spaces of abelian varieties).

We denote the set of nontrivial normalized discrete valuations on $\mathrm{Rat}(X)$ by $\mathfrak{V}(\mathrm{Rat}(X))$. We regard such valuations as a generalized notion of prime divisors (although it contains many pathological ones).

Definition 2. An $\mathbb{R}$-*base condition* on X is defined as a finite formal $\mathbb{R}$-linear combination

$$\mathcal{V} = \sum_{\nu \in \mathfrak{V}(\mathrm{Rat}(X))} \nu(\mathcal{V})[\nu], \quad \nu(\mathcal{V}) \in \mathbb{R}.$$

We denote the $\mathbb{R}$-vector space of $\mathbb{R}$-base conditions by $\mathrm{BC}_{\mathbb{R}}(X)$, and regard $\mathrm{Div}_{\mathbb{R}}(X) \subset \mathrm{WDiv}_{\mathbb{R}}(X) \subset \mathrm{BC}_{\mathbb{R}}(X)$.

A valuation $\nu \in \mathfrak{V}(\mathrm{Rat}(X))$ extends by linearity to

$$\nu : \mathrm{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}.$$

Given a $\nu \in \mathfrak{V}(\mathrm{Rat}(X))$ and a nonzero $D \in \mathrm{Div}_{\mathbb{R}}(X)$, we choose a local equation f defining D around $c_X(\nu)$ and set $\nu^{\mathrm{C}}(D) := \nu(f)$, which does not depend on a specific choice of f.

Given a pair $(\overline{D}; \mathcal{V})$ of a $\overline{D} \in \widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ and a $\mathcal{V} \in \mathrm{BC}_{\mathbb{R}}(X)$, we set

$$\widehat{\Gamma}^{\mathrm{ss}}(\overline{D}; \mathcal{V}) := \left\{\phi \in \mathrm{Rat}(X)^{\times} \ : \ (\overline{D} + \widehat{(\phi)}; \mathcal{V}) > 0\right\} \cup \{0\},$$

where $(\overline{D}; \mathcal{V}) \geq 0$ means $\overline{D} > 0$ and $\nu^{\mathrm{C}}(D) \geq \nu(\mathcal{V})$ $(\forall \nu \in \mathfrak{V}(\mathrm{Rat}(X)))$.

Definition 3. We define the *arithmetic volume* of a pair $(\overline{D}; \mathcal{V})$ as

$$\widehat{\mathrm{vol}}(\overline{D}; \mathcal{V}) := \limsup_{m \to +\infty} \frac{(\dim X + 1)!}{m^{\dim X + 1}} \log \sharp \widehat{\Gamma}^{\mathrm{ss}}(m\overline{D}; m\mathcal{V}).$$

Example 2 (Arithmetic Okounkov bodies)**.** Let $\mathbb{P}^1_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[X_0, X_1])$, $z = X_0/X_1$[1)], $0 = (1:0)$, and $\infty = (0:1)$. Note that z is a local equation defining ∞. Let $a, b > 1$. Then

$$g := -\log|z|^2 + \max\left\{\log(a^2|z|^2), \log b^2\right\}$$

is an ∞-Green function. We consider the arithmetic Cartier divisor $\overline{H} := (\infty, g)$. If we set

$$\varphi(x) := x \log a + (1 - x) \log b,$$

then one can verify by direct computation

$$\widehat{\mathrm{vol}}(\overline{H}; r\infty) = \int_r^1 \max\{\varphi, 0\}\, dx$$

for every r with $0 \leq r \leq 1$.

What is $\varphi(x)$? — $-\varphi(x)$ is known to be given as the Legendre–Fenchel transform of the metric:

$$-\varphi(x) = \sup_{t \in \mathbb{R}} \left\{xt - \max\left\{t + \log a, \log b\right\}\right\}.$$

If $(\overline{D}; \mathcal{V})$ is big, the "unique" nef adelic $\mathbb{R}$-Cartier divisor $\overline{P}(\overline{D}; \mathcal{V})$ satisfying $\overline{P}(\overline{D}; \mathcal{V}) \leq (\overline{D}; \mathcal{V})$ and $\widehat{\mathrm{vol}}(\overline{P}(\overline{D}; \mathcal{V})) = \widehat{\mathrm{vol}}(\overline{D}; \mathcal{V})$ is called the *positive part of the arithmetic Zariski decomposition* of $(\overline{D}; \mathcal{V})$.

One has

$$\overline{P}(\overline{H}; r\infty) = (1 - r)\overline{H} + r(0, \log a^2).$$

In fact, the RHS is nef,

$$(\overline{H}; r\infty) - \overline{P}(\overline{H}; r\infty) = r\left(H, g - \log a^2; \infty\right) \geq 0,$$

and

$$\widehat{\mathrm{vol}}(\overline{H}; r\infty) = \widehat{\mathrm{vol}}(\overline{P}(\overline{H}; r\infty)) = \frac{1 - r}{2} \log(a^{1+r} b^{1-r}).$$

Moreover, $\varphi(r) = h_{\overline{P}(\overline{H}; r\infty)}(\infty)$.

[1)] In the talk, I set $z = X_1/X_0$.

Continuity of the arithmetic volume function (I [3]):

Theorem 1 ([3, Main Theorem]). *Let X be a normal projective K-variety. Let $(V, \|\cdot\|^V)$ be a finite-dimensional $\mathbb{R}$-subspace of $\widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ endowed with a norm $\|\cdot\|^V$, let Σ be a finite set of points on X, and let $B > 0$ be a constant. Then, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\left|\widehat{\mathrm{vol}}(\overline{D} + (0, \boldsymbol{f}); \mathcal{V}) - \widehat{\mathrm{vol}}(\overline{E}; \mathcal{V})\right| \leq \varepsilon$$

for every $\overline{D}, \overline{E} \in \widehat{\mathrm{Div}}_{\mathbb{R}}(X)$ with $\max\left\{\|\overline{D}\|^V, \|\overline{E}\|^V\right\} \leq B$ and $\|\overline{D} - \overline{E}\|^V \leq \delta$, $\boldsymbol{f} \in \bigoplus C(X_v^{\mathrm{an}})$ with $\|\boldsymbol{f}\|_{\sup} \leq \delta$, and $\mathcal{V} \in \mathrm{BC}_{\mathbb{R}}(X)$ with $\{c_X(\nu) : \nu(\mathcal{V}) > 0\} \subset \Sigma$.

Remark 2. It gives a generalization of Moriwaki's continuity theorem.

big: $(\overline{D}; \mathcal{V})$ is *big* if $\widehat{\mathrm{vol}}(\overline{D}; \mathcal{V}) > 0$. One has

$$\text{“}(\overline{D}; \mathcal{V}) \text{ is big} \quad \Leftrightarrow \quad \exists \text{ a big } \overline{A} \text{ s.t. } \widehat{\mathrm{vol}}(\overline{D} - \overline{A}; \mathcal{V}) > 0\text{” (by [3]).}$$

psef: $(\overline{D}; \mathcal{V})$ is *psef* if $\widehat{\mathrm{vol}}(\overline{D} + \overline{A}; \mathcal{V}) > 0$ for $\forall$ big $\overline{A}$.

Definition 4 (Approximating Zariski decomposition). Let $(\overline{D}; \mathcal{V})$ be a big pair. An *approximation* of $(\overline{D}; \mathcal{V})$ is a couple $(\mu : X' \to X, \overline{M})$ having the following properties.

- μ is a birational morphism of normal projective varieties.
- $\overline{M}$ is a nef and big adelic $\mathbb{R}$-Cartier divisor on X' s.t. $\mu^*(\overline{D}; \mathcal{V}) - \overline{M}$ is psef.

We denote the set of approximations of $(\overline{D}; \mathcal{V})$ by $\widehat{\Theta}(\overline{D}; \mathcal{V})$.

Definition 5 (Arithmetic positive intersection numbers). Given big $(\overline{D}_1; \mathcal{V}_1), \dots, (\overline{D}_p; \mathcal{V}_p)$ and nef and big $\overline{D}_{p+1}, \dots, \overline{D}_{\dim X+1}$,

$$\begin{aligned}\langle(\overline{D}_1; \mathcal{V}_1) \cdots (\overline{D}_p; \mathcal{V}_p)\rangle \cdot \overline{D}_{p+1} \cdots \overline{D}_{\dim X+1} \\ := \sup_{(\mu, \overline{M}_i) \in \widehat{\Theta}(\overline{D}_i; \mathcal{V}_i)} \widehat{\deg}\left(\overline{M}_1 \cdots \overline{M}_p \cdot \mu^*\overline{D}_{p+1} \cdots \mu^*\overline{D}_{\dim X+1}\right).\end{aligned}$$

By linearity and continuity, one obtains a map

$$\widehat{\mathrm{Div}}_{\mathbb{R}}(X) \ni \overline{E} \mapsto \langle(\overline{D}; \mathcal{V})^{\cdot \dim X}\rangle \cdot \overline{E} \in \mathbb{R}.$$

Arithmetic Fujita approximation ([2]). *If $(\overline{D}; \mathcal{V})$ is big, then*

$$\widehat{\mathrm{vol}}(\overline{D}; \mathcal{V}) = \langle(\overline{D}; \mathcal{V})^{\cdot(\dim X+1)}\rangle.$$

Differentiability of the arithmetic volume function:

Theorem 2 ([2, Theorem A]). *Let X be a normal projective K-variety, let $\overline{D}, \overline{E}$ be adelic $\mathbb{R}$-Cartier divisors, and let $\mathcal{V}$ be an $\mathbb{R}$-base condition. If $(\overline{D}; \mathcal{V})$ is big, then*

$$\left.\frac{d}{dt}\widehat{\mathrm{vol}}(\overline{D} + t\overline{E}; \mathcal{V})\right|_{t=0} = (\dim X + 1)\langle(\overline{D}; \mathcal{V})^{\cdot \dim X}\rangle \cdot \overline{E}.$$

Remark 3. It gives a generalization of my previous work [1].

Corollary 1 ([2, Theorem B]). *Let $(\overline{D}; \mathcal{V}), (\overline{D}'; \mathcal{V}')$ be big pairs on X. Let $n := \dim X + 1$,*

$$\begin{aligned}&s_i := \langle(\overline{D}; \mathcal{V})^i \cdot (\overline{D}'; \mathcal{V}')^{n-i}\rangle, \quad \text{for } i = 0, \dots, n, \\ &r((\overline{D}; \mathcal{V}), (\overline{D}'; \mathcal{V}')) := \inf_{(\mu, \overline{M}) \in \widehat{\Theta}(\overline{D}'; \mathcal{V}')} \sup\{s > 0 : (\mu^*\overline{D} - s\overline{M}; \mathcal{V}) \text{ is psef}\},\end{aligned}$$

and $R((\overline{D}; \mathcal{V}), (\overline{D}'; \mathcal{V}')) := 1/r((\overline{D}'; \mathcal{V}'), (\overline{D}; \mathcal{V}))$. Then the same inequality as the Bonnesen–Diskant inequality holds true.

References

[1] H. Ikoma, On the concavity of the arithmetic volumes, Int. Math. Res. Not. IMRN 2015, no. 16, 7063–7109.

[2] H. Ikoma, Adelic Cartier divisors with base conditions and the Bonnesen–Diskant-type inequalities, preprint, 2016, arXiv:1602.02336.

[3] H. Ikoma, Adelic Cartier divisors with base conditions and the continuity of volumes, preprint, 2017, arXiv:1702.03514.