Local p-rank and semi-stable reduction of curves

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Abstract

In the present paper, we investigate the local p-ranks of coverings of stable curves. Let G be a finite p-group, $f : Y \longrightarrow X$ a morphism of stable curves over a complete discrete valuation ring with algebraically closed residue field of characteristic $p > 0$, x a singular point of the special fiber X_s of X. Suppose that the generic fiber X_η of X is smooth, and the morphism of generic fibers f_η is a Galois étale covering with Galois group G. Write Y' for the normalization of X in the function field of Y, $\psi : Y' \longrightarrow X$ for the resulting normalization morphism. Let $y' \in \psi^{-1}(x)$ be a point of the inverse image of x. Suppose that the inertia group $I_{y'} \subseteq G$ of y' is an abelian p-group. Then we give an explicit formula for the p-rank of a connected component of $f^{-1}(x)$. Furthermore, we prove that the p-rank is bounded by a constant which depends only on the order of $I_{y'}$. These results give an answer of a problem posed by M. Saïdi concerning local p-ranks of coverings of curves in the case where $I_{y'}$ is abelian.

1 Definitions and Problems

Let R be a complete valuation ring with algebraically closed residue field k of characteristic $p > 0$, K the quotient field of R, and $\overline{K}$ an algebraic closure of K. We use the notation S to denote the spectrum of R. Write $\eta, \overline{\eta}$ and s for the generic point, the geometric generic point, and the closed point corresponding to the natural morphisms $\operatorname{Spec} K \longrightarrow S$, $\operatorname{Spec} \overline{K} \longrightarrow S$, and $\operatorname{Spec} k \longrightarrow S$, respectively. Let X be a stable curve of genus g_X over S. Write X_η, $X_{\overline{\eta}}$, and X_s for the generic fiber, the geometric generic fiber, and the special fiber, respectively. Moreover, we suppose that X_η is smooth over η.

Definition 1.1. Let $f : Y \longrightarrow X$ be a morphism of stable curves over S and G a finite group. Then f is called a *stable covering* (resp. *G-stable covering*) over S if the morphism of generic fibers f_η is an étale covering (resp. an étale covering whose Galois group is isomorphic to G).

Let Y_η be a geometrically connected curve over η, $f_\eta : Y_\eta \longrightarrow X_\eta$ a finite Galois étale covering over η with Galois group G. By replacing S by a finite extension of S, we may assume that Y_η admits a stable model over S. Then f_η extends uniquely to a G-stable covering $f : Y \longrightarrow X$ over S (cf. [L, Theorem 0.2]). We are interested in understanding the structure of the special fiber Y_s of Y. If the order $\sharp G$ of G is prime to p, then by the specialization theorem for log étale fundamental groups, f_s is an admissible covering (cf. [Y1]); thus, Y_s may be obtained by gluing together tame coverings of the irreducible components of X_s. On the other hand, if $p \mid \sharp G$, then f_s is not a finite morphism in general, where $\sharp(-)$ denotes the cardinality of $(-)$. For example, if $\mathrm{char}(K) = 0$ and $\mathrm{char}(k) = p > 0$, then there exists a Zariski dense subset Z of the set of closed points of X, which may in fact be taken to be X when k is an algebraic closure of $\mathbb{F}_p$, such that for any $x \in Z$, after possibly replacing K by a finite extension of K, there exist a finite group H and an H-stable covering $f_W : W \longrightarrow X$ such that the fiber $(f_W)^{-1}(x)$ is not finite (cf. [T], [Y2]).

Definition 1.2. Let $f : Y \longrightarrow X$ be a stable covering over S. Suppose that the morphism of special fibers $f_s : Y_s \longrightarrow X_s$ is not finite. A closed point $x \in X$ is called a *vertical point associated to* f, or for simplicity, a *vertical point* when there is no fear of confusion, if $f^{-1}(x)$ is not a finite set. The inverse image $f^{-1}(x)$ is called the *vertical fiber associated to* x.

In order to investigate the properties of Y_s, we focus on a geometric invariant $\sigma(Y_s)$ which is called the p-rank of Y_s defined as follows.

Definition 1.3. Let C be a disjoint union of projective curves over an algebraically closed field of characteristic $p > 0$. We define the *p-rank of C* as follows:

$$\sigma(C) := \dim_{\mathbb{F}_p} \mathrm{H}^1_{\text{ét}}(C, \mathbb{F}_p).$$

Remark 1.3.1. Let C be a semi-stable curve over an algebraically closed field of characteristic $p > 0$. Write Γ_C for the dual graph of C, $v(\Gamma_C)$ for the set of vertices of Γ_C. Then we have

$$\sigma(C) = \sum_{v \in v(\Gamma_C)} \sigma(\widetilde{C_v}) + \mathrm{rank}_{\mathbb{Z}} \mathrm{H}^1(\Gamma_C, \mathbb{Z}),$$

where for $v \in v(\Gamma)$, $\widetilde{C}_v$ denotes the normalization of the irreducible component of C corresponding to v.

By Remark 1.3.1, to calculate $\sigma(Y_s)$, it suffices to calculate the rank of $\mathrm{H}^1(\Gamma_{Y_s}, \mathbb{Z})$ (where Γ_{Y_s} denotes the dual graph of Y_s), the p-ranks of the irreducible components of Y_s which are finite over X_s, and the p-ranks of the vertical fibers of f. In the present paper, we study the p-rank of a vertical fiber and consider the following problem:

Problem 1.4. *Let G be a finite p-group, x a vertical point associated to the G-stable covering $f : Y \longrightarrow X$, and $f^{-1}(x)$ the vertical fiber associated to x.*

(a) Does there exist a bound on the p-rank $\sigma(f^{-1}(x))$ which depends only on the order of G (note that $\sigma(f^{-1}(x))$ is always bounded by the genus of Y_s)?

(b) Does there exist an explicit formula for the p-rank $\sigma(f^{-1}(x))$?

Remark 1.4.1. If x is a singular point, Problem 1.4 (a) is an open problem posed by M. Saïdi (cf. [S, Question]).

We will answer Problem 1.4 under the assumption that G is abelian.

2 Results

If x is a nonsingular point, M. Raynaud proved the following result (cf. [R, Théorème 1]):

Theorem 2.1. *If x is a non-singular point of X_s, and G is an arbitrary p-group, then the p-rank $\sigma(f^{-1}(x))$ is equal to 0.*

By Theorem 2.1, in order to resolve Problem 1.4, it is sufficient to consider the case where x is a *singular point* of X_s. In order to explain the results obtained in the present paper, let us introduce some notations. Write X_1 and X_2 for the irreducible components of X_s which contain x, $\psi : Y' \longrightarrow X$ for the normalization of X in the function field of Y. Let $y' \in \psi^{-1}(x)$ be a point in the inverse image of x. Write $I_{y'} \subseteq G$ for the inertia group of y'. In order to calculate the p-rank of $f^{-1}(x)$, since $Y/I_{y'} \longrightarrow X$ is finite étale over x, by replacing X by the stable model of the quotient $Y/I_{y'}$ (note that $Y/I_{y'}$ is a semi-stable curve over S (cf. [R, Appendice, Corollaire])), we may assume that G is equal to $I_{y'}$.

Thus, from the point of view of resolving Problem 1.4, we may assume without loss of generality that $G = I_{y'}$. Then $f^{-1}(x)$ is connected. With regard to Problem 1.4 (a), M. Saïdi proved the following result (cf. [S, Theorem 1]), by applying Theorem 2.1:

Theorem 2.2. *If G is a cyclic p-group, then we have $\sigma(f^{-1}(x)) \leq \sharp G - 1$, where $\sharp G$ denotes the order of G.*

Next, let us explain our main theorems.

Definition 2.3. Let N be a finite p-group and H a subgroup of N. We define $I(H)$ to be a maximal set satisfied the following conditions: (i) $H \in I(H)$; (2) for any two different elements H' and H'' of $I(H)$, neither $H' \subseteq H''$ nor $H' \supseteq H''$ holds. Write $\mathrm{Sub}(N)$ for the set of the subgroups of N. We set

$$M(N) := \max\{\sharp I(N')\}_{I(N'),\ N' \subseteq \mathrm{Sub}(N)}.$$

Let A be an elementary abelian p-group such that $\sharp A = \sharp N$. We set

$$B(\sharp N) := \sharp \mathrm{Sub}(A),$$

where $\mathrm{Sub}(A)$ denotes the set of the subgroups of A. Note that $B(\sharp N)$ depends only on $\sharp N$.

Then for Problem 1.4 (a), we have the following theorem.

Theorem 2.4. *Let G be an abelian p-group. Then we have $\sigma(f^{-1}(x)) \leq M(G)\sharp G - 1 \leq B(\sharp G)\sharp G - 1$.*

Remark 2.4.1. If G is a cyclic p-group, then by the definition, we have $M(G) = 1$. Then Theorem 2.4 is a generalization of Theorem 2.2 to the case where G is abelian.

Remark 2.4.2. Note that for any finite p-group G, we have $M(G) \leq B(\sharp G)$.

Next, let us consider Problem 1.4 (b). Let us introduce some notations. Suppose that G is an abelian p-group. It follows from [R, Appendice, Corollaire], that the quotient Y/G is a semi-stable curve over S. Write X^{sst} for Y/G, g and ϕ for the resulting morphism $g : X^{\mathrm{sst}} \longrightarrow X$ and $\phi : Y \longrightarrow X^{\mathrm{sst}}$ such that $f = g \circ \phi$ induced by f. We still use the notations X_1 and X_2 to denote the strict transforms of X_1 and X_2 in X^{sst}, respectively.

By the general theory of semi-stable curves, $g^{-1}(x)_{\mathrm{red}}$ (the reduced induced closed subscheme of $g^{-1}(x)$) is a semi-stable subcurve of X_s^{sst} whose irreducible components are isomorphic to $\mathbb{P}^1_k$. Write C for the semi-stable subcurve of $g^{-1}(x)_{\mathrm{red}}$ which is a chain of projective lines $\cup_{i=1}^n P_i$ such that the following conditions: (i) P_i is not equal to P_j if $i \neq j$; (ii) $P_1 \cap X_1$ are $P_n \cap X_2$ are not empty; (iii) P_i meets P_{i+1} at only one point; (iv) $P_i \cap P_j$ is empty if j is not equal to $i-1, i$ or $i+1$. Let $\{V_i\}_{i=0}^{n+1}$ be a set of irreducible components of the special fiber Y_s of Y such that the following conditions hold: (i) $\phi_r(V_i) = P_i$ if $i \not\in \{0, n+1\}$; (ii) $\phi_r(V_0) = X_1$ and $\phi_r(V_{n+1}) = X_2$; (iii) the union $\cup_{i=0}^{n+1} V_i$ is a connected sub-semi-stable curve of the special fiber Y_s of Y. Write $I_{P_i} \subseteq G$ for the inertia subgroup of V_i. Note that since G is an abelian p-group, I_{P_i} does not depend on the choices of V_i.

If G is a cyclic p-group, Saïdi obtained an explicit formula of the p-rank $\sigma(f^{-1}(x))$ as follows (cf. [S, Proposition 1]):

Theorem 2.5. *If G is a cyclic p-group, and I_{P_0} is equal to G, then we have*

$$\sigma(f^{-1}(x)) = \sharp(G/I_{\min}) - \sharp(G/I_{P_{n+1}}),$$

where $I_{\min}$ denotes the group $\cap_{i=0}^{n+1} I_{P_i}$.

We develop a general method to compute the p-ranks and generalize Saïdi's formula to the case where G is an arbitrary abelian p-group as follows:

Theorem 2.6. *If G is an arbitrary abelian p-group, then we have*

$$\sigma(f^{-1}(x)) = \sum_{i=1}^{n} \sharp(G/I_{P_i}) - \sum_{i=1}^{n+1} \sharp(G/(I_{P_{i-1}} + I_{P_i})) + 1.$$

Lemma 2.7. *If G is a cyclic group, then there exists $0 \leq u \leq n+1$ such that*

$$I_{P_0} \supseteq I_{P_1} \supseteq I_{P_2} \supseteq \cdots \supseteq I_{P_i} \subseteq \cdots \subseteq I_{P_{n-1}} \subseteq I_{P_n} \subseteq I_{P_{n+1}}.$$

Remark 2.7.1. If G is a cyclic p-group, since G is generated by I_{P_0} and $I_{P_{n+1}}$, we may assume that $I_{P_0} = G$. Follows Lemma 2.7 below, there exists u such that

$$I_{P_0} \supseteq I_{P_1} \supseteq I_{P_2} \supseteq \cdots \supseteq I_{P_u} \subseteq \cdots \subseteq I_{P_{n-1}} \subseteq I_{P_n} \subseteq I_{P_{n+1}}.$$

Then we obtain

$$\sharp(G/I_{P_i}) - \sharp(G/(I_{P_{i-1}} + I_{P_i})) - \sharp(G/(I_{P_{i+1}} + I_{P_i})) + 1 = -\sharp(G/(I_{P_{i-1}})) + 1$$

$$(\text{resp. } \sharp(G/(I_{P_{i+1}} + I_{P_i})) - 1 = \sharp(G/(I_{P_i})) - 1)$$

if $i < u$,

$$\sharp(G/I_{P_i}) - \sharp(G/(I_{P_{i-1}} + I_{P_i})) - \sharp(G/(I_{P_{i+1}} + I_{P_i})) + 1 = -\sharp(G/(I_{P_{i+1}})) + 1$$

$$(\text{resp. } \sharp(G/(I_{P_{i+1}} + I_{P_i})) - 1 = \sharp(G/(I_{P_{i+1}})) - 1)$$

if $i > u$ and

$$\sharp(G/I_{P_i}) - \sharp(G/(I_{P_{i-1}} + I_{P_i})) - \sharp(G/(I_{P_{i+1}} + I_{P_i})) + 1 = \sharp(G/I_{P_t}) - \sharp(G/I_{P_{t-1}}) - \sharp(G/(I_{P_{t+1}}) + 1$$

$$(\text{resp. } \sharp(G/(I_{P_{i+1}} + I_{P_i})) - 1 = \sharp(G/(I_{P_{t+1}})) - 1)$$

if $i = u$. Thus, by applying Theorem 2.6, we obtain

$$\sigma(f^{-1}(x)) = \sharp(G/I_{P_u}) - \sharp(G/I_{P_{n+1}}).$$

Then Theorem 2.6 is a generalization of Theorem 2.5.

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