Euler numbers and related numbers, and their applications

Takao Komatsu (Wuhan University)

Abstract

For a nonnegative integer N, define hypergeometric Euler numbers $E_{N,n}$ by

$$\frac{1}{{}_1F_2(1;N+1,(2N+1)/2;t^2/4)}=\sum_{n=0}^\infty E_{N,n}\frac{t^n}{n!},$$

where ${}_1F_2(a;b,c;z)$ is the hypergeometric function defined by

$${}_1F_2(a;b,c;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}}{(b)^{(n)}(c)^{(n)}}\frac{z^n}{n!}.$$

Here, $(x)^{(n)}$ is the rising factorial, defined by $(x)^{(n)}=x(x+1)\cdots(x+n-1)$ $(n\ge 1)$ with $(x)^{(0)}=1$. When $N=0$, then $E_n=E_{0,n}$ are classical Euler numbers. Hypergeometric Euler numbers $E_{N,n}$ are analogues of hypergeometric Bernoulli numbers $B_{N,n}$ and hypergeometric Cauchy numbers $c_{N,n}$. In this paper, we shall consider several expressions and sums of products of hypergeometric Euler numbers. We also introduce complementary hypergeometric Euler numbers and give some characteristic properties.

1 Introduction

Euler numbers E_n are defined by the generating function

$$\frac{1}{\cosh t}=\sum_{n=0}^\infty E_n\frac{t^n}{n!}. \tag{1}$$

One of the different definitions is

$$\frac{2}{e^t+1}=\sum_{n=0}^\infty E_n\frac{t^n}{n!}$$

(see e.g. [1]). There are several generalizations have been studied based upon one of these expression. For example, one kind of poly-Euler numbers is a typical generalization, in the aspect of L-functions ([14, 15, 16]). Other generalizations can be found in [2, 12] and the reference therein.

A different type of generalization is based upon hypergeometric functions. For $N\ge 1$, define hypergeometric Bernoulli numbers $B_{N,n}$ (see [6, 7, 8]) by

$$\frac{1}{{}_1F_1(1;N+1;t)}=\frac{t^N/N!}{e^t-\sum_{n=0}^{N-1}t^n/n!}=\sum_{n=0}^\infty B_{N,n}\frac{t^n}{n!},$$

where

$${}_1F_1(a;b;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}}{(b)^{(n)}}\frac{z^n}{n!}$$

is the confluent hypergeometric function with $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ $(n \ge 1)$ and $(x)^{(0)} = 1$. When $N = 1$, $B_n = B_{1,n}$ are classical Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

In addition, define hypergeometric Cauchy numbers $c_{N,n}$ (see [9]) by

$$\frac{1}{{}_2F_1(1, N; N+1; -t)} = \frac{(-1)^{N-1} t^N / N}{\log(1+t) - \sum_{n=1}^{N-1} (-1)^{n-1} t^n / n} = \sum_{n=0}^{\infty} c_{N,n} \frac{t^n}{n!},$$

where

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)} (b)^{(n)}}{(c)^{(n)}} \frac{z^n}{n!}$$

is the Gauss hypergeometric function. When $N = 1$, $c_n = c_{1,n}$ are classical Cauchy numbers defined by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}.$$

This is a different generalization of the classical Cauchy numbers. Other kind of generalizations can be seen in [11] and the references therein.

Now, for $N \ge 0$ define *hypergeometric Euler numbers* $E_{N,n}$ $(n = 0, 1, 2, \dots)$ by

$$\frac{1}{{}_1F_2(1; N+1, (2N+1)/2; t^2/4)} = \sum_{n=0}^{\infty} E_{N,n} \frac{t^n}{n!}, \tag{2}$$

where ${}_1F_2(a; b, c; z)$ is the hypergeometric function defined by

$${}_1F_2(a; b, c; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)} (c)^{(n)}} \frac{z^n}{n!}.$$

It is seen that

$$\cosh t - \sum_{n=0}^{N-1} \frac{t^{2n}}{(2n)!} = \frac{t^{2N}}{(2N)!} \sum_{n=0}^{\infty} \frac{(2N)! n!}{(2n+2N)!} \frac{(t^2)^n}{n!} = \frac{t^{2N}}{(2N)!} {}_1F_2\Bigl(1; N+1, \frac{2N+1}{2}; \frac{t^2}{4}\Bigr). \tag{3}$$

When $N = 0$, then $E_n = E_{0,n}$ are classical Euler numbers defined in (1). We list the numbers $E_{N,n}$ for $0 \le N \le 6$ and $0 \le n \le 12$ in Table 1. From (3) we see that $E_{N,n} = 0$ if n is odd. Similarly to poly-Euler numbers ([14, 15, 16]), hypergeometric Euler numbers are rational numbers, though the classical Euler numbers are integers.

From (2) and (3), we have

$$\begin{aligned}
\frac{t^{2N}}{(2N)!} &= \left(\sum_{n=N}^{\infty} \frac{t^{2n}}{(2n)!} \right) \left(\sum_{n=0}^{\infty} E_{N,n} \frac{t^n}{n!} \right) \\
&= t^{2N} \left(\sum_{n=0}^{\infty} \frac{\frac{1+(-1)^n}{2} t^n}{(n+2N)!} \right) \left(\sum_{n=0}^{\infty} E_{N,n} \frac{t^n}{n!} \right) \\
&= t^{2N} \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{\frac{1+(-1)^{n-i}}{2}}{(2N+n-i)!} \frac{E_{N,i}}{i!} \right) t^n.
\end{aligned}$$

Table 1: The numbers $E_{N,n}$ for $0 \le N \le 6$ and $0 \le n \le 12$

n	0	2	4	6	8
$E_{0,n}$	1	-1	5	-61	1385
$E_{1,n}$	1	$-1/6$	$1/10$	$-5/42$	$7/30$
$E_{2,n}$	1	$-1/15$	$13/1050$	$-1/350$	$-31/173250$
$E_{3,n}$	1	$-1/28$	$17/5880$	$-29/362208$	$-863/6420960$
$E_{4,n}$	1	$-1/45$	$7/7425$	$53/2027025$	$-443/22052250$
$E_{5,n}$	1	$-1/66$	$25/66066$	$47/2906904$	$-16945/5300012718$
$E_{6,n}$	1	$-1/91$	$29/165620$	$1205/153728484$	$-2279/4467168888$

n	10	12
$E_{0,n}$	-50521	2702765
$E_{1,n}$	$-15/22$	$7601/2730$
$E_{2,n}$	$1343/750750$	$-6137/2388750$
$E_{3,n}$	$6499/131843712$	$6997213/156894017280$
$E_{4,n}$	$-10157/4873547250$	$558599021/126395447928750$
$E_{5,n}$	$-475767/492312292472$	$71844089/268802511689712$
$E_{6,n}$	$-6430761/25339270989032$	$-17675104079/4917799642149532320$

Hence, for $n \ge 1$, we have

$$\sum_{i=0}^{n} \frac{1+(-1)^{n-i}}{(2N+n-i)!i!} E_{N,i} = 0.$$

Thus, we have the following proposition. Note that $E_{N,n} = 0$ when n is odd.

Proposition 1.

$$\sum_{i=0}^{n/2} \frac{1}{(2N+n-2i)!(2i)!} E_{N,2i} = 0 \quad (n \ge 2 \textit{ is even})$$

and $E_{N,0} = 1$.

By using the identity in Proposition 1 or the identity

$$E_{N,n} = -n!(2N)! \sum_{i=0}^{n/2-1} \frac{E_{N,2i}}{(2N+n-2i)!(2i)!},$$

we can obtain the values of $E_{N,n}$ $(n = 0, 2, 4, \dots)$. We record the first few values of $E_{N,n}$:

$$E_{N,2} = -\frac{2}{(2N+1)(2N+2)},$$
$$E_{N,4} = \frac{2 \cdot 4!(4N+5)}{(2N+1)^2(2N+2)^2(2N+3)(2N+4)},$$
$$E_{N,6} = \frac{4 \cdot 6!(8N^3 - 2N^2 - 65N - 61))}{(2N+1)^3(2N+2)^3(2N+3)(2N+4)(2N+5)(2N+6)},$$
$$E_{N,8} = \frac{16 \cdot 8!(32N^7 - 8N^6 - 1252N^5 - 3914N^4 - 769N^3 + 12667N^2 + 18954N + 8310)}{(2N+1)^4(2N+2)^4(2N+3)^2(2N+4)^2(2N+6)(2N+7)(2N+8)}.$$

We have an explicit expression of $E_{N,n}$ for each even n:

Theorem 1. *For $N \ge 0$ and $n \ge 1$ we have*

$$E_{N,2n} = (2n)! \sum_{r=1}^{n} (-1)^r \sum_{\substack{i_1+\cdots+i_r=n \\ i_1,\dots,i_r \ge 1}} \frac{\big((2N)!\big)^r}{(2N+2i_1)! \cdots (2N+2i_r)!}.$$

The proof can be done by induction for n. However, we shall give a different proof by using the Hasse-Teichmüller derivative in the next section.

2 Hasse-Teichmüller derivative

We define the Hasse-Teichmüller derivative $H^{(n)}$ of order n by

$$H^{(n)}\left(\sum_{m=R}^{\infty} c_m z^m\right) = \sum_{m=R}^{\infty} c_m \binom{m}{n} z^{m-n}$$

for $\sum_{m=R}^{\infty} c_m z^m \in \mathbb{F}((z))$, where R is an integer and $c_m \in \mathbb{F}$ for any $m \ge R$.

The Hasse-Teichmüller derivatives satisfy the product rule [17], the quotient rule [4] and the chain rule [5]. One of the product rules can be described as follows.

Lemma 1. *For $f_i \in \mathbb{F}[[z]]$ $(i = 1, \dots, k)$ with $k \ge 2$ and for $n \ge 1$, we have*

$$H^{(n)}(f_1 \cdots f_k) = \sum_{\substack{i_1,\dots,i_k \ge 0 \\ i_1+\cdots+i_k=n}} H^{(i_1)}(f_1) \cdots H^{(i_k)}(f_k).$$

The quotient rules can be described as follows.

Lemma 2. *For $f \in \mathbb{F}[[z]] \backslash \{0\}$ and $n \ge 1$, we have*

$$H^{(n)}\left(\frac{1}{f}\right) = \sum_{k=1}^{n} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1,\dots,i_k \ge 1 \\ i_1+\cdots+i_k=n}} H^{(i_1)}(f) \cdots H^{(i_k)}(f) \tag{4}$$

$$= \sum_{k=1}^{n} \binom{n+1}{k+1} \frac{(-1)^k}{f^{k+1}} \sum_{\substack{i_1,\dots,i_k \ge 0 \\ i_1+\cdots+i_k=n}} H^{(i_1)}(f) \cdots H^{(i_k)}(f). \tag{5}$$

By using the Hasse-Teichmüller derivative of order n, we shall obtain some explicit expressions of the hypergeometric Euler numbers.

Proof of Theorem 1. Put

$$F := {}_1F_2\big(1; N+1, \frac{2N+1}{2}; \frac{t^2}{4}\big) = \sum_{n=0}^{\infty} \frac{(2N)!}{(2N+2n)!} t^{2n}$$

for simplicity. Note that

$$H^{(i)}(F)\Big|_{t=0} = \sum_{j=0}^{\infty} \frac{(2N)!}{(2N+2j)!} \binom{2j}{i} t^{2j-i} \Bigg|_{t=0} = \begin{cases} (2N)!/(2N+i)! & \text{if } i \text{ is even;} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

Hence, by using Lemma 2 (4), we have

$$
\begin{aligned}
\frac{E_{N,n}}{n!} &= H^{(n)}\left(\frac{1}{F}\right)\Big|_{t=0}\\
&=\sum_{k=1}^{n}\frac{(-1)^k}{F^{k+1}}\Big|_{t=0}\sum_{\substack{i_1,\dots,i_k\ge 1\\ i_1+\cdots+i_k=n}} H^{(i_1)}(F)\Big|_{t=0}\cdots H^{(i_k)}(F)\Big|_{t=0}\\
&=\sum_{k=1}^{n}(-1)^k\sum_{\substack{i_1,\dots,i_k\ge 1\\ 2(i_1+\cdots+i_k)=n}}\frac{\bigl((2N)!\bigr)^k}{(2N+2i_1)!\cdots(2N+2i_k)!}.
\end{aligned}
$$

□

We can express the hypergeometric Euler numbers also in terms of the binomial coefficients. In fact, by using Lemma 2 (5) instead of Lemma 2 (4) in the above proof, we obtain a little different expression from one in Theorem 1.

Proposition 2. *For $N\ge 0$ and even $n\ge 2$,*

$$
E_{N,n}=n!\sum_{k=1}^{n}(-1)^k\binom{n+1}{k+1}\sum_{\substack{i_1,\dots,i_k\ge 0\\ i_1+\cdots+i_k=n/2}}\frac{\bigl((2N)!\bigr)^k}{(2N+2i_1)!\cdots(2N+2i_k)!}.
$$

For example, when $n=4$, we get

$$
\begin{aligned}
E_4 &= 4!\left(-\binom{5}{2}\frac{1}{4!}+\binom{5}{3}\left(\frac{2}{4!}+\frac{1}{2!2!}\right)-\binom{5}{4}\left(\frac{3}{4!}+\frac{3}{2!2!}\right)+\binom{5}{5}\left(\frac{4}{4!}+\frac{6}{2!2!}\right)\right)\\
&=5,\\
E_{1,4} &= 4!\left(-\binom{5}{2}2\frac{1}{6!}+\binom{5}{3}2^2\left(\frac{2}{6!2!}+\frac{1}{4!4!}\right)\right.\\
&\qquad\left.-\binom{5}{4}2^3\left(\frac{3}{6!2!2!}+\frac{3}{4!4!2!}\right)+\binom{5}{5}2^4\left(\frac{4}{6!2!2!2!}+\frac{6}{4!4!2!2!}\right)\right)\\
&=\frac{1}{10},\\
E_{2,4} &= 4!\left(-\binom{5}{2}4!\frac{1}{8!}+\binom{5}{3}(4!)^2\left(\frac{2}{8!4!}+\frac{1}{6!6!}\right)\right.\\
&\qquad\left.-\binom{5}{4}(4!)^3\left(\frac{3}{8!4!4!}+\frac{3}{6!6!4!}\right)+\binom{5}{5}(4!)^4\left(\frac{4}{8!4!4!4!}+\frac{6}{6!6!4!4!}\right)\right)\\
&=\frac{13}{1050},\\
E_{3,4} &= 4!\left(-\binom{5}{2}6!\frac{1}{10!}+\binom{5}{3}(6!)^2\left(\frac{2}{10!6!}+\frac{1}{8!8!}\right)\right.\\
&\qquad\left.-\binom{5}{4}(6!)^3\left(\frac{3}{10!6!6!}+\frac{3}{8!8!6!}\right)+\binom{5}{5}(6!)^4\left(\frac{4}{10!6!6!6!}+\frac{6}{8!8!6!6!}\right)\right)\\
&=\frac{17}{5880}.
\end{aligned}
$$

3 Complementary hypergeometric Euler numbers

We introduce the *complementary hypergeometric Euler numbers* $\widehat{E}_{N,n}$ by

$$\frac{t^{2N+1}/(2N+1)!}{\sinh t-\sum_{n=0}^{N-1}t^{2n+1}/(2n+1)!}=\frac{1}{{}_1F_2(1;N+1,(2N+3)/2;t^2/4)}=\sum_{n=0}^{\infty}\widehat{E}_{N,n}\frac{t^n}{n!}\tag{6}$$

as an analogous concept of (2). When $n=0$, $\widehat{E}_n=\widehat{E}_{0,n}$ are the *complementary Euler numbers* defined by

$$\frac{t}{\sinh t}=\sum_{n=0}^{\infty}\widehat{E}_n\frac{t^n}{n!}$$

as an analogous concept of (1). In [13], these numbers are called *weighted Bernoulli numbers*, but this naming means different in other literatures.

From the definition (6), we have

$$\begin{aligned}\frac{t^{2N+1}}{(2N+1)!}&=\left(\sum_{n=N}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty}\widehat{E}_{N,n}\frac{t^n}{n!}\right)\\&=t^{2N+1}\left(\sum_{n=0}^{\infty}\frac{\frac{1+(-1)^n}{2}t^n}{(2N+n+1)!}\right)\left(\sum_{n=0}^{\infty}\widehat{E}_{N,n}\frac{t^n}{n!}\right)\\&=t^{2N+1}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{\frac{1+(-1)^{n-i}}{2}}{(2N+n-i+1)!}\frac{\widehat{E}_{N,i}}{i!}\right)t^n.\end{aligned}$$

Therefore, the complementary hypergeometric Euler numbers satisfy the recurrence relation for even $n\ge 2$

$$\sum_{i=0}^{n/2}\frac{\widehat{E}_{N,2i}}{(2N+n-2i+1)!(2i)!}=0$$

or for $n\ge 1$

$$\widehat{E}_{N,2n}=-(2n)!(2N+1)!\sum_{i=0}^{n-1}\frac{\widehat{E}_{N,2i}}{(2N+2n-2i+1)!(2i)!}.\tag{7}$$

By using the Hasse-Teichmüller derivative or by proving by induction, we have the following.

Theorem 2. *For $N\ge 0$ and $n\ge 1$ we have*

$$\begin{aligned}\widehat{E}_{N,n}&=n!\sum_{k=1}^{n}(-1)^k\sum_{\substack{i_1,\dots,i_k\ge 1\\ i_1+\dots+i_k=n/2}}\frac{\big((2N+1)!\big)^k}{(2N+2i_1+1)!\cdots(2N+2i_k+1)!}\\&=n!\sum_{k=1}^{n}(-1)^k\binom{n+1}{k+1}\sum_{\substack{i_1,\dots,i_k\ge 0\\ i_1+\dots+i_k=n/2}}\frac{\big((2N+1)!\big)^k}{(2N+2i_1+1)!\cdots(2N+2i_k+1)!}.\end{aligned}$$

Some initial values of $\widehat{E}_{N,n}$ $(n = 0, 2, 4, \dots)$, we have

$$\begin{aligned}
\widehat{E}_{N,2} &= -\frac{2}{(2N+2)(2N+3)},\\
\widehat{E}_{N,4} &= \frac{2\cdot 4!(4N+7)}{(2N+2)^2(2N+3)^2(2N+4)(2N+5)},\\
\widehat{E}_{N,6} &= \frac{4\cdot 6!(8N^3+10N^2-61N-93))}{(2N+2)^3(2N+3)^3(2N+4)(2N+5)(2N+6)(2N+7)},\\
\widehat{E}_{N,8} &= \frac{8\cdot 8!(32N^6+8N^5-1132N^4-3538N^3-1063N^2+7280N+6858)}{(2N+2)^4(2N+3)^4(2N+4)^2(2N+5)^2(2N+7)(2N+8)(2N+9)}.
\end{aligned}$$

4 Expressions in terms of the determinants

It is known that the Euler numbers are given by the determinant (*cf.* [3, p.52]):

$$E_{2n} = (-1)^n (2n)! \begin{vmatrix} \frac{1}{2!} & 1 & & & \\ \frac{1}{4!} & \frac{1}{2!} & 1 & & \\ \vdots & & \ddots & \ddots & \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & & \frac{1}{2!} & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}. \tag{8}$$

This can be generalized to the determinant expression of hypergeometric Euler numbers. Namely, when $N = 0$ in Theorem 3, we get (8) as a special case.

Theorem 3. *For $N \ge 0$ and $n \ge 1$, we have*

$$E_{N,2n} = (-1)^n (2n)! \begin{vmatrix} \frac{(2N)!}{(2N+2)!} & 1 & & \\ \frac{(2N)!}{(2N+4)!} & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \frac{(2N)!}{(2N+2n)!} & \cdots & \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!} \end{vmatrix}.$$

Proof. In Proposition 1, it is shown that hypergeometric Euler numbers $E_{N,n}$ satisfy the relation:

$$\sum_{i=0}^{n/2} \frac{1}{(2N+n-2i)!(2i)!} E_{N,2i} = 0 \quad (n \ge 2 \text{ is even})$$

with $E_{N,0} = 1$. The proof is done by using this relation. The detail is similar to the next theorem. □

It turns that $\widehat{E}_{N,2n}$ can be expressed by the determinant too.

Theorem 4. *For $N \ge 0$ and $n \ge 1$, we have*

$$\widehat{E}_{N,2n} = (-1)^n (2n)! \begin{vmatrix} \frac{(2N+1)!}{(2N+3)!} & 1 & & \\ \frac{(2N+1)!}{(2N+5)!} & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \frac{(2N+1)!}{(2N+2n+1)!} & \cdots & \frac{(2N+1)!}{(2N+5)!} & \frac{(2N+1)!}{(2N+3)!} \end{vmatrix}.$$

Proof. When $n = 1$, we have

$$\widehat{E}_{N,2} = -\frac{2}{(2N+1)(2N+2)}.$$

For convenience, put

$$|A_{N,2n}| = \begin{vmatrix} \frac{(2N+1)!}{(2N+3)!} & 1 & & \\ \frac{(2N+1)!}{(2N+5)!} & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \frac{(2N+1)!}{(2N+2n+1)!} & \cdots & \frac{(2N+1)!}{(2N+5)!} & \frac{(2N+1)!}{(2N+3)!} \end{vmatrix}$$

with $|A_{N,0}| = 1$. Then, using the induction with (7), we have

$$\begin{aligned} &(-1)^n (2n)! |A_{N,2n}| \\ &= (-1)^n (2n)! \left(\frac{(2N+1)!}{(2N+3)!} |A_{N,2n-2}| - \begin{vmatrix} \frac{(2N+1)!}{(2N+3)!} & 1 & & \\ \vdots & & \ddots & \\ \frac{(2N+1)!}{(2N+2n-3)!} & \cdots & \frac{(2N+1)!}{(2N+3)!} & 1 \\ \frac{(2N+1)!}{(2N+2n+1)!} & \cdots & \frac{(2N+1)!}{(2N+5)!} & \frac{(2N+1)!}{(2N+3)!} \end{vmatrix} \right) \\ &= (-1)^n (2n)! \left(\frac{(2N+1)!}{(2N+3)!} |A_{N,2n-2}| - \frac{(2N+1)!}{(2N+5)!} |A_{N,2n-4}| \right. \\ &\qquad \left. + \cdots + \frac{(-1)^n (2N+1)!}{(2N+2n-1)!} |A_{N,2}| + \frac{(-1)^{n+1} (2N+1)!}{(2N+2n+1)!} \right) \\ &= -(2n)!(2N+1)! \sum_{i=0}^{n-1} \frac{\widehat{E}_{N,2i}}{(2N+2n-2i+1)!(2i)!} \\ &= \widehat{E}_{N,2n}. \end{aligned}$$

□

When $N = 0$ in Theorem 4, we obtain the determinant expression of Euler numbers of the second kind, which corresponds with that of Euler numbers in (8).

Corollary 1. *For $n \ge 1$, we have*

$$\widehat{E}_{2n} = (-1)^n (2n)! \begin{vmatrix} \frac{1}{3!} & 1 & & \\ \frac{1}{5!} & \ddots & \ddots & \\ \vdots & & \ddots & 1 \\ \frac{1}{(2n+1)!} & \cdots & \frac{1}{5!} & \frac{1}{3!} \end{vmatrix}.$$

5 Sums of products of hypergeometric Euler numbers

Put

$$F := {}_1F_2\big(1; N+1, \frac{2N+1}{2}; \frac{t^2}{4}\big) = \sum_{n=0}^{\infty} \frac{(2N)!}{(2N+2n)!} t^{2n}$$

for simplicity. Then by

$$\frac{d}{dt}F = \sum_{n=0}^{\infty} \frac{(2n)(2N)!}{(2N+2n)!} t^{2n-1},$$

we have

$$2NF + t\frac{t}{dt}F = 2N \cdot {}_1F_2\big(1; N, \frac{2N+1}{2}; \frac{t^2}{4}\big). \tag{9}$$

Since

$$F^* := {}_1F_2\big(1; N, \frac{2N+1}{2}; \frac{t^2}{4}\big) = \sum_{n=0}^{\infty} \frac{(2N-1)!}{(2N+2n-1)!} t^{2n}$$

and

$$\frac{d}{dt}F = -F^2 \frac{d}{dt}\frac{1}{F}, \tag{10}$$

by (9) we have

$$\frac{1}{F^2} = \frac{1}{F^*}\left(\frac{1}{F} - \frac{t}{2N}\frac{d}{dt}\frac{1}{F}\right). \tag{11}$$

Since

$$\frac{1}{F^*} = \frac{t^{2N-1}}{(2N-1)! \sum_{n=0}^{\infty} t^{2N+2n-1}/(2N+2n-1)!} = \sum_{n=0}^{\infty} \widehat{E}_{N-1,n} \frac{t^n}{n!}$$

and

$$\frac{1}{F} - \frac{t}{2N}\frac{d}{dt}\frac{1}{F} = \sum_{n=0}^{\infty} E_{N,n}\frac{t^n}{n!} - \frac{t}{2N}\sum_{n=1}^{\infty} E_{N,n}\frac{t^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{2N-n}{2N} E_{N,n}\frac{t^n}{n!},$$

we have

$$\begin{aligned}\frac{1}{F^*}\left(\frac{1}{F} - \frac{t}{2N}\frac{d}{dt}\frac{1}{F}\right) &= \left(\sum_{m=0}^{\infty} \widehat{E}_{N-1,m}\frac{t^m}{m!}\right)\left(\sum_{k=0}^{\infty} \frac{2N-k}{2N} E_{N,k}\frac{t^k}{k!}\right)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \frac{2N-k}{2N} E_{N,k}\widehat{E}_{N-1,n-k}\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients, we obtain a result about the sums of products.

Theorem 5. *For $N \ge 1$ and $n \ge 0$,*

$$\sum_{i=0}^{n} \binom{n}{i} E_{N,i}E_{N,n-i} = \sum_{k=0}^{n} \binom{n}{k} \frac{2N-k}{2N} E_{N,k}\widehat{E}_{N-1,n-k}.$$

Using (10) and (11) again, we have

$$\frac{1}{F^3} = \frac{1}{F^*}\left(\frac{1}{F^2} - \frac{t}{2N}\frac{1}{F}\frac{d}{dt}\frac{1}{F}\right) = \frac{1}{F^*}\left(\frac{1}{F^2} - \frac{t}{4N}\frac{d}{dt}\frac{1}{F^2}\right).$$

Since

$$\frac{1}{F^2} - \frac{t}{4N}\frac{d}{dt}\frac{1}{F^2} = \sum_{n=0}^{\infty} \frac{4N-n}{4N} \sum_{k=0}^{n} \binom{n}{k} \frac{2N-k}{2N} E_{N,k}\widehat{E}_{N-1,n-k}\frac{t^n}{n!},$$

we have

$$\begin{aligned}
&\frac{1}{F^*}\left(\frac{1}{F^2}-\frac{t}{4N}\frac{d}{dt}\frac{1}{F^2}\right)\\
&=\left(\sum_{i=0}^\infty \widehat{E}_{N-1,i}\frac{t^i}{i!}\right)\left(\sum_{m=0}^\infty \frac{4N-m}{4N}\sum_{k=0}^m\binom{m}{k}\frac{2N-k}{2N}E_{N,k}\widehat{E}_{N-1,m-k}\frac{t^m}{m!}\right)\\
&=\sum_{n=0}^\infty\sum_{m=0}^n\sum_{k=0}^m\binom{n}{m}\binom{m}{k}\frac{(4N-m)(2N-k)}{8N^2}E_{N,k}\widehat{E}_{N-1,n-m}\widehat{E}_{N-1,m-k}\frac{t^n}{n!},
\end{aligned}$$

Comparing the coefficients, we get a result about the sums of products for trinomial coefficients.

Theorem 6. *For $N \ge 1$ and $n \ge 0$,*

$$\begin{aligned}
&\sum_{\substack{i_1+i_2+i_3=n\\ i_1,i_2,i_3\ge 0}}\binom{n}{i_1,i_2,i_3}E_{N,i_1}E_{N,i_2}E_{N,i_3}\\
&\qquad=\sum_{m=0}^n\sum_{k=0}^m\binom{n}{m}\binom{m}{k}\frac{(4N-m)(2N-k)}{8N^2}E_{N,k}\widehat{E}_{N-1,n-m}\widehat{E}_{N-1,m-k}.
\end{aligned}$$

6 Sums of products of complementary hypergeometric Euler numbers

Put

$$\widehat{F}=\sum_{n=0}^\infty\frac{(2N+1)!}{(2N+2n+1)!}t^{2n}$$

so that

$$\frac{1}{\widehat{F}}=\sum_{n=0}^\infty \widehat{E}_{N,n}\frac{t^n}{n!}.$$

Since

$$(2N+1)\widehat{F}+t\frac{d}{dt}\widehat{F}=(2N+1)F,$$

we have

$$\begin{aligned}
\frac{1}{\widehat{F}^2}&=\frac{1}{F}\left(\frac{1}{\widehat{F}}-\frac{t}{2N+1}\frac{d}{dt}\frac{1}{\widehat{F}}\right)\\
&=\left(\sum_{m=0}^\infty E_{N,m}\frac{t^m}{m!}\right)\left(\sum_{k=0}^\infty\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}\frac{t^k}{k!}\right)\\
&=\sum_{n=0}^\infty\sum_{k=0}^n\binom{n}{k}\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}E_{N,n-k}\frac{t^n}{n!}.
\end{aligned}$$

Hence, as an analogue of Theorem 5, we have the following.

Theorem 7. *For $N \ge 1$ and $n \ge 0$,*

$$\sum_{i=0}^n\binom{n}{i}\widehat{E}_{N,i}\widehat{E}_{N,n-i}=\sum_{k=0}^n\binom{n}{k}\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}E_{N,n-k}.$$

We then have

$$\frac{1}{\widehat{F}^3}=\frac{1}{F}\left(\frac{1}{\widehat{F}^2}-\frac{t}{2(2N+1)}\frac{d}{dt}\frac{1}{\widehat{F}^2}\right).$$

Since

$$\frac{1}{\widehat{F}^2}-\frac{t}{2(2N+1)}\frac{d}{dt}\frac{1}{\widehat{F}^2}=\sum_{n=0}^{\infty}\frac{4N-n+2}{2(2N+1)}\sum_{k=0}^{n}\binom{n}{k}\frac{2N-k+1}{2N+1}\widehat{E}_{N,k}E_{N,n-k}\frac{t^n}{n!},$$

we have the following result as an analogue of Theorem 6.

Theorem 8. *For $N\ge 1$ and $n\ge 0$,*

$$\sum_{\substack{i_1+i_2+i_3=n\\ i_1,i_2,i_3\ge 0}}\binom{n}{i_1,i_2,i_3}\widehat{E}_{N,i_1}\widehat{E}_{N,i_2}\widehat{E}_{N,i_3}$$
$$=\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{n}{m}\binom{m}{k}\frac{(4N-m+2)(2N-k+1)}{2(2N+1)^2}\widehat{E}_{N,k}E_{N,n-m}E_{N,m-k}.$$

One can continue to obtain the sum of four or more products, though the results seem to become more complicated.

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