Minus space of half-integral weight modular forms

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We summarize our joint work with E. M. Baruch [3] where we give a counterpart of Kohnen's newform theory of half-integral weight modular forms.

Let M be odd and square-free and k be a positive integer. In a remarkable work, Niwa [8] comparing the traces of Hecke operators proved existence of Hecke isomorphism between $S_{k+1/2}(\Gamma_0(4M))$ and $S_{2k}(\Gamma_0(2M))$.

In [5, 6] Kohnen defines plus space $S^{+}_{k+1/2}(4M)$, a subspace of the space $S_{k+1/2}(\Gamma_0(4M))$, which is given by a certain Fourier coefficient condition. Kohnen considers a new space $S^{+,\mathrm{new}}_{k+1/2}(4M)$ inside his plus space and proves that this new subspace is Hecke isomorphic to $S^{\mathrm{new}}_{2k}(\Gamma_0(M))$, the space of newforms of weight $2k$ and level M, giving a newform theory of half-integral weight.

In the case $M = 1$ the Kohnen plus space is given as an eigenspace of a certain Hecke operator considered by Niwa [8]. Niwa's operator has two eigenvalues of opposite sign and the Kohnen plus space is the eigenspace of the positive eigenvalue. In this case the Kohnen plus space itself is the newspace $S^{+,\mathrm{new}}_{k+1/2}(4)$ and is isomorphic as a Hecke module to $S_{2k}(\Gamma_0(1))$. In the case $M > 1$, the Kohnen plus space can be again given as an eigenspace of a certain operator [6], we prove that this operator at level $4M$ is an analogue of Niwa's operator at level 4.

From Kohnen's results, it is clear that the Niwa map sends the Kohnen plus space to a subspace of old forms inside $S_{2k}(\Gamma_0(2M))$. It is natural to look for a subspace of $S_{k+1/2}(\Gamma_0(4M))$ that maps Hecke isomorphically to $S^{\mathrm{new}}_{2k}(2M)$. We identify such a subspace of half-integer weight forms, we call it the "minus space" at level $4M$. We note that this space complements the Kohnen plus space, however, their sum will not give the whole space if the Kohnen plus space is nonzero.

In order to desribe our minus space we need certain operators on $S_{k+1/2}(\Gamma_0(4M))$. For that we shall consider genuine Hecke algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ modulo certain subgroup $\overline{K^p_0(p)}$ and a genuine central character for every prime p dividing $2M$. Our work follows that of Loke and Savin who studies a certain 2-adic Hecke algebra which allowed them to give a representation theoretic interpretation of the Kohnen plus space at level 4. In our p-adic Hecke algebra, we consider two p-adic operators that give rise to conjugate classical Hecke operators which when we use along with Niwa's operator and it's conjugate allow us to define our minus space at level $4M$. We give two descriptions of the minus space: One description as an orthogonal complement of a certain sum of subspaces and another description as a common -1 eigenspace of the Niwa operator, its conjugate and a certain pair of conjugate operators for each prime dividing M. This is completely analogous to our description of the space of newforms of weight $2k$ for $\Gamma_0(2M)$ given in [2]. Our main result is that our minus space of weight $k + 1/2$ at level $4M$ is isomorphic as a Hecke module to the space of newforms of weight $2k$ at level $2M$.

1 Preliminaries

Let p be any prime (including infinite place). The group $\mathrm{SL}_2(\mathbb{Q}_p)$ has a non-trivial central extension by $\mu_2 = \{\pm 1\}$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p) & \longrightarrow & \mathrm{SL}_2(\mathbb{Q}_p) & \longrightarrow & 1 \\ & & \{(I, \pm 1)\} & & (g, \pm 1) & \longmapsto & g & & \end{array}$$

We use the 2-cocycle described by Gelbart [4] to determine the double cover $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$, define

$$\tau(g) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0; \end{cases}$$

if $p = \infty$, set $s_p(g) = 1$ while for a finite prime p

$$s_p(g) = \begin{cases} (c, d)_p & \text{if } cd \neq 0 \text{ and } \mathrm{ord}_p(c) \text{ is odd}, \\ 1 & \text{else}. \end{cases}$$

Define the 2-cocycle σ_p on $\mathrm{SL}_2(\mathbb{Q}_p)$ as follows:

$$\sigma_p(g, h) = (\tau(gh)\tau(g), \tau(gh)\tau(h))_p \, s_p(g) s_p(h) s_p(gh).$$

Then the double cover $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ is the set $\mathrm{SL}_2(\mathbb{Q}_p) \times \mu_2$ with the group law:

$$(g, \ \epsilon_1)(h, \ \epsilon_2) = (gh, \ \epsilon_1 \epsilon_2 \sigma_p(g, h)).$$

For any subgroup H of $\mathrm{SL}_2(\mathbb{Q}_p)$, we shall denote by $\overline{H}$ the complete inverse image of H in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$.

We consider the following subgroups of $\mathrm{SL}_2(\mathbb{Z}_p)$:

$$K_0(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) \ : \ c \in p^n \mathbb{Z}_p \right\},$$
$$K_1(p^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) \ : \ c \in p^n \mathbb{Z}_p, \ a \equiv 1 \pmod{p^n \mathbb{Z}_p} \right\}.$$

By [4] for odd primes p, $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ splits over $\mathrm{SL}_2(\mathbb{Z}_p)$ and the center M_p of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ is direct product $\{\pm I\} \times \mu_2$. However $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ does not split over $\mathrm{SL}_2(\mathbb{Z}_2)$ but instead splits over subgroup

$$K_1(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_2) \ : \ c \equiv 0, \ a \equiv 1 \pmod{4\mathbb{Z}_2} \right\}.$$

The center M_2 of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ is a cyclic group of order 4 generated by $(-I, 1)$.

Loke and Savin [7] considered a genuine Hecke algebra for $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_2)$ corresponding to $\overline{K_0(4)}$ and a genuine central character. In the next section we shall study genuine Hecke algebras for $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ corresponding to $\overline{K_0(p)}$ and a given genuine character of M_p for general odd prime p.

We set up the following notation: For $s \in \mathbb{Q}_p$, $t \in \mathbb{Q}_p^\times$, consider the following elements of $\mathrm{SL}_2(\mathbb{Q}_p)$:

$$x(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \ y(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \ w(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}, \ h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

2 Iwahori Hecke Algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ modulo $\overline{K_0(p)}$, p odd

Fix an odd prime p. Let γ be a character of $K_0(p)$ such that it is trivial on $K_1(p)$. Since $\frac{K_0(p)}{K_1(p)} \cong (\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ we can define γ by a character of $(\mathbb{Z}/p\mathbb{Z})^\times$. We shall use the symbol γ to also denote a genuine character on $\overline{K_0(p)}$ by defining $\gamma(A,\epsilon) = \epsilon\gamma(A)$ for $A \in K_0(p)$.

Let $C_c^\infty(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p))$ denote the space of locally constant, compactly supported complex-valued functions on $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ and $H(\overline{K_0(p)}, \gamma)$ be the subalgebra defined as follows:

$$\{f \in C_c^\infty(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)) \,:\, f(\tilde{k}\tilde{g}\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}(\tilde{k}')f(\tilde{g}) \text{ for } \tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p),\ \tilde{k},\ \tilde{k}' \in \overline{K_0(p)}\}.$$

Then $H(\overline{K_0(p)}, \gamma)$ is a $\mathbb{C}$-algebra under convolution which, for any $f_1, f_2 \in H(\overline{K_0(p)}, \gamma)$, is defined by

$$f_1 * f_2(\tilde{h}) = \int_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)} f_1(\tilde{g}) f_2(\tilde{g}^{-1}\tilde{h}) d\tilde{g} = \int_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)} f_1(\tilde{h}\tilde{g}) f_2(\tilde{g}^{-1}) d\tilde{g},$$

where $d\tilde{g}$ is the Haar measure on $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ such that the measure of $\overline{K_0(p)}$ is one.

We say $H(\overline{K_0(p)}, \gamma)$ "genuine " Iwahori Hecke algebra of $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ with respect to $\overline{K_0(p)}$ and central character γ. We want to describe it using generators and relations.

Lemma 2.1. *A complete set of representatives for the double cosets of* $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ *mod* $\overline{K_0(p)}$ *are given by* $(h(p^n), 1)$, $(w(p^{-n}), 1)$ *where* n *varies over integers.*

We say that $H(\overline{K_0(p)}, \gamma)$ is supported on $\tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ if there exists a $f \in H(\overline{K_0(p)}, \gamma)$ such that $f(\tilde{g}) \neq 0$. We note that in general $H(\overline{K_0(p)}, \gamma)$ need not be supported on the double coset representatives $(h(p^n), 1)$, $(w(p^{-n}), 1)$. However we can prove the following

Proposition 2.2. *If* γ *is a quadratic character then* $H(\overline{K_0(p)}, \gamma)$ *is supported on the double cosets of* $\overline{K_0(p)}$ *represented by* $(h(p^n), 1)$ *and* $(w(p^{-n}), 1)$ *where* n *varies over integers.*

For the rest of this section we shall assume γ to be quadratic.

2.1 Decomposition of double cosets into union of single cosets and convolution formulae

We have the following decomposition that can be obtained using triangular decomposition of $K_0(p)$.

Lemma 2.3. (1) *For* $n \geq 0$,

$$K_0(p)h(p^n)K_0(p) = \bigcup_{s \in \mathbb{Z}_p/p^{2n}\mathbb{Z}_p} x(s)h(p^n)K_0(p) = \bigcup_{s \in \mathbb{Z}_p/p^{2n}\mathbb{Z}_p} K_0(p)h(p^n)y(ps).$$

(2) *For* $n \geq 1$,

$$K_0(p)h(p^{-n})K_0(p) = \bigcup_{s \in \mathbb{Z}_p/p^{2n}\mathbb{Z}_p} y(ps)h(p^{-n})K_0(p) = \bigcup_{s \in \mathbb{Z}_p/p^{2n}\mathbb{Z}_p} K_0(p)h(p^{-n})x(s).$$

(3) *For* $n \geq 1$,

$$K_0(p)w(p^{-n})K_0(p) = \bigcup_{s \in \mathbb{Z}_p/p^{2n-1}\mathbb{Z}_p} y(ps)w(p^{-n})K_0(p) = \bigcup_{s \in \mathbb{Z}_p/p^{2n-1}\mathbb{Z}_p} K_0(p)w(p^{-n})y(ps).$$

(4) *For* $n \geq 0$,

$$K_0(p)w(p^n)K_0(p) = \bigcup_{s \in \mathbb{Z}_p/p^{2n+1}\mathbb{Z}_p} x(s)w(p^n)K_0(p) = \bigcup_{s \in \mathbb{Z}_p/p^{2n+1}\mathbb{Z}_p} K_0(p)w(p^n)x(s).$$

We note the following lemmas that are used in computing convolutions in order to obtain relations amongst Hecke algebra elements.

Lemma 2.4. *Let $f_1,\ f_2 \in H(\overline{K_0(p)},\gamma)$ such that f_1 is supported on $\overline{K_0(p)}\tilde{x}\overline{K_0(p)} = \bigcup_{i=1}^{m}\tilde{\alpha}_i\overline{K_0(p)}$ and f_2 is supported on $\overline{K_0(p)}\tilde{y}\overline{K_0(p)} = \bigcup_{j=1}^{n}\tilde{\beta}_i\overline{K_0(p)}$. Then*

$$f_1 * f_2(\tilde{h}) = \sum_{i=1}^{m} f_1(\tilde{\alpha}_i)f_2(\tilde{\alpha}_i^{-1}\tilde{h})$$

where the nonzero summands are precisely for those i for which there exist a j such that $\tilde{h} \in \tilde{\alpha}_i\tilde{\beta}_j\overline{K_0(p)}$.

For $\tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ let $\mu(\tilde{g})$ denotes the number of disjoint left (right) $\overline{K_0(p)}$ cosets in the decomposition of double coset $\overline{K_0(p)}\tilde{g}\overline{K_0(p)}$.

Lemma 2.5. *Let $\tilde{g},\ \tilde{h} \in \widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$ be such that $\mu(\tilde{g})\mu(\tilde{h}) = \mu(\tilde{g}\tilde{h})$. Let f_1 and $f_2 \in H(\overline{K_0(p)},\gamma)$ are respectively supported on $\overline{K_0(p)}\tilde{g}\overline{K_0(p)}$ and $\overline{K_0(p)}\tilde{h}\overline{K_0(p)}$. Then $f_1 * f_2$ is precisely supported on $\overline{K_0(p)}\tilde{g}\tilde{h}\overline{K_0(p)}$ and $f_1 * f_2(\tilde{g}\tilde{h}) = f_1(\tilde{g})f_2(\tilde{h})$.*

2.2 Generators and relations

Let $T = \{(h(t),\epsilon)\ :\ t \in \mathbb{Q}_p^\times,\ \epsilon = \pm 1\}$ and $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)}(T)$ be the normalizer of T in $\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)$. Note that $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)}(T)$ is generated by elements $(h(t),\epsilon)$, $(w(t),\epsilon)$ for $t \in \mathbb{Q}_p^\times$. We extend the character γ of $\overline{K_0(p)}$ to the normalizer group $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)}(T)$.

Let $\varepsilon_p = 1$ or $(-1)^{1/2}$ depending on whether $p \equiv 1$ or $3 \pmod 4$. Thus $\varepsilon_p^2 = \left(\frac{-1}{p}\right)$. Define

$$\gamma((h(t),1)) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \varepsilon_p \left(\frac{u}{p}\right) & \text{if } n \text{ is odd.} \end{cases}$$

One can check that γ indeed extends to a character on T.

We now extend the character to the normalizer $N_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)}(T)$ by defining $\gamma((w(1),1)) = 1$ and extend it using the following relation

$$(w(t),1) = (h(t),1)(w(1),1)(I,\left(-1,t^{-1}\right)_p).$$

Let $t = p^n u$. We define

$$\gamma((w(t),1)) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ \varepsilon_p \left(\frac{-u}{p}\right) & \text{if } n \text{ is odd.} \end{cases}$$

We define the following elements of $H(\overline{K_0(p)},\gamma)$ supported respectively on the double cosets of $(h(p^n),1)$ and $(w(p^{-n}),1)$: For $n \in \mathbb{Z}$, and $\tilde{k},\tilde{k}' \in \overline{K_0(p)}$,

$$X_{(h(p^n),1)}(\tilde{k}(h(p^n),1)\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}((h(p^n),1))\overline{\gamma}(\tilde{k}'),$$
$$X_{(w(p^{-n}),1)}(\tilde{k}(w(p^{-n}),1)\tilde{k}') = \overline{\gamma}(\tilde{k})\overline{\gamma}((w(p^{-n}),1))\overline{\gamma}(\tilde{k}').$$

Let $\mathcal{T}_n = X_{(h(p^n),1)}$ and $\mathcal{U}_n = X_{(w(p^{-n}),1)}$. Then Proposition 2.2 implies that $\mathcal{T}_n$ and $\mathcal{U}_n$ forms a $\mathbb{C}$-basis for $H(\overline{K_0(p)},\gamma)$.

Using decomposition 2.3 and lemmas 2.4, 2.5 we obtain the following.

Proposition 2.6. *We have following relations:*

(1) *If* $mn \geq 0$ *then* $\mathcal{T}_m * \mathcal{T}_n = \mathcal{T}_{m+n}$.

(2) *For* $n \geq 0$, $\mathcal{U}_1 * \mathcal{T}_n = \mathcal{U}_{n+1}$ *and* $\mathcal{T}_{-n} * \mathcal{U}_1 = \mathcal{U}_{n+1}$.

(3) *For* $n \geq 0$, $\mathcal{U}_0 * \mathcal{T}_{-n} = \mathcal{U}_{-n}$ *and* $\mathcal{T}_n * \mathcal{U}_0 = \mathcal{U}_{-n}$.

(4) *For* $n \geq 1$, $\mathcal{U}_0 * \mathcal{U}_n = \overline{\gamma}(-1) \cdot \mathcal{T}_n$ *and* $\mathcal{U}_n * \mathcal{U}_0 = \overline{\gamma}(-1) \cdot \mathcal{T}_{-n}$.

We consider two choices for γ as a character of $(\mathbb{Z}/p\mathbb{Z})^*$, either γ is trivial or γ is given by the Kronecker symbol $\gamma = \left(\frac{\cdot}{p}\right)$. Then we have following proposition.

Proposition 2.7. *We have following relations.*

(1) $\mathcal{U}_0^2 = \begin{cases} (p-1)\mathcal{U}_0 + p & \text{if } \gamma \text{ is trivial,} \\ \left(\frac{-1}{p}\right) p & \text{if } \gamma = \left(\frac{\cdot}{p}\right). \end{cases}$

(2) $\mathcal{U}_1^2 = \begin{cases} p & \text{if } \gamma \text{ is trivial,} \\ \varepsilon_p(p-1)U_1 + \left(\frac{-1}{p}\right) p & \text{if } \gamma = \left(\frac{\cdot}{p}\right). \end{cases}$

(3) *If* γ *is trivial, then* $\mathcal{T}_1 * \mathcal{U}_1 = p\,\mathcal{U}_0$ *and* $\mathcal{T}_{-1} = 1/p \cdot \mathcal{U}_1 * \mathcal{T}_1 * \mathcal{U}_1$.

We obtain the following theorem.

Theorem 1. *The "genuine" Iwahori Hecke algebra* $H(\overline{K_0(p)}, \gamma)$ *for* γ *trivial or* $\left(\frac{\cdot}{p}\right)$ *is generated as an* $\mathbb{C}$*-algebra by* $\mathcal{U}_0$ *and* $\mathcal{U}_1$ *with the defining relations given by above proposition.*

3 Translation of adelic to classical

We briefly recall Shimura's [10] work on half-integral weight forms. Let $\mathcal{G}$ be the set of all ordered pairs $(\alpha, \phi(z))$ where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$ and $\phi(z)$ is a holomorphic function on the upper half plane $\mathbb{H}$ such that $\phi(z)^2 = t \det(\alpha)^{-1/2}(cz+d)$ with t in the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. Then $\mathcal{G}$ is a group under the following operation:

$$(\alpha, \phi(z)).(\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z)).$$

Let $k \geq 1$ and $\zeta = (\alpha, \phi(z))$. Recall the slash operator $|[\zeta]_{k+1/2}$ on functions f on $\mathbb{H}$ by $f|[\zeta]_{k+1/2}(z) = f(\alpha z)(\phi(z))^{-2k-1}$.

Let N be divisible by 4 and $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Define the automorphy factor

$$j(\alpha, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{1/2},$$

where $\varepsilon_d = 1$ or $(-1)^{1/2}$ according as $d \equiv 1$ or $3 \pmod 4$ and $\left(\frac{c}{d}\right)$ is as in [10]. Let

$$\Delta_0(N) := \{\alpha^* = (\alpha, j(\alpha, z)) \in \mathcal{G} : \alpha \in \Gamma_0(N)\} \leq \mathcal{G}.$$

Let ξ be an element of $\mathcal{G}$ such that $\Delta_0(N)$ and $\xi^{-1}\Delta_0(N)\xi$ are commensurable. We have an operator $|[\Delta_0(N)\xi\Delta_0(N)]_{k/2}$ on $S_{k+1/2}(\Gamma_0(N))$ defined by

$$f|[\Delta_0(N)\xi\Delta_0(N)]_{k+1/2} = \det(\xi)^{(2k-3)/4} \sum_v f|[\xi_v]_{k+1/2}$$

where $\Delta_0(N)\xi\Delta_0(N) = \bigcup_v \Delta_0(N)\xi_v$.

Let $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ be the adele ring of $\mathbb{Q}$ and $\widetilde{\mathrm{SL}}_2(\mathbb{A}) = \mathrm{SL}_2(\mathbb{A}) \times \{\pm 1\}$ with the group law: for $g = (g_\nu),\ h = (h_\nu) \in \mathrm{SL}_2(\mathbb{A})$ and $\epsilon_1,\ \epsilon_2 \in \{\pm 1\}$

$$(g, \epsilon_1)(h, \epsilon_2) = (gh,\ \epsilon_1\epsilon_2\sigma(g,h)),\ \text{where } \sigma(g,h) = \prod_\nu \sigma_\nu(g_\nu, h_\nu).$$

The group $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ splits over $\mathrm{SL}_2(\mathbb{Q})$ and the splitting is given by

$$s_{\mathbb{Q}} : \mathrm{SL}_2(\mathbb{Q}) \longrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{A}),\ g \mapsto (g, s_{\mathbb{A}}(g)) \text{ where } s_{\mathbb{A}}(g) = \prod_\nu s_\nu(g).$$

For $\tilde{g} = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \epsilon) \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ and $z \in \mathbb{H}$, define

$$\tilde{g}(z) = \frac{az+b}{cz+d} \quad \text{and} \quad J(\tilde{g}, z) = \epsilon(cz+d)^{1/2}.$$

Note that $J(\tilde{g}, z)$ satisfies automorphy condition i.e.,

$$J(\tilde{g}\tilde{h}, z) = J(\tilde{g}, \tilde{h}z)J(\tilde{h}, z).$$

For $\theta \in \mathbb{R}$, let $k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Define $\tilde{K}_\infty := \{\tilde{k}(\theta) : \theta \in (-2\pi, 2\pi]\}$ where

$$\tilde{k}(\theta) = \begin{cases} (k(\theta), 1) & \text{if } -\pi < \theta \leq \pi, \\ (k(\theta), -1) & \text{if } -2\pi < \theta \leq -\pi \text{ or } \pi < \theta \leq 2\pi. \end{cases}$$

Then $\tilde{K}_\infty$ is a maximal compact subgroup of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ and $\tilde{k}(\theta) \mapsto e^{i\frac{2k+1}{2}\theta}$ is a genuine character of $\tilde{K}_\infty$. Let

$$K_1(N) = \prod_{q<\infty} \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_q) : c \equiv 0, \text{ and } a, d \equiv 1 \pmod{N\mathbb{Z}_q}\}.$$

We follow the notation of Waldspurger [12]. Let χ be an even Dirichlet character modulo N. Write $\chi_0 = \chi\left(\frac{-1}{\cdot}\right)^k$. Define $\tilde{\gamma_2}$ on $\mathbb{Z}_2^\times$ as

$$\tilde{\gamma_2}(t) = \begin{cases} 1 & \text{if } t \equiv 1 \pmod{4\mathbb{Z}_2}, \\ -i & \text{if } t \equiv 3 \pmod{4\mathbb{Z}_2}, \end{cases}$$

and for $k_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0^2(4)$, define

$$\tilde{\epsilon}_2(k_0) = \begin{cases} \tilde{\gamma_2}(d)^{-1}\,(c,d)_2\, s_2(k_0) & \text{if } c \neq 0, \\ \tilde{\gamma_2}(d) & \text{if } c = 0. \end{cases}$$

Let χ_0 also denote the idelic character (of $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times$) corresponding to the Dirichlet character χ_0 (it will be clear from the context when we consider χ_0 to be idelic or Dirichlet character). Let $A_{k+1/2}(N, \chi_0)$ denote the set of functions $\Phi : \widetilde{\mathrm{SL}}_2(\mathbb{A}) \to \mathbb{C}$ satisfying the following properties:

(1) $\Phi(s_{\mathbb{Q}}(\alpha)\tilde{g}(k_1, 1)) = \Phi(\tilde{g})$ for all $k_1 \in \prod_{q \nmid N} \mathrm{SL}_2(\mathbb{Z}_q), \alpha \in \mathrm{SL}_2(\mathbb{Q}), \tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$.

(2) Φ is genuine, i.e., $\Phi((I, \zeta)\tilde{g}) = \zeta\Phi(\tilde{g})$ for $\zeta \in \mu_2$.

(3) For odd primes p such that $p^n \| N$, $\Phi(\tilde{g}(k_0,1)) = \chi_{0,p}(d)\Phi(\tilde{g})$, $k_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0^p(p^n)$.
(4) If $2^n \| N$ $(n \geq 2)$, $\Phi(\tilde{g}(k_0,1)) = \tilde{\epsilon}_2(k_0)\chi_{0,2}(d)\Phi(\tilde{g})$ for all $k_0 \in K_0^2(2^n)$.
(5) $\Phi(\tilde{g}\tilde{k}(\theta)) = e^{i\frac{2k+1}{2}\theta}\Phi(\tilde{g})$ for all $\tilde{k}(\theta) \in \tilde{K}_\infty$.
(6) Φ is smooth as a function of $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ and satisfies $\Delta\Phi = -[(2k+1)/4 \cdot (2k-3)/4]\Phi$ where Δ is the Casimir operator.
(7) Φ is square integrable, that is $\int_{s_{\mathbb{Q}}(\mathrm{SL}_2(\mathbb{Q}))\backslash \widetilde{\mathrm{SL}}_2(\mathbb{A})/\mu_2} |\Phi(\tilde{g})|^2 d\tilde{g} < \infty$.
(8) Φ is cuspidal, that is $\int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \Phi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\tilde{g}\right) da = 0$ for all $\tilde{g} \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$.

In the above $\chi_{0,p}$ is the p-component of idelic character χ_0.

By Waldspurger [12, Proposition 3] there exists an isomorphism between

$$A_{k+1/2}(N,\chi_0) \to S_{k+1/2}(\Gamma_0(N),\chi)$$

given by $\Phi \mapsto f_\Phi$ where for $z \in \mathbb{H}$,

$$f_\Phi(z) = \Phi(\tilde{g}_\infty)J(\tilde{g}_\infty, i)^{2k+1}$$

where $\tilde{g}_\infty \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$ is such that $\tilde{g}_\infty(i) = z$. The inverse map is given by $f \mapsto \Phi_f$ where for $g \in \widetilde{\mathrm{SL}}_2(\mathbb{A})$ if $\tilde{g} = (\alpha, s_{\mathbb{A}}(\alpha))\tilde{g}_\infty(k_1,1)$ (by the strong approximation theorem for $\widetilde{\mathrm{SL}}_2(\mathbb{A})$),

$$\Phi_f(\tilde{g}) = f(\tilde{g}_\infty(i))J(\tilde{g}_\infty, i)^{-2k-1}.$$

This isomorphism induces a ring isomorphism of spaces of linear operators,

$$q : \mathrm{End}_{\mathbb{C}}(A_{k+1/2}(N,\chi_0)) \to \mathrm{End}_{\mathbb{C}}(S_{k+1/2}(\Gamma_0(N),\chi))$$

given by

$$q(\mathcal{T})(f) = f_{\mathcal{T}(\Phi_f)}.$$

3.1 $N = 4M$, M odd and $p \| M$

We shall now use the map q to translate the elements $\mathcal{T}_1$, $\mathcal{U}_1$ and $\mathcal{U}_0$ in the p-adic Hecke algebra to classical operators on $S_{k+1/2}(\Gamma_0(4M),\chi)$. We will restrict ourselves to χ being the trivial character modulo $4M$. In this case $\chi_0 = \left(\frac{-1}{\cdot}\right)^k$ is of conductor 1 or 4 and so $\chi_{0,p}$ is trivial on $\mathbb{Z}_p^\times$ while $\chi_{0,2}$ acts by $\chi_0^{-1} = \chi_0$ on $\mathbb{Z}_2^\times$.

Let γ be character on $(\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ induced by $\chi_{0,p}|\mathbb{Z}_p^\times$. Then Iwahori Hecke algebra $H(\overline{K_0(p)},\gamma)$ is a subalgebra of $\mathrm{End}_{\mathbb{C}}(A_{k+1/2}(N,\chi_0))$ via the following action: for $\mathcal{T} \in H(\overline{K_0(p)},\gamma)$ and $\Phi \in A_{k+1/2}(N,\chi_0)$,

$$\mathcal{T}(\Phi)(\tilde{g}) = \int_{\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p)} \mathcal{T}(\tilde{x})\Phi(\tilde{g}\tilde{x})d\tilde{x}.$$

We prove the following proposition.

Proposition 3.1. *Let χ be the trivial character modulo $4M$ with M as above and γ be induced by $\chi_{0,p}$. Let $\mathcal{T}_1$, $\mathcal{U}_1$ and $\mathcal{U}_0 \in H(\overline{K_0(p)},\gamma)$ and $f \in S_{k+1/2}(\Gamma_0(4M),\chi)$. Then,*

(1)

$$q(\mathcal{T}_1)(f)(z) = \left(\frac{-1}{p}\right)^k p^{-k-1/2} \sum_{s=0}^{p^2-1} f\left(\frac{z+s}{p^2}\right).$$

(2)

$$q(\mathcal{U}_1)(f)(z) = \overline{\varepsilon_p}\left(\frac{-1}{p}\right)\left(\frac{M/p}{p}\right)\sum_{s=0}^{p-1} f|[(\alpha_s, \phi_{\alpha_s})]_{k+1/2}(z),$$

where $\alpha_s = \begin{pmatrix} p^2n - 4Mms & m \\ 4pM(1-s) & p \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z})$ *is of determinant* p^2 *and* $m,\ n \in \mathbb{Z}$ *are such that* $pn - (4M/p)m = 1$, *and* $\phi_{\alpha_s}(z) = (4M(1-s)z+1)^{1/2}$.

(3)

$$q(\mathcal{U}_0)(f)(z) = \sum_{s=0}^{p-1} f|[(\beta_s, \phi_{\beta_s})]_{k+1/2}(z),$$

where $\beta_s = \begin{pmatrix} 1 & m-s \\ 4M_1 & np - 4M_1 s \end{pmatrix} \in \Gamma_0(4M_1)$ *with* $M_1 = M/p$ *and* $m,\ n \in \mathbb{Z}$ *are chosen as above and* $\phi_{\beta_s} = (4M_1 z + (np - 4M_1 s))^{1/2}$.

Recall by [10, Proposition 1.5], for $m \mid M$, the map U_m given by

$$U_m\left(\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{n=1}^{\infty} a_{mn} q^n = m^{(2k-3)/4}\sum_{s=0}^{m-1} f|[(\begin{pmatrix} 1 & s \\ 0 & m \end{pmatrix}, m^{1/4})]_{k+1/2}(z)$$

sends $S_{k+1/2}(\Gamma_0(4M), \chi)$ to $S_{k+1/2}(\Gamma_0(4M), \chi\left(\frac{m}{\cdot}\right))$. Thus by (1) of the above proposition, we have $q(\mathcal{T}_1)(f)(z) = \left(\frac{-1}{p}\right)^k p^{(3-2k)/2} U_{p^2}(f)$.

Let us denote $q(p^{-1/2}\mathcal{U}_1)$ by $\widetilde{W}_{p^2}$ and $q(\mathcal{U}_0)$ by $\widetilde{Q}_p$.

Corollary 3.2. *On* $S_{k+1/2}(\Gamma_0(4M))$ *we have*

(1) $\widetilde{W}_{p^2}$ *is an involution.*

(2) $(\widetilde{Q}_p - p)(\widetilde{Q}_p + 1) = 0$.

(3) $\widetilde{Q}_p = \left(\frac{-1}{p}\right)^k p^{1-k} U_{p^2}\widetilde{W}_{p^2}$.

(4) *If* $f \in S_{k+1/2}(\Gamma_0(4M/p))$ *then* $\widetilde{Q}_p(f) = pf$.

Proof. The proof of (1)-(3) follows by using Proposition 2.7. For (4) we use Proposition 3.1(3). □

We further define an operator $\widetilde{Q}'_p$ on $S_{k+1/2}(\Gamma_0(4M))$ to be the conjugate of $\widetilde{Q}_p$ by $\widetilde{W}_{p^2}$, i.e., $\widetilde{Q}'_p = \widetilde{W}_{p^2}\widetilde{Q}_p\widetilde{W}_{p^2}$. Thus $\widetilde{Q}'_p$ satisfies same quadratic as $\widetilde{Q}_p$ and we have $\widetilde{Q}'_p = \left(\frac{-1}{p}\right)^k p^{1-k}\widetilde{W}_{p^2}U_{p^2}$.

Remark 1. We note that for a prime q such that $(q, 2M) = 1$, one can similarly obtain the usual Hecke operator T_{q^2} on $S_{k+1/2}(\Gamma_0(4M))$. In particular, if we take $\mathcal{T}_1 := X_{(h(q),1)} \in H(\overline{\mathrm{SL}_2(\mathbb{Z}_q)}, \gamma_q)$ then $q(\mathcal{T}_1) = \left(\frac{-1}{p}\right)^k p^{(3-2k)/2} T_{q^2}$.

Moreover if p and q are distinct primes such that $p^n,\ q^m$ strictly divides N then the operators $\mathcal{S} \in H(\overline{K_0^p(p^n)}, \gamma_p)$ and $\mathcal{T} \in H(\overline{K_0^q(q^m)}, \gamma_q)$ in $\mathrm{End}_{\mathbb{C}}(S_{k+1/2}(\Gamma_0(N))$ commute.

In particular the operators $\widetilde{Q}_p$, $\widetilde{W}_{p^2}$ on $S_{k+1/2}(\Gamma_0(4M))$ that we defined above commute with T_{q^2} for primes q coprime to $2M$.

We prove that the operators we obtained from $\mathcal{U}_0$ and $\mathcal{U}_1$ are self-adjoint.

Theorem 2. *The operators* $\widetilde{W}_{p^2}$, $\widetilde{Q}_p$ *and* $\widetilde{Q}'_p$ *on* $S_{k+1/2}(\Gamma_0(4M))$ *are self-adjoint with respect to Petersson inner product.*

4 Local Hecke algebra at prime 2 and Kohnen's plus space

In the previous sections we consider genuine Iwahori Hecke algebra for the case when p is an odd prime and translate some of the Hecke algebra elements to obtain classical operators on $S_{k+1/2}(\Gamma_0(4M))$ when $p\|M$. We shall now consider the case $p = 2$. We recall the work of Loke and Savin [7] on genuine local Hecke algebra of $\widetilde{\mathrm{SL}}(\mathbb{Q}_2)$ with respect to $\overline{K_0(4)}$ and central character of M_2.

Let γ be a genuine character of $\overline{K_0(4)}$ determined by its value on $(-I,1)$ such that it is trivial on $K_1(4)$. This character can be extended to the normalizer $N_{\widetilde{\mathrm{SL}}(\mathbb{Q}_2)}(T)$ by defining $\gamma((h(2^n),1)) = 1$ for all integers n and $\gamma((w(1),1)) = (1+\gamma((-I,1)))/\sqrt{2} =: \zeta_8$, a primitive 8th root of unity. Let $H(\overline{K_0(4)},\gamma)$ be genuine Hecke algebra of $\widetilde{\mathrm{SL}}(\mathbb{Q}_2)$ with respect to $\overline{K_0^2(4)}$ and γ. Let $\mathcal{T}_n = X_{(h(2^n),1)}$, $\mathcal{U}_n = X_{(w(2^{-n}),1)} \in H(\overline{K_0^2(4)},\gamma)$ (defined as in the odd prime case).

Theorem 3 (Loke-Savin [7]). *For $m,n \in \mathbb{Z}$,*
(1) *If $mn \geq 0$ then $\mathcal{T}_m * \mathcal{T}_n = \mathcal{T}_{m+n}$.*
(2) *$\mathcal{U}_1 * \mathcal{T}_n = \mathcal{U}_{n+1}$ and $\mathcal{T}_n * \mathcal{U}_1 = \mathcal{U}_{1-n}$.*
(3) *$\mathcal{U}_1 * \mathcal{U}_n = \mathcal{T}_{n-1}$ and $\mathcal{U}_n * \mathcal{U}_1 = \mathcal{T}_{1-n}$.*
The Hecke algebra $H(\overline{K_0^2(4)},\gamma)$ is generated by $\mathcal{U}_0$ and $\mathcal{U}_1$ modulo relations $(\mathcal{U}_0 - 2\sqrt{2})(\mathcal{U}_0+\sqrt{2}) = 0$ and $\mathcal{U}_1^2 = 1$.

In particular, $\mathcal{U}_2 = \mathcal{U}_1 * \mathcal{T}_1 = \mathcal{U}_1 * \mathcal{U}_0 * \mathcal{U}_1$ and so $(\mathcal{U}_2 - 2\sqrt{2})(\mathcal{U}_2 + \sqrt{2}) = 0$.

We set $\gamma((-I,1)) = -i^{2k+1}$. Let χ be the trivial character modulo 4, so $\chi_0 = \left(\frac{-1}{\cdot}\right)^k$. Then, for any $k_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0^2(4)$ we check that $\tilde{\epsilon}_2(k_0)\chi_{0,2}(d) = \gamma((k_0,1))$. Using the ring isomorphism q, Loke-Savin obtained following classical operators on $S_{k+1/2}(\Gamma_0(4),\chi)$.

Proposition 4.1 (Loke-Savin [7]). *For $f \in S_{k+1/2}(\Gamma_0(4),\chi)$,*

(1)
$$q(\mathcal{T}_1)(f)(z) = 2^{(3-2k)/2}U_4(f)(z).$$

(2)
$$q(\mathcal{U}_1)(f)(z) = \left(\frac{2}{2k+1}\right) W_4(f)(z)$$

where the operator W_4 is given by $W_4(f)(z) = (-2iz)^{-k-1/2}f(-1/4z)$.

Niwa [8] considered operator $R = W_4U_4$ on $S_{k+1/2}(\Gamma_0(4),\chi)$, proved that it is self-adjoint and that $(R-\alpha_1)(R-\alpha_2) = 0$ where $\alpha_1 = \left(\frac{2}{2k+1}\right)2^k$, $\alpha_2 = -\frac{\alpha_1}{2}$. Kohnen [5] defined his plus space $S^+(4)$ at level 4 to be the α_1-eigenspace of R in $S_{k+1/2}(\Gamma_0(4))$. It follows from the above proposition that $S^+(4)$ is the 2-eigenspace of $q(\mathcal{U}_1)q(\mathcal{T}_1)/\sqrt{2}$ and hence that of $q(\mathcal{U}_2)/\sqrt{2}$.

In the case level is $4M$ with M odd and χ is a ivial character modulo $4M$, Kohnen [6] defines a classical operator Q on $S_{k+1/2}(\Gamma_0(4M),\chi)$ in order to obtain his plus space. The operator Q is defined by

$$Q := [\Delta_0(4M)\rho\Delta_0(4M)] \quad \text{where } \rho = (\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{\pi i/4}).$$

By [6, Proposition 1] Q is self-adjoint and satisfies $(Q-\alpha)(Q-\beta) = 0$ where $\alpha = (-1)^{[(k+1)/2]}2\sqrt{2}$, $\beta = -\alpha/2$, and the plus space $S^+_{k+1/2}(4M)$ is the α-eigenspace of Q.

In our work we need that Kohnen's plus space at level $4M$ is the 2-eigenspace of the product $q(\mathcal{U}_1)q(\mathcal{T}_1)/\sqrt{2}$. We have the following proposition.

Proposition 4.2. *Let $f \in S_{k+1/2}(\Gamma_0(4M))$ with M odd. Then we have*

$$Q(f) = \left(\frac{2}{2k+1}\right) q(\mathcal{U}_2)(f) = \left(\frac{2}{2k+1}\right) q(\mathcal{U}_1)q(\mathcal{T}_1)(f).$$

Consequently $S^+_{k+1/2}(4M)$ is the 2-eigenspace of $q(\mathcal{U}_1)q(\mathcal{T}_1)/\sqrt{2}$.

We can translate $\mathcal{T}_1$, $\mathcal{U}_1$, $\mathcal{U}_0 \in H(\overline{K_0^2(4)}, \gamma)$ to classical operators on $S_{k+1/2}(\Gamma_0(4M))$.

Proposition 4.3. *For $f \in S_{k+1/2}(\Gamma_0(4M))$,*

(1)

$$q(\mathcal{T}_1)(f)(z) = 2^{(3-2k)/2} U_4(f)(z).$$

(2)

$$q(\mathcal{U}_1)(f)(z) = \overline{\zeta_8} \left(\frac{2}{M}\right) \left(\frac{-1}{M}\right)^{k+3/2} f|[W, \phi_W(z)]_{k+1/2}(z)$$

where $W = \begin{pmatrix} 4n & m \\ 4M & 2 \end{pmatrix}$ with $a, b \in \mathbb{Z}$ such that $2n - mM = 1$ and $\phi_W(z) = (2Mz+1)^{1/2}$.

(3)

$$q(\mathcal{U}_0)(f)(z) = \overline{\zeta_8} \left(\frac{-1}{M}\right)^{k+3/2} \sum_{s=0}^{3} f|[A_s, \phi_{A_s}(z)]_{k+1/2}(z)$$

where $A_s = \begin{pmatrix} n & -ns+m \\ M & -Ms+4 \end{pmatrix}$ with $m, n \in \mathbb{Z}$ such that $4n - mM = 1$ and $\phi_W(z) = (Mz + 4 - Ms)^{1/2}$.

Define $\widetilde{Q}_2 := q(\mathcal{U}_0)/\sqrt{2} = q(\mathcal{T}_1)q(\mathcal{U}_1)/\sqrt{2}$ and $\widetilde{W}_4 := q(\mathcal{U}_1)$. Further define $\widetilde{Q}'_2$ to be the conjugate of $\widetilde{Q}_2$ by $\widetilde{W}_4$, so $\widetilde{Q}'_2 = q(\mathcal{U}_1)q(\mathcal{T}_1)/\sqrt{2}$. Then the Kohnen's plus space at level $4M$ is the 2-eigenspace of $\widetilde{Q}'_2$.

5 The minus space of half-integral weight forms

Let M be odd and square-free. In this section we shall define the minus space $S^-_{k+1/2}(4M)$ of weight $k + 1/2$ and level $4M$ and state our main theorem.

We shall first start with defining the minus space at level 4 and then at level $4p$ for p an odd prime. We can then extend our definition to level $4M$ where M is as above.

5.1 Eigenvalues of U_p

In order to define our minus space we need to look at eigenvalues of operator U_p on the space of old forms $S^{\text{old}}_{2k}(\Gamma_0(N))$ where N is any positive integer such that $p \| N$.

For every positive integer n and a modular form F we denote by

$$F_n(z) := V(n)F(z) = F(nz).$$

Recall that $V(n)$ is the shift operator sending $S_{2k}(\Gamma_0(L))$ to $S_{2k}(\Gamma_0(nL))$.

Let M be a positive integer such that $p \nmid M$. Let $F \in S_{2k}(\Gamma_0(M))$ be a primitive Hecke eigenform and a_p is the p-th Fourier coefficient of F. We note the following proposition.

Proposition 5.1. *U_p stabilizes the two dimensional subspace spanned by F_n and F_{np} for $(p,n)=1$ and the eigenvalues of U_p on the subspace $\langle F_n, F_{np}\rangle$ are*

$$a_p+\beta=\frac{a_p\pm\sqrt{a_p^2-4p^{2k-1}}}{2}.$$

Consequently the eigenvalues of $(U_p)^2$ on the two dimensional subspace spanned by F_n and F_{np} are either complex or $\pm p^{2k-1}$.

5.2 Minus space for $\Gamma_0(4)$

We recall the following theorem of Niwa which was obtained by proving equality of traces of Hecke operators.

Theorem 4 (Niwa [8]). *Let M be odd and square-free. There exists an isomorphism of vector spaces $\psi : S_{k+1/2}(\Gamma_0(4M)) \to S_{2k}(\Gamma_0(2M))$ satisfying*

$$T_p(\psi(f))=\psi(T_{p^2}(f)) \text{ for all primes } p \text{ coprime to } 2M.$$

Moreover if $f\in S_{k+1/2}(\Gamma_0(4))$ then we further have

$$U_2(\psi(f))=\psi(U_4(f)),\quad T_p(\psi(f))=\psi(T_{p^2}(f)) \text{ for all odd primes } p.$$

We also recall Shimura lift [10]: For t a positive square-free integer, there is a linear map $\mathrm{Sh}_t : S_{k+1/2}(\Gamma_0(4M)) \to S_{2k}(\Gamma_0(2M))$ given by

$$\mathrm{Sh}_t\left(\sum_{n=1}^{\infty}a_nq^n\right)=\sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-1}{d}\right)^k\left(\frac{t}{d}\right)d^{k-1}a\left(t\frac{n^2}{d^2}\right)\right)q^n.$$

Sh_t commutes with all Hecke operators, i.e., $T_p(\mathrm{Sh}_t(f)) = \mathrm{Sh}_t(T_{p^2}(f))$ for all primes p coprime to $2M$ and $U_p(\mathrm{Sh}_t(f)) = \mathrm{Sh}_t(U_{p^2}(f))$ for all primes p dividing $2M$.

We need the following theorem of Kohnen.

Theorem 5 (Kohnen [5]). (1) $\dim(S^+(4)) = \dim(S_{2k}(\Gamma_0(1))$.

(2) *$S^+(4)$ has a basis of eigenforms for all the operators T_{p^2}, p odd.*

(3) *If f is such an eigenform then $\psi(f)$ is an old form and $\psi(f)=\lambda F+\beta F_2$ where $F\in S_{2k}(\Gamma_0(1))$ is a primitive eigenform determined by the eigenvalues of f.*

Define $A^+(4)=\widetilde{W}_4S^+(4)$. We know that $S^+(4)$ is the 2-eigenspace of $\widetilde{Q}_2'$, hence $A^+(4)$ is the 2-eigenspace of $\widetilde{Q}_2$. Following above theorem of Kohnen we have $\dim(A^+(4)) = \dim(S_{2k}(\Gamma_0(1))$ and following observation.

Corollary 5.2. (1) *$A^+(4)$ has a basis of eigenforms under T_{p^2} for all p odd.*

(2) *ψ maps $A^+(4)$ into the space of old forms in $S_{2k}(\Gamma_0(2))$.*

We prove the following proposition.

Proposition 5.3. *$S^+(4)\bigcap A^+(4)=\{0\}$.*

Proof. Suppose there is a nonzero $f\in S^+(4)\bigcap A^+(4)$. We can assume that f is an eigenform under T_{p^2} for all p odd (as such T_{p^2} stabilizes the intersection $S^+(4)\bigcap A^+(4)$). Then $\widetilde{Q}_2(f)=2f=\widetilde{Q}_2'(f)$. This implies that $U_4\widetilde{W}_4(f)=2^kf=\widetilde{W}_4U_4(f)$. Thus

$$(U_4)^2(f)=2^kU_4\widetilde{W}_4(f)=2^{2k}f.$$

Applying ψ to the above equation we get that $(U_2)^2(\psi(f))=2^{2k}\psi(f)$. Now $\psi(f)\in\langle F,F_2\rangle$ for some primitive form $F\in S_{2k}(\Gamma_0(1))$ and by Proposition 5.1, the eigenvalues of $(U_2)^2$ on this subspace are either complex or 2^{2k-1}. This is a contradiction. □

Define $S^-_{k+1/2}(4)$ to be the orthogonal complement of $S^+(4) \oplus A^+(4)$. That is, $S^-_{k+1/2}(4)$ is the common eigenspace with the eigenvalue -1 of the operators $\widetilde{Q}_2$ and $\widetilde{Q}'_2$. We shall write $S^-_{k+1/2}(4)$ simply by $S^-(4)$. Thus

$$S_{k+1/2}(\Gamma_0(4)) = S^+(4) \oplus A^+(4) \oplus S^-(4).$$

We now state our main theorem for the minus space at level 4.

Theorem 6. *$S^-(4)$ has a basis of eigenforms for all the operators T_{p^2}, p odd; these eigenforms are also eigenfunctions under U_4. If two eigenforms in $S^-(4)$ share the same eigenvalues for all T_{p^2} then they are a scalar multiple of each other. ψ induces a Hecke algebra isomorphism:*

$$S^-_{k+1/2}(4) \cong S^{\mathrm{new}}_{2k}(\Gamma_0(2)).$$

5.3 Minus space for $\Gamma_0(4p)$ for p an odd prime

In order to define minus space for $\Gamma_0(4p)$ we need the involution $\widetilde{W}_{p^2}$, the operators U_{p^2}, $\widetilde{Q}_p$ and $\widetilde{Q}'_p = \widetilde{W}_{p^2}\widetilde{Q}_p\widetilde{W}_{p^2}$ on $S_{k+1/2}(\Gamma_0(4p))$ that we defined in Section 3.

Consider the subspace $\mathcal{V}(1)$ of $S_{2k}(\Gamma_0(2p))$ coming from the old forms at level 1, that is,

$$\mathcal{V}(1) = S_{2k}(\Gamma_0(1)) \oplus V(2)S_{2k}(\Gamma_0(1)) \oplus V(p)S_{2k}(\Gamma_0(1)) \oplus V(2p)S_{2k}(\Gamma_0(1)).$$

We consider the eigenvalues of $(U_p)^2$ on $\mathcal{V}(1)$.

Lemma 5.4. *The operator U_p stabilizes $\mathcal{V}(1)$. If an eigenvalue λ of $(U_p)^2$ on this space is real then $\lambda = \pm p^{2k-1}$.*

Let $R := S^+(4) \oplus A^+(4)$. Then we have

Proposition 5.5. $R \bigcap \widetilde{W}_{p^2} R = \{0\}$.

Remark 2. The proof of the above proposition is similar to proof of Proposition 5.3. In Proposition 5.3 we use the fact that Niwa's map ψ commutes with U_4, but we do not know whether ψ commutes with U_{p^2} when p divides the level. However Shimura lift Sh_t does commute with U_{p^2} for primes p dividing the level and so we instead use Sh_t in Proposition 5.5 and then apply Lemma 5.4 to arrive at a contradiction.

In the lines of Corollary 5.2(2) we have the following.

Corollary 5.6. *Niwa's map ψ maps $R \oplus \widetilde{W}_{p^2} R$ isomorphically onto $\mathcal{V}(1)$.*

Next we consider the following subspace of $S_{2k}(\Gamma_0(2p))$ coming from the old forms at level 2,

$$\mathcal{V}(2) = S^{\mathrm{new}}_{2k}(\Gamma_0(2)) \oplus V(p)S^{\mathrm{new}}_{2k}(\Gamma_0(2)).$$

This space is a direct sum of two dimensional subspaces spanned by F and F_p where F is a primitive Hecke eigenform in $S^{\mathrm{new}}_{2k}(\Gamma_0(2))$. Using Proposition 5.1 we have the following lemma.

Lemma 5.7. *If an eigenvalue λ of $(U_p)^2$ on $\mathcal{V}(2)$ is real then $\lambda = \pm p^{2k-1}$.*

Since (by Theorem 6) ψ maps $S^-_{k+1/2}(4)$ isomorphically onto $S^{\mathrm{new}}_{2k}(\Gamma_0(2))$, it follows that ψ maps $\widetilde{W}_{p^2} S^-_{k+1/2}(4)$ into the space $\mathcal{V}(2)$. The proof of the following is identical to that of Proposition 5.5.

Proposition 5.8. $S^-_{k+1/2}(4) \bigcap \widetilde{W}_{p^2} S^-_{k+1/2}(4) = \{0\}$.

Corollary 5.9. *ψ maps $S^-_{k+1/2}(4) \oplus \widetilde{W}_{p^2} S^-_{k+1/2}(4)$ isomorphically onto $\mathcal{V}(2)$.*

Finally, we consider the following subspace of $S_{2k}(\Gamma_0(2p))$ coming from the old forms at level p,

$$\mathcal{V}(p) = S_{2k}^{\text{new}}(\Gamma_0(p)) \oplus V(2) S_{2k}^{\text{new}}(\Gamma_0(p)).$$

This space is a direct sum of two dimensional subspaces spanned by F and F_2 where F is a primitive Hecke eigenform in $S_{2k}^{\text{new}}(\Gamma_0(p))$. We have

Lemma 5.10. *If an eigenvalue λ of $(U_2)^2$ on $\mathcal{V}(p)$ is real then $\lambda = \pm 2^{2k-1}$.*

Let $S^{+,\text{new}}_{k+1/2}(4p)$ be the new space inside the plus space in $S_{k+1/2}(\Gamma_0(4p))$. Kohnen [6, Theorem 2] proved that ψ maps $S^{+,\text{new}}_{k+1/2}(4p)$ into $\mathcal{V}(p)$ and the dimension of $S^{+,\text{new}}_{k+1/2}(4p)$ equals the dimension of $S_{2k}^{\text{new}}(\Gamma_0(p))$. Then as before ψ maps $\widetilde{W}_4 S^{+,\text{new}}_{k+1/2}(4)$ into the space $\mathcal{V}(p)$ and we have the following proposition and corollary.

Proposition 5.11. $S^{+,\text{new}}_{k+1/2}(4p) \bigcap \widetilde{W}_4 S^{+,\text{new}}_{k+1/2}(4p) = \{0\}$.

Corollary 5.12. *ψ maps $S^{+,\text{new}}_{k+1/2}(4p) \oplus \widetilde{W}_4 S^{+,\text{new}}_{k+1/2}(4p)$ isomorphically onto $\mathcal{V}(p)$.*

We define the following subspace of $S_{k+1/2}(\Gamma_0(4p))$,

$$E = R \oplus \widetilde{W}_{p^2} R \oplus S^-_{k+1/2}(4) \oplus \widetilde{W}_{p^2} S^-_{k+1/2}(4) \oplus S^{+,\text{new}}_{k+1/2}(4p) \oplus \widetilde{W}_4 S^{+,\text{new}}_{k+1/2}(4p).$$

By Corollary 5.6, 5.9 and 5.12, we get that ψ maps the space E isomorphically onto the old space $S_{2k}^{\text{old}}(\Gamma_0(2p))$. We define the minus space to be the orthogonal complement of E under the Petersson inner product. That is,

$$S^-_{k+1/2}(4p) := E^\perp.$$

Theorem 7. *$S^-_{k+1/2}(4p)$ has a basis of eigenforms for all the operators T_{q^2}, q an odd prime different than p, uniquely determined up to a non-zero scalar multiplication. ψ maps the space $S^-_{k+1/2}(4p)$ isomorphically to the space $S_{2k}^{\text{new}}(\Gamma_0(2p))$.*

Let $f \in S^-_{k+1/2}(4p)$ be a Hecke eigenform for all the operators T_{q^2} as above. It follows that $F := \psi(f)$ is a Hecke eigenform in $S_{2k}^{\text{new}}(\Gamma_0(2p))$ for all the operators T_q, $(q, 2p) = 1$. Also, $U_p(F) = -p^{k-1}\lambda(p)F$ where $\lambda(p) = \pm 1$ and $U_2(F) = -2^{k-1}\lambda(2)F$ where $\lambda(2) = \pm 1$. We have the following

Proposition 5.13. *Let $f \in S^-_{k+1/2}(4p)$ be a Hecke eigenform for all the operators T_{q^2}, q prime and $(q, 2p) = 1$. Then*

$$U_{p^2}(f) = -p^{k-1}\lambda(p)f, \quad U_4(f) = -2^{k-1}\lambda(2)f$$

where $\lambda(p) = \pm 1$ and $\lambda(2) = \pm 1$ are defined as above. Further, $\widetilde{W}_{p^2} f = \beta(p) f$, $\widetilde{W}_4 f = \beta(2) f$ where $\beta(p) = \pm 1$, $\beta(2) = \pm 1$.

Corollary 5.14. *Let $f \in S^-_{k+1/2}(4p)$. Then $\widetilde{Q}_p(f) = -f = \widetilde{Q}'_p(f)$, $\widetilde{Q}_2(f) = -f = \widetilde{Q}'_2(f)$.*

Consequently, we have the following characterization of our minus space as an intersection of -1-eigenspaces.

Theorem 8. *Let $f \in S_{k+1/2}(4p)$. Then $f \in S^-_{k+1/2}(4p)$ if and only if $\widetilde{Q}_p(f) = -f = \widetilde{Q}'_p(f)$ and $\widetilde{Q}_2(f) = -f = \widetilde{Q}'_2(f)$.*

5.4 Minus space for $\Gamma_0(4M)$ for M odd and square-free

Let $M \neq 1$ be an odd and square-free natural number. We write $M = p_1 p_2 \cdots p_k$. For each $i = 1, \dots, k$ define $M_i = M/p_i$. Since $S_{k+1/2}(\Gamma_0(4M_i))$ is contained in the p_i-eigenspace of $\widetilde{Q}_{p_i}$ (Corollary 3.2(5)), following the proof of Proposition 5.5 we obtain that

Proposition 5.15. $S_{k+1/2}(\Gamma_0(4M_i)) \bigcap \widetilde{W}_{p_i^2} S_{k+1/2}(\Gamma_0(4M_i)) = \{0\}$.

Corollary 5.16. *The Niwa map*

$$\psi : S_{k+1/2}(\Gamma_0(4M)) \to S_{2k}(\Gamma_0(2M))$$

maps $S_{k+1/2}(\Gamma_0(4M_i)) \oplus \widetilde{W}_{p_i^2} S_{k+1/2}(\Gamma_0(4M_i))$ *isomorphically onto* $S_{2k}(\Gamma_0(2M_i)) \oplus V(p_i) S_{2k}(\Gamma_0(2M_i))$.

Let $S^{+,\mathrm{new}}_{k+1/2}(4M)$ be the new space inside the Kohnen plus subspace of $S_{k+1/2}(4M)$. Then similarly we have

Proposition 5.17. $S^{+,\mathrm{new}}_{k+1/2}(4M) \bigcap \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(4M) = \{0\}$.

Corollary 5.18. *ψ maps $S^{+,\mathrm{new}}_{k+1/2}(4M) \oplus \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(4M)$ isomorphically onto $S^{\mathrm{new}}_{2k}(\Gamma_0(M)) \oplus V(2) S^{\mathrm{new}}_{2k}(\Gamma_0(M))$.*

We let $B_i = S_{k+1/2}(\Gamma_0(4M_i)) \oplus \widetilde{W}_{p_i^2} S_{k+1/2}(\Gamma_0(4M_i))$, $i = 1, \dots, k$. Define

$$E = \sum_{i=1}^{k} B_i \oplus S^{+,\mathrm{new}}_{k+1/2}(4M) \oplus \widetilde{W}_4 S^{+,\mathrm{new}}_{k+1/2}(4M).$$

Proposition 5.19. *Under ψ the space E maps isomorphically onto the old space $S^{\mathrm{old}}_{2k}(\Gamma_0(2M))$.*

We now define the minus space to be the orthogonal complement of E,

$$S^{-}_{k+1/2}(4M) := E^{\perp}.$$

Theorem 9. *The space $S^{-}_{k+1/2}(4M)$ has a basis of eigenforms for all the operators T_{q^2} where q is an odd prime satisfying $(q, M) = 1$. Under ψ, the space $S^{-}_{k+1/2}(4M)$ maps isomorphically onto the space $S^{\mathrm{new}}_{2k}(\Gamma_0(2M))$. If two forms in $S^{-}_{k+1/2}(4M)$ have the same eigenvalues for all the operators T_{q^2}, $(q, 2M) = 1$, then they are same up to a scalar factor.*

As before we can characterize our minus space as intersection of -1-eigenspaces.

Theorem 10. *Let $f \in S_{k+1/2}(4M)$. Then $f \in S^{-}_{k+1/2}(4M)$ if and only if $\widetilde{Q}_p(f) = -f = \widetilde{Q}'_p(f)$ for every prime p dividing M and $\widetilde{Q}_2(f) = -f = \widetilde{Q}'_2(f)$.*

5.5 An example

The space $S_{7/2}(\Gamma_0(12))$ is three dimensional and is spanned by

$$g_1 = q - 4q^4 - 6q^6 + 8q^7 + 9q^9 + 4q^{10} + 12q^{12} - 20q^{13} - 24q^{15} + \cdots,$$
$$g_2 = q^3 - 2q^4 + 2q^7 - 2q^{12} - 6q^{15} + 12q^{16} - 10q^{19} + 12q^{24} + \cdots,$$
$$g_3 = q^2 - q^3 - 4q^5 + 3q^6 + 4q^8 + 2q^{11} - 4q^{12} - 8q^{14} + 16q^{17} + \cdots.$$

We have two primitive forms of weight 6 and level dividing 6, namely F_3 of level 3 and F_6 of level 6. Using algorithm in [9] we have

$$S_{7/2}(\Gamma_0(12)) = S_{7/2}(12, F_3) \oplus S_{7/2}(12, F_6) = \langle g_1,\ g_2 \rangle \oplus \langle g_3 \rangle.$$

Note that g_2 is in the plus space, so we have $S^+_{7/2}(12) = \langle g_2 \rangle$. The minus space $S^-_{7/2}(12) = S_{7/2}(12, F_6) = \langle g_3 \rangle$.

Remark 3. (1) In general, $S^-_{k+1/2}(4M) = \bigoplus_F S_{k+1/2}(4M, F)$ where F runs through all primitive forms of weight $2k$ and level $2M$.

(2) The Kohnen plus space is given by well-known Fourier coefficient condition. But we do not expect any such Fourier coefficient condition for forms in our minus space. This is also evident from the above examples.

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