A p-adic analytic family of the D-th Shintani lifting for a Coleman family and congruences between the central L-values

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1 Introduction

These notes are based on my talk in the 10th Fukuoka Symposium on Number Theory. See [4] for the details of the notes.

For a positive integer M, we put

$$\Gamma_0(M) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{M} \right\}.$$

We denote by $S_k^{\mathrm{new}}(M,\varepsilon)$ the orthogonal complement of the subspace of old forms of level M in the space $S_k(M,\varepsilon)$ of cusp forms of weight k with respect to $\Gamma_0(M)$ with character ε and by $S_k(M,\varepsilon)_\alpha$ the subspace of $S_k(M,\varepsilon)$ spanned by the generalized eigenspaces for eigenvalues λ of the Hecke operator T_p at p with $\mathrm{ord}_p(\lambda) = \alpha$. Let $\mathbb{Z}[\varepsilon]$ be the ring generated by the values of ε over $\mathbb{Z}$. For a $\mathbb{Z}[\varepsilon]$-algebra R, we put

$$S_k(M,\varepsilon;R)_\alpha := S_k(M,\varepsilon)_\alpha \cap \mathbb{Z}[\varepsilon][[q]] \otimes_{\mathbb{Z}[\varepsilon]} R,$$
$$S_k^{\mathrm{new}}(M,\varepsilon;R)_\alpha := S_k^{\mathrm{new}}(M,\varepsilon) \cap S_k(M,\varepsilon;\mathbb{Z}[\varepsilon])_\alpha \otimes_{\mathbb{Z}[\varepsilon]} R.$$

Throughout the notes, we fix an odd prime p, a positive integer N satisfying $(N,2p) = 1$ and $Np \geq 4$, and $\alpha \in \mathbb{Q}_{\geq 0}$. We assume that N is square-free for simplicity. Let $f \in S_{2k_0+2}^{\mathrm{new}}(N,\chi^2)_\alpha$ be a primitive form with $2k_0+1 > \alpha \neq (2k_0+1)/2$, $f^* \in S_{2k_0+2}(Np,\chi^2)_\alpha$ its p-stabilization, and K the p-adic completion of the field obtained by adjoining $\sqrt{\chi(-1)}$, $\sqrt{c_\chi}$ and the values of χ to the *Hecke field* $\mathbb{Q}_{f^*} := \mathbb{Q}(\{a_n(f^*)\}_{n\geq 1})$, where $a_n(f^*)$ is the n-th Fourier coefficient of f^*. Then we have a *Coleman family* $\{f_{2k+2}^*\}_k$ *passing through* f^*, where the index k runs over all positive integers k satisfying $2k+1 > \alpha$ and $k \equiv k_0 \pmod{(p-1)p^{m_f}}$ for some positive integer m_f and f_{2k+2}^* is the p-stabilization of a primitive form $f_{2k+2} \in S_{2k+2}^{\mathrm{new}}(N,\chi^2;\mathcal{O}_K)_\alpha$. Namely, the family satisfies $f_{2k_0+2} = f$ and for some positive integer m_0 and any integer $r > m_0$,

$$f_{2k+2}^* \equiv f^* \pmod{p^{r-m_0}\mathcal{O}_K} \text{ if } k \equiv k_0 \pmod{(p-1)p^r}$$

(see [2] and [9]). For a non-zero integer a, we let χ_a denote the *Kronecker symbol*

$$\chi_a(b) := \left(\frac{a}{b}\right)$$

defined by [5, (3.1.9)]. Let D a fundamental discriminant (i.e., 1 or the discriminant of a quadratic field over $\mathbb{Q}$) with $\chi(-1)(-1)^{k_0+1}D > 0$ and $(D,Np) = 1$. Then we have the *D-th Shintani lifting*

$$\theta_{k,\chi,D}^{Np} : S_{2k_0+2}(Np,\chi^2) \to S_{k_0+2/3}^{+}(4Np,\tilde{\chi}),$$

where the target space is the *Kohnen plus space* with $\tilde{\chi} := \chi_{\chi(-1)}\chi$ (see [3]). Let $\Omega^{-}(f^{*}_{2k+2}) \in \mathbb{C}_p^{\times}$ be the canonical period attached to f^{*}_{2k+2} in the sense of [8], which is uniquely defined up to a p-adic unit. Similarly to [7], by the virtue of cohomological interpretation of the D-th Shintani lifting using the Eichler-Shimura isomorphism and the group of modular symbols (see Section 3), we can define the *algebraic D-th Shintani lifting*

$$\theta^{Np,\mathrm{alg}}_{k,\chi,D}(f^{*}_{2k+2}) := \frac{\theta^{Np}_{k,\chi,D}(f^{*}_{2k+2})}{\Omega^{-}(f^{*}_{2k+2})} \in \mathcal{O}_K[[q]],$$

where we use our hypothesis $Np \geq 4$ to ensure that $\Gamma_0(Np)$ is torsion-free. We will interpolate a family $\{\theta^{Np,\mathrm{alg}}_{k,\chi,D}(f^{*}_{2k+2})\}_k$, p-adically. When $D = 1$, the interpolation has already done by [7] for $\alpha = 0$ and by [6] for any $\alpha \in \mathbb{Q}_{\geq 0}$ with some error term that is not necessarily a p-adic unit.

2 Main results

Theorem 1. *Let the notation be the same as Section* 1. *Then there exists a positive integer m_0 such that for any $r > m_0$ and an integer $k > (\alpha-1)/2$ satisfying $k \equiv k_0 \pmod{(p-1)p^r}$, we have the following*:

(i) $e_k\theta^{Np,\mathrm{alg}}_{k,\chi,D}(f^{*}_{2k+2}) \equiv \theta^{Np,\mathrm{alg}}_{k,\chi,D}(f^{*}) \pmod{p^{r-m_0}\mathcal{O}_K}$ *for some $e_k \in \mathcal{O}_K^{\times}$.*

(ii) *We further assume that χ is the trivial character $\mathbf{1}$ and that $\chi_D(\ell)$ coincides with the eigenvalue of f for the Atkin-Lehner involution for any prime $\ell \mid N$. Then,*

$$e_{k,D}L^{\mathrm{alg}*}\left(k+1, f_{2k+2}\otimes\chi_D\right) \equiv L^{\mathrm{alg}*}\left(k_0+1, f\otimes\chi_D\right) \pmod{p^{r-m_0}\mathcal{O}_K}$$

for some $e_{k,D} \in \mathcal{O}_K^{\times}$, where

$$L^{\mathrm{alg}*}\left(k+1, f_{2k+2}\otimes\chi_D\right) := \frac{k!L\left(k+1, f_{2k+2}\otimes\chi_D\right)}{\pi^{k+1}\Omega^{-}(f^{*}_{2k+2})} \in \mathcal{O}_K.$$

Assume $\alpha = 0$ for the remainder of this section.

Theorem 2. *Let $f \in S^{\mathrm{new}}_{2k+2}(N,\mathbf{1})_0$ and $g \in S^{\mathrm{new}}_{2k'+2}(N,\mathbf{1})_0$ be primitive forms with $k, k' \geq 0$ and $\mathcal{O}$ the integer ring of the p-adic completion of the composite field of the Hecke fields of f^* and g^*. Assume that $f^* \equiv g^* \pmod{p^{r_0}\mathcal{O}}$ for some positive integer r_0 and that $k \equiv k' \pmod{(p-1)p^r}$ for a sufficiently large integer r. We further assume that $\chi_D(\ell)$ coincides with the eigenvalue of f for the Atkin-Lehner involution for any prime $\ell \mid N$ and that the Galois representation attached to f^* is residually irreducible. Then there exist $u \in \mathcal{O}^{\times}$ such that we have*

$$L^{\mathrm{alg}}\left(k+1, f^*\otimes\chi_D\right) \equiv uL^{\mathrm{alg}}\left(k'+1, g^*\otimes\chi_D\right) \pmod{p^{r_0}\mathcal{O}},$$

where

$$L^{\mathrm{alg}}\left(k+1, f^*\otimes\chi_D\right) := \frac{G(\chi_D)k!L\left(k+1, f^*\otimes\chi_D\right)}{(-2\pi\sqrt{-1})^{k+1}\Omega^{-}(f^*)} \in \mathcal{O}$$

with the Gauss sum $G(\chi_D)$ of χ_D.

Remark 1. When $k = k'$, we can take $u = 1$ in the theorem above by [8, Corollary 1.11].

Assume $p = 3$ (and hence $N \geq 3$). Then we have the following:

Theorem 3. *Let E be an elliptic curve over $\mathbb{Q}$ of conductor $N \geq 3$. Assume that E has a rational potint of order 3 and good ordinary reduction at 3. If any prime $\ell \mid N$ at which E has nonsplit multiplicative reduction satisfies $\ell \equiv 2 \pmod 3$, then, for a sufficiently large integer r and any positive integer $k \equiv 0 \pmod{2 \cdot 3^r}$, there exists a primitive form $f \in S_{2k+2}^{\mathrm{new}}(N, \mathbf{1})_0$ satisfying*

$$M_f(X) := \sharp\{|D| \leq X \mid L(k, f \otimes \chi_D) \neq 0\} \gg X,$$

i.e., there exists a positive constant c such that for sufficiently large X, we have $M_f(X) \geq cX$.

Remark 2. Let f_E be the primitive form attached to the elliptic curve E as in the theorem above. Then $M_{f_E}(X) \gg X$ is due to [8, Corollary 3.5]. In addition, the elliptic curve given by the equation $y^2 + y = x^3 + x^2 - 9x - 15$ satisfies the assumption of the theorem above (see [8, Example 3.7]).

3 Key ingredients for interpolation

For a $\mathbb{Z}[\Gamma_0(Np)]$-module M, the group of M-valued *modular symbols* over $\Gamma_0(Np)$ is defined by

$$\mathrm{Symb}_{\Gamma_0(Np)}(M) := \mathrm{Hom}_{\mathbb{Z}[\Gamma_0(Np)]}(\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})), M)$$

For a ring R, we denote by $L(2k, \chi^2; R)$ the $R[\Gamma_0(Np)]$-module of homogeneous polynomials in (X, Y) of degree $2k$ with coefficients in R endowed with the χ^2-twisted action, i.e., for $\gamma \in \Gamma_0(Np)$ and $P(X, Y) \in L(2k, \chi^2; R)$,

$$(\gamma P)(X, Y) := \chi^2(\gamma) P((X, Y)^t\gamma),$$

where $\chi^2(\gamma)$ is the value of χ^2 at the lower right entry of γ. For a compact p-adic manifold $W (= \mathbb{Z}_p^\times$ or $\mathbb{Z}_p \times \mathbb{Z}_p^\times)$, we denote by $\mathcal{A}(W)$ the space of K-valued locally analytic functions on W and by

$$\mathcal{D}(W) := \mathrm{Hom}_K^{\mathrm{cont}}(\mathcal{A}(W), K)$$

the space of K-valued locally analytic distributions on W endowed with the strong topology. For a rigid analytic variety X over K, we denote by $A(X)$ the ring of rigid analytic functions on X. Let $\mathcal{W}$ be the *weight space* attached to $\mathcal{O}_K[\![\mathbb{Z}_p^\times]\!]$. According to [1, Theorem 3.4.3 and pp.25–26], there exists the canonical isomorphism (referred to as the *p-adic Fourier transform*) of locally convex K-algebras

$$\mathcal{D}(\mathbb{Z}_p^\times) := \mathcal{D}(\mathbb{Z}_p^\times, K) \xrightarrow{\sim} A(\mathcal{W});\ \nu \mapsto \hat{\nu},$$

where $\hat{\nu}(k) := \int_{\mathbb{Z}_p^\times} t^k d\nu(t)$ for any $k \in \mathcal{W}(K)$ and notice that $\mathcal{W}(K)$ is contained in $\mathcal{A}(\mathbb{Z}_p^\times)$ by [1, Lemma 3.2.3]. For an open affinoid subvariety Ω of $\mathcal{W}$, we define $k_\Omega \in \mathcal{W}(A(\Omega))$ by

$$t^{k_\Omega} := k_\Omega(t) := \mathrm{res}_\Omega(\widehat{\delta_t})$$

for any $t \in \mathbb{Z}_p^\times$, where $\mathrm{res}_\Omega : A(\mathcal{W}) \to A(\Omega)$ is the restriction homomorphism and $\delta_t \in \mathcal{D}(\mathbb{Z}_p^\times)$ is the *Dirac delta distribution* at t. We denote by $\mathcal{A}_{\chi^2}$ the $K[\Gamma_0(Np)]$-module $\mathcal{A}(\mathbb{Z}_p \times \mathbb{Z}_p^\times)$ endowed with χ^2-twisted action; we let $\gamma \in \Gamma_0(Np)$ act on $f \in \mathcal{A}_{\chi^2}$ by

$$(\gamma \cdot f)(x, y) = \chi^2(\gamma) f((x, y)^t\gamma).$$

We set $\mathcal{D}_{\chi^2} := \mathrm{Hom}_K^{\mathrm{cont}}(\mathcal{A}_{\chi^2}, K)$ and let $\gamma \in \Gamma_0(Np)$ act on $\mu \in \mathcal{D}_{\chi^2}(K)$ by

$$(\mu|\gamma)(f) := \mu(\gamma \cdot f)$$

for $f \in \mathcal{A}_{\chi^2}$. We let $\nu \in \mathcal{D}(\mathbb{Z}_p^\times)$ act on $\mu \in \mathcal{D}_{\chi^2}$ by

$$(\nu \cdot \mu)(f) := \int_{\mathbb{Z}_p} \left(\int_{\mathbb{Z}_p^\times} f(\lambda z) d\nu(\lambda) \right) d\mu(z)$$

for $f \in \mathcal{A}_{\chi^2}$. Since this action commutes with the $\Gamma_0(Np)$-action, we may consider $\mathcal{D}_{\chi^2}$ as a $\mathcal{D}(\mathbb{Z}_p^\times)[\Gamma_0(Np)]$-module. By [1, Theorem 3.5.4], there exists a unique element $k_\Omega^* \in \mathrm{Hom}_{K\text{-alg}}^{\mathrm{cont}}(\mathcal{D}(\mathbb{Z}_p^\times), A(\Omega))$ such that $k_\Omega^*(\delta_t) = t^{k_\Omega}$. We define the $A(\Omega)[\Gamma_0(Np)]$-module

$$\mathcal{D}_{\Omega,\varepsilon} := \mathcal{D}_{\chi^2} \hat{\otimes}_{\mathcal{D}(\mathbb{Z}_p^\times)} A(\Omega)$$

as the complete tensor product with respect to k_Ω^*.

Lemma 4 ([1]). *Let $h \in \mathbb{Q}_{\geq 0}$. For any $\kappa \in \mathcal{W}(K)$, there exists an open K-affinoid subvariety Ω in $\mathcal{W}$ containing κ such that an $A(\Omega)$-module $\mathrm{Symb}_{\Gamma_0(Np)}(\mathcal{D}_{\Omega,\chi^2})^\pm$ admits a slope $\leq h$ decomposition with respect to the Hecke operator T_p. Moreover, for any integer $k > (h-1)/2$, we have the following control theorem:*

$$\mathrm{Symb}_{\Gamma_0(Np)}(\mathcal{D}_{\Omega,\chi^2})^\pm_{\leq h} \otimes_{A(\Omega)} A(\Omega)/P_{2k} \xrightarrow{\sim} \mathrm{Symb}_{\Gamma_0(Np)}(L(2k, \chi^2; K))^\pm_{\leq h},$$

where the subscript $\leq h$ means the T_p-slope $\leq h$-part (see [1, Definitions 4.6.1 and 4.6.3] *for the definition of slope $\leq h$ decomposition).*

Similarly to [7], we have a cohomological interpretation of D-th Shintani lifting as follows:

$$\begin{array}{ccccc}
H_c^1(\Gamma_0(Np), L(2k, \chi^2; \mathbb{C}))^- & \xrightarrow[\text{Ash-Stevens}]{\sim} & \mathrm{Symb}_{\Gamma_0(Np)}(L(2k, \chi^2; \mathbb{C}))^- & \xrightarrow{\Theta_{k,\chi,D}^{Np}} & \mathbb{C}[[q]] \\
\text{Manin-Drinfeld} \uparrow & & & & \uparrow q\text{-expansion} \\
H_p^1(\Gamma_0(Np), L(2k, \chi^2; \mathbb{C}))^- & \xleftarrow[\text{Eichler-Shimura}]{\sim} & S_{2k+2}(Np, \chi^2) & \xrightarrow{\theta_{k,\chi,D}^{Np}} & S_{k+3/2}^+(4Np, \tilde{\chi}),
\end{array}$$

where the superscript $-$ is the (-1)-eigenspace for some involution and all arrows are Hecke equivariant. From this point of view, it suffices to interpolate $\Theta_{k,\chi,D}^{Np}$, i.e., construct a Hecke equivariant morphism $\mathrm{Symb}_{\Gamma_0(Np)}(\mathcal{D}_{\Omega,\chi^2})^\pm_{\leq h} \to A(\Omega)[[q]]$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Symb}_{\Gamma_0(Np)}(\mathcal{D}_{\Omega,\chi^2})^-_{\leq h} & \longrightarrow & A(\Omega)[[q]] \\
\downarrow & & \downarrow \\
\mathrm{Symb}_{\Gamma_0(Np)}(L(2k, \chi^2; K))^-_{\leq h} & \xrightarrow{\Theta_{k,\chi,D}^{Np}} & K[[q]]
\end{array}$$

with varying k, where the left vertical arrow is the specialization obtained by the lemma above and the right vertical arrow is modulo P_k. In fact, we make such a diagram of semi-simple and integral part.

4 Outline of the proof of Theorem 1

We denote by $K\langle X\rangle$ the ring of restricted power series over K and by $B_K[a,r]$ the affinoid disk of radius r about a (see [2]). Let

$$\sigma : K\left\langle (X-2k_0)/p^{m_0}\right\rangle \xrightarrow{\sim} K\left\langle (X-(2k_0+2))/p^{m_0}\right\rangle ;\ X\mapsto X-2$$

be an isometric K-algebra isomorphism with respect to the supremum semi-norm and

$${}^{a}\sigma : B := B_K[2k_0+2, p^{-m_0}] \xrightarrow{\sim} B_K[2k, p^{-m_0}] =: B_\sigma;\ \mathfrak{m}\mapsto \sigma^{-1}(\mathfrak{m})$$

the corresponding isomorphism of K-affinoid varieties. Via σ, the specialization at $2k$ on B_σ corresponds to the specialization at $2k+2$ on B. We apply Lemma 4 to $h := \alpha$ and $\Omega := B_\sigma$. Roughly speaking, we can make a Hecke equivariant isomorphism between a subspace of overconvergent families over B and a subspace of $\mathrm{Symb}_{\Gamma_0(Np)}(\mathcal{D}_{\Omega,\chi^2})^-_{\le\alpha}$, which interpolates an isomorphism between their specialized spaces by shrinking B about the center if necessary. Consequently, we see that there exists $\Phi \in \mathrm{Symb}_{\Gamma_0(Np)}(\mathcal{D}_{B_\sigma,\chi^2})^{\pm}_{\le\alpha}$ such that for any integer $k > (\alpha-1)/2$ with $k \equiv k_0 \pmod{(p-1)p^{m_0}}$, the image of Φ under the specialization at $2k$ coincides with $e_k\Omega^-(f^*_{2k+2})^{-1}\Phi_{f^*_{2k+2}}$ for some $e_k \in \mathcal{O}_K^\times$, where $\Phi_{f^*_{2k+2}} \in \mathrm{Symb}_{\Gamma_0(Np)}(L(2k,\chi^2;\mathbb{C}))^-$ is the modular symbol attached to f^*_{2k+2}. Similarly to [7], we can construct $\Theta(\Phi) \in A(B_\sigma)[[q]]$ such that for any integer $k > (\alpha-1)/2$ with $k \equiv k_0 \pmod{(p-1)p^{m_0}}$, the image of $\Theta(\Phi)$ under the specialization at $2k$ coincides with $e_k\theta^{Np,\mathrm{alg}}_{k,\chi,D}(f^*_{2k+2})$. Since we can check that the $|D|$-th Fourier coefficient of $\theta^{Np}_{k,\chi,D}(f^*_{2k+2})$ equals the central L-value of $f_{2k+2}\otimes\chi_D$ with explicit multiples, the second assertion of the theorem follows.

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