Iwasawa Main Conjecture for abelian varieties over global field of characteristic p

Fabien Trihan (Sophia University)

Abstract

We discuss the Iwasawa Main conjecture for semistable abelian varieties over non-commutative unramified extensions of a global field of positive characteristic.

1 Preliminaries

The non-commutative Iwasawa theory was introduced by Coates, Fukaya, Kato, Sujatha and Venjakob in [8] for number fields. Here we study the function field analogue of their theory. In this section we shall, for the reader's convenience, recall some basic background material relating to relative algebraic K-theory, refined Euler characteristics, the Iwasawa algebras of certain kinds of non-commutative p-adic Lie groups and relevant aspects of the theory of pro-coverings.

1.1 Relative algebraic K-theory and Iwasawa algebras

1.1.1

Throughout this article, modules are always to be understood, unless explicitly stated otherwise, as left modules.

For any associative, unital, left noetherian ring R we write $D(R)$ for the derived category of R-modules. We also write $D^-(R)$, $D^+(R)$ and $D^{\mathrm{p}}(R)$ for the full triangulated subcategories of $D(R)$ comprising complexes that are isomorphic to an object of the categories $C^-(R)$, $C^+(R)$ and $C^{\mathrm{p}}(R)$ of bounded above complexes of projective R-modules, bounded below complexes of injective R-modules and bounded complexes of finitely generated projective R-modules.

For any homomorphism $R \to R'$ of rings as above we write $K_0(R, R')$ for the relative algebraic K_0-group that is defined in terms of explicit generators and relations by Swan in [18, p. 215]. We recall in particular that this group fits into a canonical exact sequence of abelian groups of the form

$$K_1(R) \to K_1(R') \xrightarrow{\partial_{R,R'}} K_0(R, R') \to K_0(R) \to K_0(R'). \tag{1}$$

Here, for any ring A, we write $K_1(A)$ for its Whitehead group and $K_0(A)$ for the Grothendieck group of the category of finitely generated projective A-modules, and the first and last arrows in (1) denote the homomorphisms that are naturally induced by the given ring homomorphism $R \to R'$. (For more details about this sequence, and a proof of its exactness, see [18, Chap. 15]).

It is well known that for any object $C^\bullet$ of $D^{\mathrm{p}}(R)$ one can define a canonical Euler characteristic $\chi_R(C^\bullet)$ in the group $K_0(R)$. We recall further that for any such complex $C^\bullet$ and any (bounded) exact sequence of R'-modules of the form

$$\epsilon : \ 0 \to \cdots \to R' \otimes_R H^i(C^\bullet) \to R' \otimes_R H^{i+1}(C^\bullet) \to R' \otimes_R H^{i+2}(C^\bullet) \to \cdots \to 0,$$

one can define a canonical pre-image $\chi^{\mathrm{ref}}_{R,R'}(C^\bullet,\epsilon)$ of $\chi_R(C^\bullet)$ under the connecting homomorphism $K_0(R,R') \to K_0(R)$ that occurs in the sequence (1). (For more details of this natural 'refined Euler characteristic' construction see either [6] or [5]).

In particular, if $C^\bullet$ belongs to $D^{\mathrm{p}}(R)$ and the complex $R'\otimes^{\mathbb{L}}_R C^\bullet$ is acyclic, then we shall set

$$\chi^{\mathrm{ref}}_{R,R'}(C^\bullet) := \chi^{\mathrm{ref}}_{R,R'}(C^\bullet,\epsilon_{R'})$$

where $\epsilon_{R'}$ denotes the exact sequence of zero R'-modules.

In the case that R' is the total quotient ring of R, then we often abbreviate the connecting homomorphism $\partial_{R,R'}$ in the exact sequence (1) to ∂_R and the Euler characteristics $\chi^{\mathrm{ref}}_{R,R'}(-,-)$ and $\chi^{\mathrm{ref}}_{R,R'}(-)$ to $\chi^{\mathrm{ref}}_R(-,-)$ and $\chi^{\mathrm{ref}}_R(-)$ respectively. In those cases that R and R' are both clear from context we also sometimes abbreviate the notation $\partial_{R,R'}$ and $\chi^{\mathrm{ref}}_{R,R'}(-,-)$ to ∂ and $\chi^{\mathrm{ref}}(-,-)$ respectively.

1.1.2

For any profinite group G and any finite extension $\mathcal{O}$ of $\mathbb{Z}_\ell$ (for any prime ℓ) we write $\Lambda_{\mathcal{O}}(\mathcal{G})$ for the $\mathcal{O}$-Iwasawa algebra $\varprojlim_U \mathcal{O}[G/U]$ of G, where U runs over the set of open normal subgroups of G (partially ordered by inclusion) and the limit is taken with respect to the obvious transition homomorphisms. We also write $\mathrm{Q}_{\mathcal{O}}(\mathcal{G})$ for the total quotient ring of $\Lambda_{\mathcal{O}}(\mathcal{G})$, and if $\mathcal{O}=\mathbb{Z}_p$, then we omit the subscripts '$\mathcal{O}$' from both $\Lambda_{\mathcal{O}}(\mathcal{G})$ and $\mathrm{Q}_{\mathcal{O}}(\mathcal{G})$.

Motivated by the approach of Coates, Fukaya, Kato, Sujatha and Venjakob in [8], we now assume that G lies in a group extension of the form

$$\{1\}\to H\to G\xrightarrow{\pi_G}\Gamma\to\{1\} \tag{2}$$

in which Γ is (topologically) isomorphic to the additive group of p-adic integers $\mathbb{Z}_p$. We also fix an algebraic closure $\mathbb{Q}_p^c$ of $\mathbb{Q}_p$ and write $\mathcal{O}$ for the valuation ring of a finite extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$.

We write S for the subset of $\Lambda_{\mathcal{O}}(G)$ comprising elements f for which the quotient $\Lambda_{\mathcal{O}}(G)/\Lambda_{\mathcal{O}}(G)f$ is finitely generated as a module over the ring $\Lambda_{\mathcal{O}}(H)$ and we also set $S^* := \bigcup_{i\ge 0} p^{-i}S$.

We recall that in [8, §2] it is shown that S and S^* are both multiplicatively closed left and right Ore sets of non-zero divisors and so we can write $\Lambda_{\mathcal{O}}(G)_S$ and $\Lambda_{\mathcal{O}}(G)_{S^*} = \Lambda_{\mathcal{O}}(G)_S[\frac{1}{p}]$ for the corresponding localisations of $\Lambda_{\mathcal{O}}(G)$ (we however caution the reader that this notation does not explicitly indicate that S and S^* depend upon both $\mathcal{O}$ and the chosen extension (2)).

We often use the fact that the long exact sequence (1) is compatible with the formulation of scalar extensions in the sense that there exists a commutative diagram

$$\begin{array}{ccccc} K_1(\Lambda_{\mathcal{O}}(G)) & \longrightarrow & K_1(\Lambda_{\mathcal{O}}(G)_{S^*}) & \xrightarrow{\partial_{\mathcal{O},G,S^*}} & K_0(\Lambda_{\mathcal{O}}(G),\Lambda_{\mathcal{O}}(G)_{S^*}) \\ \| & & \uparrow & & \uparrow \\ K_1(\Lambda_{\mathcal{O}}(G)) & \xrightarrow{\alpha_{\mathcal{O},G}} & K_1(\Lambda_{\mathcal{O}}(G)_S) & \xrightarrow{\partial_{\mathcal{O},G,S}} & K_0(\Lambda_{\mathcal{O}}(G),\Lambda_{\mathcal{O}}(G)_S) \end{array} \tag{3}$$

in which we set $\partial_{\mathcal{O},G,\Sigma} := \partial_{\Lambda_{\mathcal{O}}(G),\Lambda_{\mathcal{O}}(G)_\Sigma}$ for both $\Sigma=S$ and $\Sigma=S^*$.

1.1.3

If Σ denotes either of the Ore sets S or S^* defined above, then we shall write $D^{\mathrm{p}}_\Sigma(\Lambda_{\mathcal{O}}(G))$ for the full triangulated subcategory of $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ comprising those (perfect) complexes $C^\bullet$ for which $\Lambda_{\mathcal{O}}(G)_\Sigma\otimes_{\Lambda_{\mathcal{O}}(G)} C^\bullet$ is acyclic.

We note, in particular, that any object $C^\bullet$ of $D^{\mathrm{p}}_{\Sigma}(\Lambda_{\mathcal{O}}(G))$ gives rise to an Euler characteristic element

$$\chi^{\mathrm{ref}}(C^\bullet) := \chi^{\mathrm{ref}}_{\Lambda_{\mathcal{O}}(G),\Lambda_{\mathcal{O}}(G)_\Sigma}(C^\bullet, \epsilon_{\Lambda_{\mathcal{O}}(G)_\Sigma})$$

in $K_0(\Lambda_{\mathcal{O}}(G), \Lambda_{\mathcal{O}}(G)_\Sigma)$ that depends only upon $C^\bullet$.

We recall that if G has no element of order p, and Σ denotes either S or S^*, then the group $K_0(\Lambda_{\mathcal{O}}(G), \Lambda_{\mathcal{O}}(G)_\Sigma)$ is naturally isomorphic to the Grothendieck group of the category of finitely generated $\Lambda_{\mathcal{O}}(G)$-modules M with the property that $\Lambda_{\mathcal{O}}(G)_\Sigma \otimes_{\Lambda_{\mathcal{O}}(G)} M$ vanishes (for an explicit description of this isomorphism see, for example, [7, §1.2]).

In particular, if G is also abelian then the determinant functor induces a natural isomorphism between $K_0(\Lambda_{\mathcal{O}}(G), \Lambda_{\mathcal{O}}(G)_\Sigma)$ and the multiplicative group of invertible $\Lambda_{\mathcal{O}}(G)$-lattices in $\Lambda_{\mathcal{O}}(G)_\Sigma$. For any (finitely generated torsion) $\Lambda_{\mathcal{O}}(G)$-module M as above, this isomorphism sends the element $\chi^{\mathrm{ref}}(M[0])$ defined above to the (classical) characteristic ideal $\mathrm{ch}_{\Lambda_{\mathcal{O}}(G)}(M)$ of M.

In the sequel we will often abbreviate the Euler characteristic $\chi^{\mathrm{ref}}_{\Lambda_{\mathcal{O}}(G),\mathrm{Q}_{\mathcal{O}}(G)}(-,-)$ to $\chi^{\mathrm{ref}}_{\mathcal{O},G}(-,-)$.

1.1.4

We now fix a topological generator γ of the group Γ that occurs in the extension (2) and also an Ore set $\Sigma \in \{S, S^*\}$.

Then for the valuation ring $\mathcal{O}'$ of any finite extension of $\mathbb{Q}_p$ in $\mathbb{Q}_p^c$ which contains $\mathcal{O}$, and any continuous homomorphism of the form

$$\rho : G \to \mathrm{GL}_n(\mathcal{O}') \tag{4}$$

there is an induced ring homomorphism

$$\Lambda_{\mathcal{O}}(G)_\Sigma \to \mathrm{M}_n(\mathcal{O}') \otimes_{\mathcal{O}'} \mathrm{Q}_{\mathcal{O}'}(\Gamma) \cong \mathrm{M}_n(\mathrm{Q}_{\mathcal{O}'}(\Gamma))$$

that sends every element g of G to $\rho(g) \otimes \pi_G(g)$. This ring homomorphism in turn induces a homomorphism of abelian groups

$$\Phi_{\mathcal{O},G,\Sigma,\rho} : K_1(\Lambda_{\mathcal{O}}(G)_\Sigma) \to K_1(\mathrm{M}_n(\mathrm{Q}_{\mathcal{O}'}(\Gamma))) \cong K_1(\mathrm{Q}_{\mathcal{O}'}(\Gamma)) \cong \mathrm{Q}_{\mathcal{O}'}(\Gamma)^\times \cong \mathrm{Q}(\mathcal{O}'[[u]])^\times \tag{5}$$

where we write $\mathcal{O}'[[u]]$ for the ring of power series over $\mathcal{O}'$ in the formal variable u, the first isomorphism is induced by Morita equivalence, the second by taking determinants (over the ring $\mathrm{Q}_{\mathcal{O}'}(\Gamma)$) and the last by sending $\gamma - 1$ to u. In the sequel we shall often not distinguish between the homomorphisms $\Phi_{\mathcal{O},G,S,\rho}$ and $\Phi_{\mathcal{O},G,S^*,\rho}$ and also abbreviate $\Phi_{\mathcal{O},G,\Sigma,\rho}$ to $\Phi_{G,\rho}$ if we feel that $\mathcal{O}$ and Σ are both clear from context.

For any element ξ of the Whitehead group $K_1(\Lambda_{\mathcal{O}}(G)_\Sigma)$ and any representation ρ as in (4), one then defines the 'value' $\xi(\rho)$, resp. the 'leading term' $\xi^*(\rho)$, of ξ at ρ to be the value, resp. the leading term, at $u = 0$ of the series $\Phi_{\mathcal{O},G,\Sigma,\rho}(\xi)$. In particular, one has $\xi^*(\rho) \in \mathbb{Q}_p^c \setminus \{0\}$ for all ρ and one regards the value $\xi(\rho)$ to be equal to '∞' if the algebraic order of $\Phi_{\mathcal{O},G,\Sigma,\rho}(\xi)$ at $u = 0$ is strictly negative.

We recall finally that a continuous representation ρ as in (4) (with $\mathcal{O} = \mathbb{Z}_p$) is said to be an 'Artin representation' if its image $\rho(G)$ is finite.

1.2 Pro-coverings

We quickly review some standard material concerning pro-coverings.

For any prime power n we write $\mathbb{F}_n$ for the finite field of cardinality n. We also now fix a prime p and a strictly positive integral power q of p.

1.2.1

Let X be a connected scheme over $\mathbb{F}_q$ and write $\mathbf{FEt}/X$ for the category of X-schemes that are finite and étale over X.

Then for any geometric point $\overline{x}$ of X the functor that takes each scheme Y to the set $F_{\overline{x}}(Y) := \mathrm{Hom}_X(\overline{x}, Y)$ of geometric points of Y that lie over $\overline{x}$ gives an equivalence of categories between projective systems in $\mathbf{FEt}/X$ and the category of projective systems of finite sets upon which the group $\pi_1(X, \overline{x})$ acts continuously (on the left).

In particular, for any continuous quotient G of $\pi_1(X, \overline{x})$ there exists a corresponding pro-covering of X of group G which we denote by $(f : Y \to X, G)$, or more simply by $f : Y \to X$.

1.2.2

As above, we fix an integral power q of a given prime number p. In addition, we now fix a separated variety X that is of finite type over $\mathbb{F}_q$ and a geometric point $\overline{x}$ of X. We also fix a separable closure $\mathbb{F}_q^c$ of $\mathbb{F}_q$.

We then consider compact p-adic Lie groups G which lie in a commutative diagram of continuous homomorphisms of the form

$$\begin{array}{ccc} \pi_1(X, \overline{x}) & \xrightarrow{\pi_{X,\overline{x}}} & \pi_1^p(\mathbb{F}_q, \mathbb{F}_q^c) \\ & {\scriptstyle \pi'_G} \searrow \quad \nearrow {\scriptstyle \pi_G} & \\ & G & \end{array} \tag{6}$$

where $\pi_{X,\overline{x}}$ is the canonical homomorphism to the maximal pro-p quotient $\pi_1^p(\mathbb{F}_q, \mathbb{F}_q^c)$ of $\pi_1(\mathbb{F}_q, \mathbb{F}_q^c)$ and π'_G is surjective.

The composite homomorphism

$$\pi_1(\mathbb{F}_q, \mathbb{F}_q^c) \to \mathrm{Gal}(\mathbb{F}_q^c/\mathbb{F}_q) \to \mathbb{Z}_p,$$

where the first map is the canonical isomorphism and the second sends the Frobenius automorphism $z \mapsto z^q$ in $\mathrm{Gal}(\mathbb{F}_q^c/\mathbb{F}_q)$ to 1, induces an identification of $\mathrm{im}(\pi_{X,\overline{x}}) = \mathrm{im}(\pi_G)$ with a subgroup of $\mathbb{Z}_p$ of finite index, d_X say. This identification in turn gives rise to a canonical group extension

$$\{1\} \to \ker(\pi_G) \to G \xrightarrow{\pi_G} \mathrm{im}(\pi_G) \to \{1\}$$

of the form (2).

In the rest of this paper, we shall set

$$H := \ker(\pi_G) \quad \text{and} \quad \Gamma := \mathrm{im}(\pi_{X,\overline{x}}) = \mathrm{im}(\pi_G)$$

and write γ for the topological generator of Γ that is given by the d_X-th power of the image of the Frobenius automorphism $z \mapsto z^q$ under the natural projection map $\mathrm{Gal}(\mathbb{F}_q^c/\mathbb{F}_q) \cong \pi_1(\mathbb{F}_q, \mathbb{F}_q^c) \to \pi_1^p(\mathbb{F}_q, \mathbb{F}_q^c)$.

2 Semistable abelian varieties over unramified extensions

There is an extensive existing literature concerning the Iwasawa theory of abelian varieties over compact ℓ-adic Lie extensions of global function fields of characteristic p, with $p \neq \ell$. In this regard we mention, in particular, the interesting work of Ellenberg [9], Sechi [17], Bandini and Longhi [1] and Pacheco [16].

In the next sections, however, we shall focus on results that concern the Iwasawa theory of abelian varieties over compact p-adic Lie extensions of such function fields (of characteristic p).

In this first section we state (in Theorem 2.1) a result that was recently proved by Vauclair and the present author in [20] and concerns the Selmer modules of semistable abelian varieties over certain unramified compact p-adic Lie-extensions.

2.1 Hypotheses and notations

We assume to be given a global function field K of characteristic p and field of constants $\mathbb{F}_q$. We then write $K_{\rm ar}$ for the constant $\mathbb{Z}_p$-extension of K.

We also fix an abelian variety A over K that has semistable reduction and write Z for the set of places at which A has bad reduction. We write C for the proper smooth curve over $\mathbb{F}_q$ which has field of functions K and $\mathcal{A}$ for the Néron model over C of A over K. We denote by $D^{\log}(A)$ the Dieudonné crystal defined by Kato and the author [12] on the log-scheme $C^{\#}$ which has underlying scheme C and log-structure defined by the smooth divisor Z.

We recall that the p-Selmer group of A over any algebraic field extension F of K is defined to be

$$\mathrm{Sel}_{p^\infty}(A/F) := \ker\big(\mathrm{H}^1_{\rm fl}(F, A_{p^\infty}) \longrightarrow \prod_v \mathrm{H}^1_{\rm fl}(F_v, A)\big),$$

where A_{p^∞} is the p-divisible group associated to A, $\mathrm{H}^1_{\rm fl}(-,-)$ denotes flat cohomology and in the product v runs over all places of F.

In the sequel we shall study the Pontrjagin dual

$$X_p(A/F) := \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{Sel}_{p^\infty}(A/F), \mathbb{Q}_p/\mathbb{Z}_p)$$

of $\mathrm{Sel}_{p^\infty}(A/F)$. If F/K is Galois, then we endow $X_p(A/F)$ with the natural contragredient action of $\mathrm{Gal}(F/K)$.

In the rest of this section we always assume that A satisfies the following hypothesis which, for convenience, we refer to as '$\mu_A \sim 0$'

- A is isogeneous to an abelian variety A' such that $X_p(A'/K_{\rm ar})$ has trivial μ-invariant as a $\Lambda(\Gamma)$-module.

We fix a compact p-adic Lie extension K_∞ of K that is unramified everywhere and contains $K_{\rm ar}$ and then set $G := \mathrm{Gal}(K_\infty/K)$. This gives a canonical group extension

$$\{1\} \to \mathrm{Gal}(K_\infty/K_{\rm ar}) \to G \xrightarrow{\pi_G} \mathrm{Gal}(K_{\rm ar}/K) \to \{1\}$$

of the form (2) in which π_G is the natural restriction map.

In the sequel we therefore set

$$H := \mathrm{Gal}(K_\infty/K_{\rm ar}) \quad \text{and} \quad \Gamma := \mathrm{Gal}(K_{\rm ar}/K)$$

and use the canonical Ore sets S and S^* in $\Lambda(G)$ that are discussed in §1.1.2 (with respect to this choice of data).

The extension K_∞/K corresponds to a pro-étale covering $C_\infty \to C$ (in the sense of §1.2). For convenience we also fix a cofinal system G_n of finite quotients of G, identify G with the inverse limit $\varprojlim_n G_n$ and for each n write $C_n \to C$ for the finite layer of C_∞ that corresponds to the group G_n.

2.2 Statement of the main results

In this section we write $\Gamma(-,-)$ for the global section functor for the flat (rather than étale) topology.

2.2.1

In [20] the authors first define a canonical element of the relative algebraic K_0-group $K_0(\Lambda(G), \Lambda(G)_{S^*})$ that plays the role of a (non-commutative) characteristic ideal in the formulation of the main conjecture.

To do this they define complexes of $\Lambda(G)$-modules by setting

- $N_\infty = N_{\infty,C,Z} := R\varprojlim_{n,k} R\Gamma^Z(C_n, \mathcal{A}_{p^k})$, where Γ^Z is the functor of global sections (for the flat topology) that vanish at Z;
- $L_\infty := R\varprojlim_n R\Gamma(C_n, \mathrm{Lie}(\mathcal{A})(-Z))$.

Under the hypothesis that $\mu_A \sim 0$, it can then be shown that these complexes belong to the category $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ that was introduced in §1.1.3 (see, for example, Proposition 3.9).

One thereby obtains a well-defined element of $K_0(\Lambda(G), \Lambda(G)_{S^*})$ by setting

$$\chi(A/K_\infty) := \chi^{\mathrm{ref}}(N_\infty) + \chi^{\mathrm{ref}}(L_\infty),$$

where $\chi^{\mathrm{ref}}(-)$ is the Euler characteristic that was discussed in §1.1.3.

2.2.2

The authors of [20] then associate a canonical p-adic L-function to the arithmetic of A over K.

To recall their construction we write $D^{\log,0}$ for the crystalline sheaf that is obtained as the kernel of the canonical map $D^{\log}(A) \to \mathrm{Lie}(\mathcal{A})$ that is constructed in [12] and note that one obtains well-defined objects of $D^{\mathrm{p}}(\Lambda(G))$ by setting

- $P_\infty := R\varprojlim_n R\Gamma(C_n^{\#}/\mathbb{Z}_p, D^{\log}(A)(-Z))$,
- $I_\infty := R\varprojlim_n R\Gamma(C_n^{\#}/\mathbb{Z}_p, D^{\log,0}(A)(-Z))$.

For convenience, for each Y in $\{I, L, N, P\}$ we denote by Y_0 and Y_{ar} the complex Y_∞ when $K_\infty = K$ and $K_\infty = K_{\mathrm{ar}}$ respectively.

Now, under the stated hypotheses on A, there are morphisms in $D^{\mathrm{p}}(\Lambda(G))$

$$\mathbf{1} : I_\infty \to P_\infty,$$
$$\varphi : I_\infty \to P_\infty$$

which have the following property: after inverting p, the map $\mathbf{1}$ coincides with the identity on $I_0[1/p] = P_0[1/p]$ and the map φ coincides with p^{-1} times the endomorphism on $P_0[1/p]$ that is induced by the Frobenius operator of $D^{\log}(A)$.

Next write $S_\infty = S_{\infty,C^{\#}}$ for a complex that lies in an exact triangle in $D^{\mathrm{p}}(\Lambda(G))$ of the form

$$S_\infty \to I_\infty \xrightarrow{\mathbf{1}-\varphi} P_\infty \to S_\infty[1]$$

and then define the complexes S_0 and S_{ar} in just the same way as above. We recall in particular that the complex S_0 was first defined in [12] where it was referred to as the 'syntomic complex' of A over K.

It is clear that the scalar extension $\mathbf{1}_{S^*}$ of the morphism $\mathbf{1}$ is an isomorphism in $D^{\mathrm{p}}(\Lambda(G)_{S^*})$ and this is in fact also true of the morphism $\mathbf{1}-\varphi$ (see Corollary 3.12). Given this observation, the article [20] defines a p-adic L function for A over K_∞ by setting

$$\mathcal{L}_{A/K_\infty} := \mathrm{Det}_{\Lambda(G)_{S^*}}((\mathbf{1}-\varphi)_{S^*}\circ(\mathbf{1}_{S^*})^{-1}),$$

where $\mathrm{Det}_{\Lambda(G)_{S^*}}$ is the universal determinant functor of [13] (see also [11]) which, in particular, induces a multiplicative map from the set of automorphisms of $D^{\mathrm{p}}(\Lambda(G)_{S^*})$ to the Whitehead group $K_1(\Lambda(G)_{S^*})$.

2.2.3

We can now collect together the main results of [20] in the statement of the following result.

This result verifies a natural analogue for the abelian variety A and the compact p-adic Lie extension K_∞ of K of the main conjecture formulated by Coates et al in [8].

Theorem 2.1. *Fix a semistable abelian variety A over K and an unramified p-adic Lie extension K_∞ of K as above.*

(i) *Then for every Artin representation ρ of G whose image has coefficients in a finite totally ramified extension of $\mathbb{Q}_p$, one has*

$$\rho(\mathcal{L}_{A/K_\infty}) = L_Z(A,\rho,1),$$

where the value of $\mathcal{L}_{A/K_\infty}$ at ρ is as defined in §1.1.4.

(ii) *In $K_0(\Lambda(G),\Lambda(G)_{S^*})$ one has an equality*

$$\partial_{G,S^*}(\mathcal{L}_{A/K_\infty}) = \chi(A/K_\infty).$$

For a brief discussion of the key ideas which underlie the proof of the interpolation formula given in Theorem 2.1(i) see §3.5. The proof of Theorem 2.1(ii) relies, on the other hand, mainly on the existence of certain natural exact triangles in $D^{\mathrm{p}}(\Lambda(G))$ of the form

$$I_\infty \xrightarrow{\mathbf{1}} P_\infty \to L_\infty \to I_\infty[1] \tag{7}$$

and

$$N_\infty \to I_\infty \xrightarrow{\mathbf{1}-\varphi} P_\infty \to N_\infty[1]. \tag{8}$$

In fact, whilst the existence of the triangle (7) follows essentially directly from the definitions of the complexes I_∞, P_∞ and L_∞, the existence of (8) is equivalent to the existence of an isomorphism in $D^{\mathrm{p}}(\Lambda(G))$ between S_∞ and N_∞ and proving that such an isomorphism exists is a difficult task. It was first proved by Kato and the author in [12] when G is the trivial group. However, constructing such an isomorphism more generally requires a close re-examination of the constructions made in Chapters 4 and 5 of [12] in order to extend them into the necessary Iwasawa-theoretic setting, and much of [20] is taken up with this rather detailed work.

3 The Proof of Theorem 2.1

In this section we explain in greater detail some of the key steps that are involved in the proof of Theorem 2.1.

As in §2.2, we shall continue to write $\Gamma(-,-)$ for the global section functor for the flat topology.

3.1 The complexes N_0 and S_0

In both this subsection and in §3.2, we shall discuss the construction of an isomorphism in $D(\Lambda(G))$ between the complexes S_∞ and N_∞ (or equivalently, of an exact triangle of the form (8)).

We start by explaining how to show that the complexes N_0 and S_0 are isomorphic by means of a careful dévissage argument.

3.1.1

We first discuss the appropriate dévissage for N_0.

At each place v in Z at which A has semistable reduction we use the natural 'Raynaud extensions'

$$0 \longrightarrow T_v \longrightarrow G_v \longrightarrow B_v \longrightarrow 0,$$
$$0 \longrightarrow T_v^* \longrightarrow G_v^* \longrightarrow B_v^* \longrightarrow 0.$$

Here T_v and T_v^* are tori and B_v and B_v^* are abelian varieties over the ring of integers $\mathcal{O}_v$ of the completion K_v of K at v. At each such v we also write Γ_v for the Cartier dual of T_v^*.

Theorem 3.1. *There exists a natural exact triangle in $D^+(\mathbb{Z}_p)$ of the form*

$$N_{0,C,Z} \to N_{0,U,\emptyset} \oplus \prod_{v\in Z} N_{0,\mathcal{O}_v,k(v)} \to N_{0,K_v,\emptyset} \to N_{0,C,Z}[1].$$

For each place v in Z there is also a natural isomorphism

$$N_{0,\mathcal{O}_v,k(v)} \simeq R\varprojlim_k R\Gamma^{k(v)}(O_v, G_{v,p^k})$$

and an exact triangle in $D^+(\mathbb{Z}_p)$

$$N_{0,\mathcal{O}_v,k(v)} \to R\varprojlim_k R\Gamma(\mathcal{O}_v, \mathcal{A}_{v,p^k}) \to R\varprojlim_k R\Gamma(k(v), \overline{A}_{k(v),p^k}) \to N_{0,\mathcal{O}_v,k(v)}[1].$$

Note that, modulo proving the acyclicity of $R\Gamma^{k(v)}(O_v, \Gamma_v \otimes \mathbb{Q}_p/\mathbb{Z}_p)$, the main point of the proof of Theorem 3.1 is the construction of a natural Mayer-Vietoris triangle.

3.1.2

We next discuss the dévissage that is necessary to deal with S_0.

Let H be a p-divisible group on $X \in \{C, \mathcal{O}_v\}$ that is endowed with the log structure induced by either $T = Z$ or $T = k(v)$ or even the trivial log-structure. We then write $S_{0,X^\#,H}$ for a complex that is defined by means of an exact triangle in $D^+(\mathbb{Z}_p)$ of the form

$$S_{0,X^\#,H} \to R\Gamma(X^\#/\mathbb{Z}_p, D^{\log,0}(H)(-T)) \xrightarrow{1-\varphi} R\Gamma(X^\#/\mathbb{Z}_p, D^{\log}(H)(-T)) \to S_{0,X^\#,H}[1],$$

where $D^{\log}(H)$ denotes the inverse image of the crystalline Dieudonné crystal of H/X under the natural morphism of topoi $(X^\#/\mathbb{Z}_p)_{\mathrm{Crys}} \to (X/\mathbb{Z}_p)_{\mathrm{Crys}}$.

Theorem 3.2. *There exists a natural exact triangle in $D^+(\mathbb{Z}_p)$*

$$S_{0,C^\#} \to S_{0,U} \oplus \prod_{v\in Z} S_{0,\mathcal{O}_v^\#} \to \prod_{v\in Z} S_{0,K_v} \to S_{0,C^\#}[1].$$

For each place v in Z there are also exact triangles in $D^+(\mathbb{Z}_p)$

$$S_{0,\mathcal{O}_v^\#} \to S_{0,\mathcal{O}_v^\#,\Gamma_{v,p^\infty}} \to S_{0,\mathcal{O}_v^\#,G_{v,p^\infty}}[1] \to S_{0,C^\#}[1]$$

and for $H \in \{\Gamma_{v,p^\infty}, G_{v,p^\infty}\}$ also

$$S_{0,\mathcal{O}_v^\#,H} \to S_{0,\mathcal{O}_v,H} \to S_{0,k(v),\bar{H}} \to S_{0,\mathcal{O}_v^\#,H}[1],$$

where $\bar{H}$ denotes the reduction of H modulo p.

The proof of this result relies on the construction of the Dieudonné crystal $D^{\log}(A)$, an explicit computation on de Rham complexes and a natural adaptation of the Hyodo-Kato construction to the setting of the diagram of schemes that underlies the associated Mayer-Vietoris triangle.

3.1.3

We can now deduce the existence of the required isomorphism between the complexes N_0 and S_0.

Theorem 3.3. *The complexes N_0 and S_0 are isomorphic.*

Proof. After taking account of the results of Theorems 3.1 and 3.2 this is reduced to proving all of the following claims:

- $S_{0,\mathcal{O}_v^\#,\Gamma_{v,p^\infty}}$ is acyclic;
- $S_{0,\mathcal{O}_v^\#,G_{v,p^\infty}}$ is naturally isomorphic to the complex $\ker(\mathcal{A}(\mathcal{O}_v) \to \mathcal{A}(k(v)))[1]$;
- If H is a p-divisible group on a scheme X having finite p-bases, then $S_{0,X,H}$ is naturally isomorphic to $R\varprojlim_k R\Gamma(X, H_{p^k})$.

The first two claims here are verified by means of an explicit computation of the syntomic complexes. The final claim is proved by applying the syntomic topology (as in [10] and [4]) to each of the p-divisible groups $G_{v,p^\infty}/\mathcal{O}_v$, $\overline{G}_{v,p^\infty}/k(v)$, $A_{v,p^\infty}/K_v$ and $\mathcal{A}_{U,p^\infty}/U$. □

Remark 3.4. There is a rather delicate technical difficulty that we have, for simplicity, ignored in the above reasoning: the mapping cone construction is not functorial in the derived category. To overcome this difficulty the authors of [20] are forced to work with an appropriate derived category of diagrams.

3.2 Extending to the complexes S_∞ and N_∞.

It is shown in [20] that the constructions described above are functorial both in the category of semistable abelian varieties as well as with respect to étale base change. This fact then allows the authors to construct objects $\mathcal{N}$ and $\mathcal{S}$ in $D^+(C_{\text{ét}})$ that are respectively associated to the cohomology theory of N and S.

For example, the complex $\mathcal{N}$ is defined to be $R\varprojlim_k R\epsilon_*^Z \mathcal{A}_{p^k}$ where the functor ϵ^Z : $C_{\text{FL}} \to C_{\text{ét}}$ is defined as $\epsilon^Z(F) := \ker(\epsilon_* F \to \epsilon_* i_* i^* F)$ with i the closed immersion $Z \subset C$ and ϵ is the natural morphism $C_{\text{FL}} \to C_{\text{ét}}$ of change of topologies.

Having constructed $\mathcal{N}$ and $\mathcal{S}$ in this way, the authors of loc. cit. are then able to prove the following result.

Theorem 3.5. *There exists an isomorphism in $D^+(C_{\text{ét}})$ of the form*

$$\mathcal{N} \simeq \mathcal{S}$$

which induces, upon applying the global section functor over C, the isomorphism

$$N_0 \simeq S_0$$

that is constructed in Theorem 3.3.

Deducing the existence of an isomorphism of complexes $N_\infty \cong S_\infty$ from the above isomorphism $\mathcal{N} \simeq \mathcal{S}$ is now a rather formal process involving the theory of normic systems.

Recall that a 'normic system' for the group G is a collection (M_n), with each M_n a $\mathbb{Z}_p[G_n]$-module, together with transition maps $M_n \to M_{n+1}$ and $M_{n+1} \to M_n$ that satisfy certain natural compatibilities. In particular, one can define a 'normic section functor' from the derived category of étale sheaves of $\mathbb{Z}_p$-modules on C to the category of normic systems for G by associating to each abelian sheaf F of $\mathbb{Z}_p$-modules on the small étale site of C the collection $(F(C_n))$ together with its natural restriction and corestriction maps (in this regard note that C_n is an étale C-scheme because the extension K_n/K is unramified and that the action of G_n on C_n endows $F(C_n)$ with a natural structure as $\mathbb{Z}_p[G_n]$-module).

For the present purposes it is actually sufficient to consider the projective system of $\Lambda(G)$-modules that underlies this normic system and in this way one constructs a well-defined exact functor

$$D^+(C_{\text{ét}}) \to D^+(\mathcal{C}^{\mathbb{N}}_{\Lambda(G)}) \to D^+(\Lambda(G))$$

where $D^+(\mathcal{C}^{\mathbb{N}}_{\Lambda(G)})$ denotes the derived category of projective systems of $\Lambda(G)$-modules and the second arrow is the natural 'passage to inverse limit' functor.

The required isomorphism $N_\infty \cong S_\infty$ is then obtained by applying this functor to the isomorphism $\mathcal{N} \simeq \mathcal{S}$ in Theorem 3.5.

3.3 The complex N_{ar}

We now write k_∞ for the $\mathbb{Z}_p$-extension $\bigcup_{n\geq 0} k_n$ of $\mathbb{F}_q$ (and note that $K_{\text{ar}} = Kk_\infty$).

Proposition 3.6. (i) *For any complex $Y \in \{I, P, L\}$, there is a canonical isomorphism in $D(\Lambda(\Gamma))$ of the form $W(k_\infty) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} Y_0 \simeq Y_{\text{ar}}$.*

(ii) *The induced morphism $\mathbf{1} - \varphi : I_{\text{ar}} \to P_{\text{ar}}$ is $W(k_\infty) - \sigma$-linear.*

Proof. Claim (i) results from the base change theorem for log crystalline cohomology and claim (ii) is straightforward to verify. □

We recall that $Q(\Gamma)$ denotes the total quotient ring of $\Lambda(\Gamma)$.

Corollary 3.7. *The complex $Q(\Gamma) \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} N_{\text{ar}}$ is acyclic.*

Proof. The result of Proposition 3.6(i) combines with the exact triangle (8) to imply that it is enough to prove that the morphism $Q(\Gamma) \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} I_{\text{ar}} \to Q(\Gamma) \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} P_{\text{ar}}$ in $D^{\text{p}}(Q(\Gamma))$ that is induced by $\mathbf{1} - \varphi$ is an isomorphism. Further, since $\text{Lie}(D)$ is a finite dimensional $\mathbb{F}_p$-vector space, $\mathbf{1}$ gives an isomorphism $\mathbb{Q}_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p} I_{\text{ar}} \to \mathbb{Q}_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p} P_{\text{ar}}$. The required result can thus be proved by using the fact that crystalline cohomology over a proper log-smooth base is finite dimensional and applying the following easy fact from σ-linear algebra: if ψ is a linear endomorphism of a finite dimensional $\mathbb{Q}_p$-vector space M then any map of the form $\text{id} - \sigma \otimes \psi$ on $W(k_\infty) \otimes_{\mathbb{Z}_p} M$ is surjective and has kernel a finite dimensional $\mathbb{Q}_p$-vector space. □

The connection between N_{ar} and the arithmetic invariants of the abelian variety A is described in the next result.

Proposition 3.8. (i) *The complex N_{ar} is concentrated in degrees 1, 2 and 3. There is also a canonical exact sequence*

$$0 \to H^1(N_{\mathrm{ar}}) \to \varprojlim_{k,n} \mathrm{Sel}_{p^k}(A/Kk_n) \to \varprojlim_{k,n} \prod_{v\in Z} \mathcal{A}(k(v)k_n)/p^k \to 0$$

and canonical isomorphisms $H^2(N_{\mathrm{ar}}) \simeq X_p(A^t/K_{\mathrm{ar}})$ and $H^3(N_{\mathrm{ar}}) \simeq A^t_{p^\infty}(K_{\mathrm{ar}})$.

(ii) *The following conditions are equivalent:*

(*a*) *The complex N_{ar} belongs to $D^p(\mathbb{Z}_p)$.*

(*b*) *The module $X_p(A^t/K_{\mathrm{ar}})$ is finitely generated over $\mathbb{Z}_p$.*

(*c*) *The module $X_p(A/K_{\mathrm{ar}})$ is finitely generated over $\mathbb{Z}_p$.*

Proof. Claim (i) is proved by an explicit computation using the definition of N_{ar}. Claim (ii) then follows easily from the descriptions in claim (i) and the natural isogeny between A and A^t. □

3.4 The complex N_∞

Proposition 3.9. (i) *The descriptions of Proposition 3.8(i) remain valid if one replaces the complex N_{ar} by N_∞.*

(ii) *Under the hypothesis $\mu_A \sim 0$ the complex N_∞ belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(G))$.*

Proof. For each natural number m we choose a finite extension K'_m of K inside K_∞ such that $K_\infty = \bigcup_{m,i} K'_m k_i$. Then, since each of the modules $H^q(N_{K'_m k_i})$ is compact, the projective system $H^q(N_{K'_m k_i})$ is $\varprojlim$-acyclic and so there are natural isomorphisms

$$H^q(N_\infty) \cong \varprojlim_{m,i} H^q(N_{K'_m k_i}) \cong \varprojlim_{m} H^q(N_{K'_m k_\infty}).$$

The descriptions in claim (i) can therefore be obtained by passing to the inverse limit over m of the descriptions in Proposition 3.8(i) with N_{ar} replaced by $N_{K'_m k_\infty} = N_{K'_{m,\mathrm{ar}}}$

Next we combine the hypothesis $\mu_A \sim 0$ with the result of Proposition 3.8(ii) to deduce that the complex $\Lambda(\Gamma) \otimes^{\mathbb{L}}_{\Lambda(G)} N_\infty \simeq N_{\mathrm{ar}}$ belongs to $D^p(\mathbb{Z}_p)$.

Claim (ii) therefore follows from the easy algebraic fact that any complex M in $D^{\mathrm{p}}(\Lambda(G))$ belongs to $D^{\mathrm{p}}_S(\Lambda(G))$, and hence also to $D^{\mathrm{p}}_{S^*}(\Lambda(G))$, if the complex $\Lambda(\Gamma) \otimes^{\mathbb{L}}_{\Lambda(G)} M$ belongs to $D^p(\mathbb{Z}_p)$. □

Remark 3.10. Propositions 3.8(ii) and 3.9(ii) combine to imply that the module $X_p(A/K_\infty)$ is $\Lambda(G)$-torsion. We are aware of two situations in which this observation has been either strengthened or generalised.

(i) Let L be any $\mathbb{Z}_p$-power extension of K that is unramified outside a finite set Σ of places of K at each of which A has ordinary reduction, and set $G := \mathrm{Gal}(L/K)$. Then in this case Tan [19] has proved the following strengthening of Proposition 3.9(ii):

- $X_p(A/L)$ is finitely generated as a $\Lambda(G)$-module if and only if the group $H^1(G_v, A(L))$ is cofinitely generated as a $\mathbb{Z}_p$-module for all v in Σ, where G_v denotes the decomposition group of G at v.

- If $X_p(A/L)$ is finitely generated as a $\Lambda(G)$-module and $K_{\rm ar} \subseteq L$, then $X_p(A/L)$ is a torsion $\Lambda(G)$-module.

- If $X_p(A/L)$ is a torsion $\Lambda(G)$-module, then there exists a finite set T of proper intermediate $\mathbb{Z}_p$-power extensions of L/K with the following property: for each $\mathbb{Z}_p$-power extension M of K inside L the $\Lambda(\mathrm{Gal}(M/K))$-module $X_p(A/M)$ is torsion unless M is a subfield of some field in T.

(ii) Let now L be any compact p-adic Lie extension of K which contains $K_{\rm ar}$, is unramified outside a finite set of places Σ and is such that $G := \mathrm{Gal}(L/K)$ contains no element of order p. Then in [15] Ochiai and the author prove that $X_p(A/L)$ is a finitely generated $\Lambda(\mathrm{Gal}(L/K_{\rm ar}))$-module for any abelian variety A over K which satisfies both of the following conditions:

- A has good reduction at all places outside Σ and ordinary reduction at all places in Σ;

- The (classical) μ-invariant of the module $X_p(A/K_{\rm ar})$ vanishes.

Note also that this latter μ-invariant always vanishes if A is constant ordinary.

Finally we note that the recent preprint [3] of Bandini and Valentino uses a natural generalisation of Mazur's Control Theorem to prove similar results concerning the structure of the module $X_p(A/L)$.

Remark 3.11. If one does not assume that L contains $K_{\rm ar}$, then the $\Lambda(G)$-module $X_p(A/L)$ need not be torsion. For an explicit example (taken from the Appendix of [14]), fix a quadratic extension K/k and a non-isotrivial semistable elliptic curve A over k that has analytic rank zero and split multiplicative reduction at a given place v_0. Then $X_p(A/L)$ fails to be a torsion $\Lambda(\mathrm{Gal}(L/K))$-module whenever L is a $\mathbb{Z}_p$-extension of K that is dihedral over k, totally ramified above v_0 and unramified at all other places.

3.5 The Main Conjecture for A over K_∞

We first record an easy consequence of Proposition 3.9.

Corollary 3.12. *Under the hypothesis that $\mu_A \sim 0$, the morphisms in $D^p(\Lambda(G)_{S^*})$*

$$\mathbf{1}_{S^*} : (I_\infty)_{S^*} \to (P_\infty)_{S^*} \quad \textit{and} \quad (\mathbf{1}-\varphi)_{S^*} : (I_\infty)_{S^*} \to (P_\infty)_{S^*}$$

that are induced by the respective scalar extensions of $\mathbf{1}$ and $\mathbf{1}-\varphi$ are isomorphisms.

Proof. For the morphism $\mathbf{1}_{S^*}$ this claim follows directly from the scalar extension of the triangle (7) and the fact that L_∞ is annihilated by p. For the morphism $(\mathbf{1}-\varphi)_{S^*}$, the claim is a direct consequence of Proposition 3.9(ii) and scalar extension of the triangle (8). □

This result leads naturally to the definition of the "p-adic L-function" which occurs in the statement of Theorem 2.1.

Definition 3.13. *Under the hypothesis $\mu_A \sim 0$, the p-adic L-function $\mathcal{L}_{A/K_\infty}$ for A over K_∞ is the element of $K_1(\Lambda(G)_{S^*})$ that is defined by setting*

$$\mathcal{L}_{A/K_\infty} := \mathrm{Det}_{\Lambda(G)_{S^*}}((\mathbf{1}-\varphi)_{S^*} \circ (\mathbf{1}_{S^*})^{-1}).$$

Given this definition of the p-adic L-function, the interpolation property stated in Theorem 2.1(i) is proved by comparing log-crystalline cohomology to rigid cohomology and then using the base change theorems and Künneth formula in rigid cohomology together with a description of the Hasse-Weil L function of A twisted by an Artin representation in terms of rigid cohomology. The additional hypothesis on the Artin representation (it should have coefficients in a totally ramified extension of $\mathbb{Q}_p$) is necessary to identify the representation with a unit F-crystal on the curve hence allowing to get a p-adic expression of the twisted Hasse-Weil L-function.

In view of Corollary 3.12, and the explicit definition of $\mathcal{L}_{A/K_\infty}$ given above, the equality of Theorem 2.1(ii) is then obtained by applying the following (straightforward) algebraic observation to the exact triangles of (7) and (8) (so that one has $R = \Lambda(G)$ and $\Sigma = S^*$).

Lemma 3.14. *Let R be an associative unital left noetherian ring and Σ a left Ore set of non-zero divisors of R. Let*

$$C \xrightarrow{a} C' \to C(a) \to C[1]$$

and

$$C \xrightarrow{b} C' \to C(b) \to C[1]$$

be exact triangles in $D^{\mathrm{p}}(R)$ which have the property that the complexes $R_\Sigma \otimes_R C(a)$ and $R_\Sigma \otimes_R C(b)$ are acyclic.

Then in $K_0(R, R_\Sigma)$ one has an equality

$$\partial_{R,R_\Sigma}(\mathrm{Det}_{R_\Sigma}(a_\Sigma \circ b_\Sigma^{-1})) = \chi^{\mathrm{ref}}_{R,R_\Sigma}(C(a)) - \chi^{\mathrm{ref}}_{R,R_\Sigma}(C(b))$$

with $a_\Sigma := R_\Sigma \otimes_R a$ and $b_\Sigma := R_\Sigma \otimes_R b$.

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