# Kernels of twisted symmetric square of elliptic modular forms

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I would like to talk about exact evaluation of special $L$-values of elliptic modular forms. To explain my approach, I discuss on the simplest case of twisted symmetric square.

**Notation.** We use some standard noation. $H_1 = \{\tau = u+iv; v > 0\}$ is the upper half-plane. The action of $SL_2(\mathbf{R})$ on $H_1$ is denoted by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = \frac{a\tau+b}{c\tau+d}$. The branch of $z^\alpha$ is taken so that $-\pi < \arg z \leq \pi$. The Petersson inner product is

$$< f, g >_N = \int_{D_N} f(\tau)\overline{g(\tau)} v^k d\mu, \quad d\mu = \frac{dvdu}{v^2},$$

where $D_N = \Gamma_0(N) \setminus H_1$ is a fundamental domain. Let us use this notation for functions $f$ and $g$, whenever it is well-defined and has a finite value. $\Gamma_1 = SL_2(\mathbf{Z})$ is the full modular group. $M_k(N)$, $S_k(N)$ are the space of modular forms and cusp forms of weight $k$ on $\Gamma_0(N)$. As usual, $e(x) = e^{2\pi i x}$ and $j(\gamma, \tau) = c\tau+d$ is the automorphic factor for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{R})$.

## 1 Twisted symmetric square

For an even integer $k \geq 4$, let

$$f(\tau) = \sum_{n=1}^{\infty} b(n)e(n\tau) \in S_k(1)$$

be a Hecke eigen cusp form of weight $k$ and level 1. To such an $f$ and a Dirichlet character $\chi$, we associate the twisted symmetric square $L$-function defined by

$$L_2(s, f, \chi) := L(2s, \chi^2) \sum_{n=1}^{\infty} \frac{\chi(n)b(n^2)}{n^{s+k-1}} \quad (\Re(s) = \sigma > 1).$$

By Shimura, it can be continued meromorphically to all $s$. The holomorphy is also investigated by him.

**Shimura (1975).** $L_2(s, f, \chi)$ has a meromorphic continuation to $\mathbf{C}$. More precisely,

$$s(s-1)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(s+1-\eta\right) L_2(s, f, \chi)$$

is holomorphic on $s \in \mathbf{C}$, where $\eta \in \{0, 1\}$ such that $\chi(-1) = (-1)^\eta$.

Sturm got the following algebraicity result on special values.

**Sturm (1980, 1989).** Assume $\chi$ is even and primitive. For any odd $l$ $(1 \leq l \leq k-1)$

$$\frac{\pi}{3} \cdot \frac{L_2(l, f, \chi)}{\pi^{k+2l} < f, f >_1} \in \overline{\mathbf{Q}}.$$

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*Outline of proof.* Sturm uses the following integral representation due to Shimura:

$$L_2(s,f,\chi) =< \theta_\chi H(\cdot,s), f >_{4r^2},$$

where

$r$ : the conductor of $\chi$,
$\theta_\chi(\tau)$ : theta series with character,
$H(\tau,s)$ : a real analytic Eisenstein series of half-integral weight,
$\theta_\chi(\tau)H(\tau,s)$ is a real analytic modular form of weight $k$ on $\Gamma(4r^2)$.

Analytic properties of $H(\tau,s)$ implies those of $L_2(s,f,\chi)$. Algebraicity result follows from specializing $s=l$:
$\Longrightarrow \theta_\chi(\tau)H(\tau,l)$ is a nearly holomorphic modular form;
$\Longrightarrow L_2(l,f,\chi) =< \theta_\chi H(\cdot,l), f >_{4r^2} \in \pi^{k+2l-1} < f,f >_1 \overline{\mathbf{Q}}$ by Shimura. $\square$

### Obstacle to get exact values

In order to compute exact values $L_2(l,f,\chi)$ at $s=l$, we need to analyze $M_k(4r^2)$ in order to describe the nearly holomorphic modular form in termes of basis of level $4r^2$. But

$$\dim M_k(4r^2) \gg \dim S_k(1).$$

It is desirable to work out needed computations inside the space of lower level, for example $S_k(1)$.

## 2 Known methods

We shall introduce some known methods to compute the exact $L$-values. Let us start from the simplest case, namely the case without twisting by characters. Suppose that $f \in S_k(1)$ is a Hecke eigen cusp form of weight $k \geq 4$ and level 1, and $\chi_0$ is the principal character mod 1.

**Zagier (1976, $\chi = \chi_0$).** Using Hecke eigen basis $\{f_j(\tau)\}_{1\leq j\leq \dim S_k(1)} \subset S_k(1)$, Zagier constructed a holomorphic kernel function of the symmetric square $L$-function by

$$\Phi_s(\tau) := \sum_{j=1}^{\dim S_k(1)} \frac{L_2(s,f_j,\chi_0)}{< f_j, f_j >_1} f_j(\tau) \in S_k(1).$$

Therefore $< \Phi_s, f_j >_1 = L_2(s,f_j,\chi_0)$.

He expresses the $n$-th Fourier coefficient $c_n(s)$ of $\Phi_s$ as

$$c_n(s) \risingdotseq \int_{D_1} \omega_n(\tau,-\overline{\tau})E(\tau,s)v^k d\mu \quad (s\in\mathbf{C}),$$

$$\omega_n(\tau,\tau') = \sum_{\substack{a,b,c,d\in\mathbf{Z}\\ ad-bc=n}} \frac{1}{(c\tau\tau'+d\tau'+a\tau+b)^k} \quad (\tau,\tau'\in H_1),$$

where $E(\tau,s)$ is the real analytic Eisenstein series on $\Gamma_1$.

In fact, by Petersson, we have $< f, \omega_n(*, -\overline{\tau'}) >_1 \doteqdot f|T(n)(\tau')$. Therefore

$$\omega_n(\tau, \tau') \doteqdot \sum_{j=1}^{\dim S_k(1)} \frac{b_j(n)}{< f_j, f_j >_1} f_j(\tau) f_j(\tau').$$

Since $L_2(s, f_j, \chi_0) \doteqdot \sum_{n=1}^{\infty} \chi_0(n) b_j(n)^2 n^{-(s+k-1)}$ for Hecke eigenforms $f_j$, the well-known integral representation of the Rankin convolution $\sum_{n=1}^{\infty} \chi_0(n) b_j(n)^2 n^{-s}$ implies the desired formula.

Then, Zagier calculates the integral explicitly. An explicit form of $c_n(s)$ is

$$c_n(s) = (-1)^{\frac{k}{2}} n^{\frac{k-1}{2}} \pi \sum_{r \in \mathbf{Z}} I(r, n, s) \frac{L_{r^2-4n}(s)}{\zeta(2s)} + \delta(n = m^2)(\cdots),$$

where

$I(r, n, s)$ : a certain special function,

$L_D(s)$ : a quadratic $L$-function,

$$\delta(n = m^2) := \begin{cases} 1 & \text{if } n \text{ is square of a natural number } m, \\ 0 & \text{otherwise.} \end{cases}$$

For the precise definition of $L_D(s)$ and $I(r, n, s)$, see section 3 of this report. The infinite sum is absolutely convergent, if $2 - k < \Re(s) < k - 1$. As a result, he computed the special $L$-values by the following manner.

Specialize $s = l$ with $1 \leq l$ : odd $< k - 1$:

$\Longrightarrow$ the infinite sum turns out to be a finite sum;

$\Longrightarrow$ evaluate $c_n(l)$ exactly as a numerical value;

$\Longrightarrow$ an explicit description of $\Phi_l(\tau)$ in terms of Hecke basis;

$\Longrightarrow$ exact evaluation of $L_2(l, f_j, \chi_0)$ (just a coefficient in the Hecke basis expression).

There are other applications of Zagier's kernel function:

(1) Trace formula of the Hecke operators (Zagier);

(2) reconstruction of Cohen's modular forms (Zagier);

(3) estimation of Ramanujan's $\tau(n)$ (Hashim-Murty);

(4) non-vanishing of symmetric square on average (Kohnen-Sengupta);

(5) estimation of central value of symmetric square on average (Kohnen-Sengupta).

**Mizumoto (1985, $\chi = \chi_0$).** Following a suggestion given in Zagier's paper, Mizumoto constructed the kernel function by using infinite sum of the Poincare series. Here the Poincare series is defined for $n \geq 1$ by

$$P_n(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1} \frac{e(n \cdot \gamma\tau)}{j(\gamma, \tau)^k} \in S_k(1).$$

Recall that the $n$-th Poincare series is a kernel function of the $n$-th Fourier coefficient, $< f, P_n >_1 \doteqdot b(n)$. He considers the function

$$\Psi_s(\tau) := \sum_{n \geq 1} n^{k-1-s} P_{n^2}(\tau) \in S_k(1) \quad (\sigma > 1).$$

Therefore $< f, \Psi_s >_1 \doteqdot \sum_{n\geq 1} b(n^2)n^{-(s+k-1)}$.

Using the Fourier expansion of the Poincare series combined with the Poisson summation formula, he calculates the Fourier coefficients to get a formula similar to Zagier's. Mizumoto also worked out his calculation in the case of Hilbert modular forms.

So far, we discuss only the case without twisting. An actual twisted case is discussed by Stopple.

**Stopple (1996, $\chi = \chi_p$).** Here $\chi_p = \left(\frac{p}{\cdot}\right)$ is Kronecker's symbol and a prime $p \equiv 1 \pmod 4$. Stopple follows Mizumoto's method in order to construct a kernel function of the symmetric square twisted by $\chi_p$,

$$\Psi_s^{\chi_p}(\tau) := \sum_{n\geq 1} \chi_p(n) n^{k-1-s} P_{n^2}(\tau) \in S_k(1).$$

Therefore $< f, \Psi_s^{\chi_p} >_1 \doteqdot \sum_{n\geq 1} \chi_p(n) b(n^2) n^{-(s+k-1)}$.

Compared with Sturm, a remarkable point is that the kernel function is a modular form of level one. In fact, Stopple determined some special values of the twisted symmetric square exactly. See section 3 of this report.

**Goto (1998), Hiraoka (2000).** They used Hida's identity. They gave three exact values of the twisted symmetric square and some values of other type of $L$-functions.

**Katsurada (2005).** He applies the pullback of degree 2 level $p^2$ involuted Siegel-Eisenstein series $E_k^{(2)}|_W(Z,s)$ due to Böcherer-Schmidt combined with Ibukiyama's differential operator. He computed many exact values of the twisted symmetric square.

**Panchishkin (1979).** He used the trace operator, but not clear for me whether the ready to compute formula is obtained or not.

# 3 Our approach

Now my construction of the kernel function of twisted symmetric square is as follows. For simplicity, assume that a prime $p$ is congruent to 1 modulo 4, $\chi$ is even primitive Dirichlet character modulo $p$. Let $f \in S_k(1)$ be a Hecke eigen cusp form of weight $k \geq 4$ and level 1.

**Jacobi Eisenstein series** (see Guerzhoy). $2\sigma + k > 3$, $(\tau, z) \in H_1 \times \mathbf{C}$,

$$E^{\chi}_{k,p^2}(\tau, z, s) = \frac{v^s}{2} \sum_{\substack{c,d\in\mathbf{Z}\\(c,d)=1}} \sum_{\lambda\in\mathbf{Z}} \chi(\lambda) \frac{e(\lambda^2 \frac{a\tau+b}{c\tau+d} + 2p\lambda\frac{z}{c\tau+d} - p^2\frac{cz^2}{c\tau+d})}{(c\tau+d)^k |c\tau+d|^{2s}}.$$

**Restriction** $z = 0$. Note the followings (see Heim):

$$E^{\chi}_{k,p^2}(\tau, 0, s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_1} \frac{\theta_\chi(\gamma\tau)}{j(\gamma,\tau)^k} \Im(\gamma\tau)^s, \qquad \theta_\chi(\tau) = \sum_{\lambda\in\mathbf{Z}} \chi(\lambda) e(\lambda^2\tau),$$

$$< E^{\chi}_{k,p^2}(\cdot, 0, s), f > = \int_0^\infty \int_0^1 \theta_\chi(\tau)\overline{f(\tau)} v^{s+k-2} du dv = \frac{2\Gamma(s+k-1)}{(4\pi)^{s+k-1}} \sum_{\lambda=1}^{\infty} \frac{\chi(\lambda)\overline{b(\lambda^2)}}{\lambda^{2s+2k-2}}.$$

Hence, if we put

$$E_s^J(\tau) := \frac{\Gamma(k-1)(4\pi)^{\frac{s-k+1}{2}}}{2\Gamma(\frac{s+k-1}{2})} E^{\chi}_{k,p^2}\left(\tau, 0, \frac{s-k+1}{2}\right),$$

then

$$E_s^J(\gamma\tau) = (c\tau+d)^k E_s^J(\tau) \quad (\forall\gamma\in\Gamma_1),$$
$$< E_s^J, f >_1 = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \cdot \frac{L_2(s,f,\chi)}{L(2s,\chi^2)},$$

since $b(n)$ are real. In other words, $E_s^J$ is a <u>non-holomorphic</u> kernel of the twisted symmetric square.

**Fourier expansion** (One of our main results. See also Eichler-Zagier, Guerzhoy)**.**

$$E_{k,p^2}^{\chi}(\tau,z,s) = v^s \sum_{\lambda\in\mathbf{Z}} \chi(\lambda) e(\lambda^2\tau + 2p\lambda z)$$
$$+ \frac{v^s}{(2i)^{1/2}p} \sum_{n,r\in\mathbf{Z}} e_\chi(n,r,s)\tau_{n-\frac{r^2}{4p^2}}\left(v, s+k-\frac{1}{2}, s\right) e^{-2\pi\frac{r^2}{4p^2}v} e(nu+rz).$$

$\underline{\tau_l(v,\alpha,\beta)} = \int_{-\infty}^{\infty} e^{-2\pi i l u}\tau^{-\alpha}\overline{\tau}^{-\beta}du$ is the analytic part:

$$\tau_l(v,\alpha,\beta) = i^{\beta-\alpha}\cdot\begin{cases} \dfrac{\pi^{\frac{\alpha+\beta}{2}} l^{\frac{\alpha+\beta}{2}-1}}{\Gamma(\alpha)} v^{-\frac{\alpha+\beta}{2}} W_{\frac{\alpha-\beta}{2},\frac{\alpha+\beta-1}{2}}(4\pi l v) & \text{if } l>0,\\ \dfrac{\pi^{\frac{\alpha+\beta}{2}} |l|^{\frac{\alpha+\beta}{2}-1}}{\Gamma(\beta)} v^{-\frac{\alpha+\beta}{2}} W_{\frac{\beta-\alpha}{2},\frac{\beta+\alpha-1}{2}}(4\pi |l| v) & \text{if } l<0,\\ 2^{2-\alpha-\beta}\pi\dfrac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)} v^{1-\alpha-\beta} & \text{if } l=0.\end{cases}$$

Here

$$W_{\kappa,\mu}(v) := \frac{v^{\mu+\frac{1}{2}}e^{-\frac{v}{2}}}{\Gamma(\mu-\kappa+\frac{1}{2})}\int_0^{\infty} e^{-vt}t^{\mu-\kappa-\frac{1}{2}}(1+t)^{\mu+\kappa-\frac{1}{2}}dt$$

is the Whittaker function, initially defined for $\Re\left(\mu-\kappa+\frac{1}{2}\right)>0$ and its holomorphic continuation to all $(\kappa,\mu)\in\mathbf{C}^2$. Note that $W_{\kappa,\mu}(v) = v^{\mu+\frac{1}{2}}e^{-\frac{v}{2}}\Psi\left(\mu-\kappa+\frac{1}{2}, 2\mu+1; v\right)$ using the notation of Lebedev.

$\underline{e_\chi(n,r,s)} = B(n,r,s)\cdot C(n,r,s)$ is the arithmetic part. Explicit forms are as follows:

$$\underline{B(n,r,s)} = \frac{L_{r^2-4p^2n}(2s+k-1,\chi)}{L(2(2s+k-1),\chi^2)}.$$

Here

$$L_D(s,\chi) := \begin{cases} L(2s-1,\chi^2) & \text{if } D=0,\\ L(s,\chi_K\chi)\displaystyle\sum_{a|f}\mu(a)\chi_K(a)\chi(a)a^{-s}\sigma_{1-2s,\chi^2}(f/a) & \text{if } D\neq 0,\ D\equiv 0,1 \pmod 4,\\ 0 & \text{if } D\equiv 2,3 \pmod 4,\end{cases}$$

where

$$\begin{aligned} &K=\mathbf{Q}(\sqrt{D}),\\ &d_K : \text{discriminant of } K,\\ &f\in\mathbf{N} : D=d_K f^2,\\ &\chi_K : \text{the Kronecker symbol},\\ &\mu : \text{the Möbius function},\\ &\sigma_{s,\chi}(n) = \sum_{d|n}\chi(d)d^s.\end{aligned}$$

Note here that Zagier's $L_D(s)$ is $L_D(s,\chi_0)$ in our notation:

$$\begin{aligned}
\underline{C(n,r,s)} &= W(\chi)\cdot\begin{cases}\overline{\chi}(r) & \text{if } p\nmid r,\\ \displaystyle\sum_{l=1}^{\infty}\frac{\tilde{H}_{n,r}(l)}{p^{(2s+k-1)l}} & \text{if } p\mid r,\end{cases}\\
W(\chi) &= \sum_{\lambda=1}^{p}\chi(\lambda)e\left(\frac{\lambda}{p}\right),\\
\tilde{H}_{n,r}(l) &= \sum_{\lambda(p^l),Q_p(\lambda)\equiv 0\,(p^{l-1})}\chi(\lambda)\overline{\chi}\left(\frac{Q_p(\lambda)}{p^{l-1}}\right),\\
Q_p(\lambda) &= \lambda^2+\frac{r}{p}\lambda+n.
\end{aligned}$$

In the case $p|r=p\hat{r}$, we put

$$\underline{F_{\hat{r},n}(T)} := \sum_{l=1}^{\infty}\tilde{H}_{n,r}(l)T^l.$$

**an explicit form of $F_{\hat{r},n}(T)$.** Put

$$\alpha=\tilde{H}_{n,r}(1)=\sum_{\lambda(p)}\chi(\lambda)\overline{\chi}\left(Q_p(\lambda)\right), \qquad \beta=\chi(-2\hat{r})\sum_{\lambda(p)}\overline{\chi}\left(\lambda^2\right),$$

$$\gamma=\chi(-2\hat{r})\sum_{\lambda(p)}\overline{\chi}\left(\lambda^2-D_Kf_0^2\right), \qquad \delta=\chi(-2\hat{r})\overline{\chi}\left(-\frac{D_K}{p}f_0^2\right),$$

where $D_K$ and $f_0$ will be introduced in the following.

$\underline{\hat{r}^2-4n\neq 0}$

$\triangleright$ If $p\nmid \hat{r}^2-4n \implies F_{\hat{r},n}(T)=\alpha T$.

$\triangleright$ If $p\mid \hat{r}^2-4n$, denote $\hat{r}^2-4n=D_Kf^2$, $p^m||f=p^mf_0$.

$$\begin{aligned}
\text{If } p\nmid D_K \implies F_{\hat{r},n}(T) &= \alpha T+\beta\sum_{e=1}^{m-1}p^eT^{2e+1}+\gamma p^mT^{2m+1}\\
&= \frac{T}{1-pT^2}\{\alpha-(\alpha-\beta)pT^2+p^mT^{2m}(\gamma-\beta-p\gamma T^2).\\
\text{If } p\mid D_K \implies F_{\hat{r},n}(T) &= \alpha T+\beta\sum_{e=1}^{m}p^eT^{2e+1}+\delta p^{m+1}T^{2m+2}\\
&= \frac{T}{1-pT^2}\{\alpha-(\alpha-\beta)pT^2-p^{m+1}T^{2m+1}(\delta pT^2+\beta T-\delta).
\end{aligned}$$

$$\underline{\hat{r}^2-4n=0} \implies F_{\hat{r},n}(T)=\alpha T+\beta\sum_{e=1}^{\infty}p^eT^{2e+1}=\frac{T}{1-pT^2}\{\alpha-(\alpha-\beta)pT^2\}.$$

*Remark.* If $\chi=\chi_p$, we can compute the character sums in more explicit forms (see Small). In fact, if $p|\hat{r}^2-4n$, then $\alpha=-\chi_p(-2\hat{r})$, $\beta=\chi_p(-2\hat{r})(p-1)=-\alpha(p-1)$, $\gamma=-\chi_p(-2\hat{r})=\alpha$, $\delta=-\alpha\chi_p\left(-\frac{D_K}{p}\right)$.

**Fourier expansion of $E^J_s$.** Put

$$E^J_s(\tau) = \frac{\Gamma(k-1)(4\pi)^{\frac{s-k+1}{2}}}{2\Gamma(\frac{s+k-1}{2})} E^{\chi}_{k,p^2}\left(\tau, 0, \frac{s-k+1}{2}\right) =: \sum_{n \in \mathbf{Z}} c_n(v) e(nu).$$

Then

$$c_n(v) = \frac{\Gamma(k-1)(4\pi)^{\frac{s-k+1}{2}}}{\Gamma(\frac{s+k-1}{2})} \Big\{ \delta(n = m^2) v^{\frac{s-k+1}{2}} \chi(m) e(m^2 iv) \\ + \sum_{r \in \mathbf{Z}} \frac{1}{(2i)^{1/2} 4^s} \cdot \frac{A(n,r,s)}{p} v^{\frac{s-k+1}{2}} \tau_{4n - \frac{r^2}{p^2}} \left(\frac{v}{4}, \frac{s+k}{2}, \frac{s-k+1}{2}\right) e^{-2\pi \frac{r^2}{4p^2} v} \Big\}.$$

We defined

$$\underline{A(n,r,s)} := e_{\chi}\left(n, r, \frac{s-k+1}{2}\right) = \frac{L_{r^2 - 4p^2 n}(s, \chi)}{L(2s, \chi^2)} \times W(\chi) \cdot \begin{cases} \overline{\chi}(r) & \text{if } p \nmid r, \\ F_{\hat{r}, n}(p^{-s}) & \text{if } p \mid r. \end{cases}$$

**Holomorphic projection lemma** (Zagier, Sturm)**.** $\Phi : H_1 \to \mathbf{C}$ : a continuous function having the Fourier expansion $\Phi(\tau) = \sum_{n=-\infty}^{\infty} c_n(v) e(nu)$ for all $\tau = u + iv \in H_1$. Suppose

(1) $\exists k > 2$ (even) such that $\Phi(\gamma\tau) = (c\tau + d)^k \Phi(\tau)\ \forall \gamma \in \Gamma_1$,

(2) $\exists \epsilon > 0$ such that $\Phi(\tau) = O(v^{-\epsilon})$ (uniform w.r.t $u$) as $v \to \infty$.

Define

$$\phi(\tau) = \sum_{n=1}^{\infty} a_n e(n\tau), \\ a_n = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \int_0^{\infty} c_n(v) e^{-2\pi n v} v^{k-2} dv.$$

Then

(a) $\phi \in S_k(1)$,

(b) $< \phi, f >_1 = < \Phi, f >_1 \quad \forall f \in S_k(1)$.

**Notation.** Write $\pi_{\mathrm{hol}}(\Phi) = \phi$, and call it a holomorphic projection of $\Phi$.

*Remark.* (2) can be weakened as $\Phi(\tau) = c_0 + O(v^{-\epsilon})$ by subtracting Eisenstein series (Zagier).

**Holomorphic kernel.** Suppose $1/2 < \sigma < k-1, s \neq 1$. Then, there exists $\mathcal{E}^J_s(\tau) := \pi_{\mathrm{hol}}(E^J_s) \in S_k(1)$. By (b),

$$< \mathcal{E}^J_s, f >_1 = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \cdot \frac{L_2(s, f, \chi)}{L(2s, \chi^2)}.$$

By definition, the Fourier expansion of $\mathcal{E}^J_s$ is

$$\mathcal{E}^J_s(\tau) = \sum_{n=1}^{\infty} a(n,s) e(n\tau), \\ a(n,s) = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \int_0^{\infty} c_n(v) e^{-2\pi n v} v^{k-2} dv.$$

Explicitly, if $1/2 < \sigma < k-1, s \neq 1$,

$$a(n,s) = \delta(n=m^2)(\chi(m)m^{k-s-1} + G_{2pm}(s) + G_{-2pm}(s)) + \sum_{r\in\mathbf{Z}, r^2\neq 4p^2n} G_r(s),$$

where

$$\underline{G_r(s)} = (-1)^{\frac{k}{2}}\pi n^{\frac{k-1}{2}} p^{-s} I(r, np^2, s) A(n,r,s),$$

$$\underline{I(r,n,s)} = \begin{cases} n^{\frac{s-1}{2}} 2^{s-1}\pi^{s-1}\dfrac{\Gamma(\frac{k-s}{2})}{\Gamma(\frac{s+k}{2})} F\left(\dfrac{k-s}{2}, 1-\dfrac{s+k}{2}; \dfrac{1}{2}; \dfrac{r^2}{4n}\right) & \text{if } 4n > r^2, \\ n^{\frac{k-1}{2}} 2^s \pi^{s-1} r^{s-k} \dfrac{\Gamma(k-s)\cos(\pi\frac{s-k}{2})}{\Gamma(k)} F\left(\dfrac{k-s}{2}, \dfrac{k-s+1}{2}; k; \dfrac{4n}{r^2}\right) & \text{if } 4n < r^2, \\ n^{\frac{s-1}{2}} 2^{s-1}\pi^{s-\frac{1}{2}} \dfrac{\Gamma(\frac{k-s}{2})\Gamma(s-\frac{1}{2})}{\Gamma(\frac{s+k}{2})\Gamma(\frac{1-k+s}{2})\Gamma(\frac{k+s-1}{2})} & \text{if } 4n = r^2. \end{cases}$$

Here $F(a,b;c;z)$ is Gauss's hypergeometric function

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad (a)_0 := 1, \quad (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}.$$

The infinite sum multiplied by $s(1-s)$ is absolutely convergent, if $1/2 < \sigma < k-1$. Also our computations can be justified. We used the followings in order to deduce these facts:

(behaviour of $W_{\kappa,\mu}(v)$)

$$W_{\kappa,\mu}(v) \sim v^{\kappa} e^{-\frac{v}{2}} \text{ as } v \to \infty, \quad W_{\kappa,\mu}(v) = \begin{cases} O(v^{\frac{1}{2}-|\Re(\mu)|}) & \text{if } \mu \neq 0, \\ O(v^{\frac{1}{2}}|\log v|) & \text{if } \mu = 0, \end{cases} \text{ as } v \to 0;$$

(integral transform) $a > 0$, $\beta > 0$, $\Re(s) > |\Re(\mu)| - 1/2$,

$$\int_0^{\infty} e^{-av} W_{\kappa,\mu}(\beta v) v^{s-1} dv$$
$$= \frac{\beta^{\mu+\frac{1}{2}}\Gamma(\mu+s+\frac{1}{2})\Gamma(-\mu+s+\frac{1}{2})}{\Gamma(s-\kappa+1)(a+\frac{\beta}{2})^{\mu+s+\frac{1}{2}}} \cdot F\left(\mu+s+\frac{1}{2}, \mu-\kappa+\frac{1}{2}; s-\kappa+1; \frac{2a-\beta}{2a+\beta}\right);$$

(transformation formula)

$$F(a,b;c;z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right);$$

(a consequence of Rademacher's Phragmen-Lindelof theorem) One has

$$|L_\Delta(s)| \leq \left(\frac{|\Delta|}{2\pi}\right)^{\frac{3}{4}-\frac{\sigma}{2}} |1+s|^{\frac{3}{4}-\frac{\sigma}{2}} \zeta(3/2)^2 \zeta(2)$$

for all $\Delta \neq \square, s \in S(-1/2,\ 3/2) = \{s = \sigma + it : -1/2 \leq \sigma \leq 3/2\}$.

$$|s(1-s)L_\Delta(s)| \leq \left(\frac{|\Delta|}{6\pi}\right)^{\frac{3}{4}-\frac{\sigma}{2}} |1+s|^{\frac{5}{4}+\frac{\sigma}{2}} \zeta(3/2)^2 \zeta(2)$$

for all $\Delta = \square \neq 0, s \in S(-1/2,\ 3/2)$.

*Remark.* For all $\Delta \neq 0$ and $s \in \mathbf{C}$ with $\Re(s) = \sigma > 1$, one has $|L_\Delta(s)| \leq \zeta(\sigma)^2 \zeta(2\sigma+1)$.

*Remark.* Recall $L_\Delta(s) = L_\Delta(s, \chi_0)$. If $\chi$ is quadratic, the above type estimate is sufficient for our use. For non-quadratic $\chi$, we can prove similar estimations of $L_\Delta(s,\chi)$ using its functional equation and Rademacher's Phragmen-Lindelof theorem.

**Special $L$-values.** Let $1 \le s = l$ : odd $< k-1$. If $4n < r^2$, then $I(r,n,l) = 0$ in view of $\cos\left(\pi\frac{s-k}{2}\right)$. If $4n > r^2$, then $F\left(\frac{k-s}{2}, 1-\frac{s+k}{2}; \frac{1}{2}; x\right)$ is just a polynomial of $x$. In fact,

$$I(r,n,l) = (-1)^{\frac{k-l-1}{2}}(2\pi\sqrt{n})^{l-1}2^{2l-1}\frac{\Gamma(k-l)\Gamma(l)}{\Gamma(l+k-1)}\left(1-\frac{r^2}{4n}\right)^{l-\frac{1}{2}} C^l_{k-l-1}\left(\frac{r}{2\sqrt{n}}\right),$$

where

$$C^l_m(x) = \sum_{j=0}^{[m/2]} \frac{(-1)^j (l)_{m-j}}{j!(m-2j)!}(2x)^{m-2j}$$

is the Gegenbauer polynomial. It turned out that $s=1$ is a removable singularity. Following the manner similar to Zagier, we can compute some special $L$-values.

Specialize $s = l$ with $1 \le l$ : odd $< k-1$:
$\Longrightarrow$ the infinite sum turns out to be a finite sum;
$\Longrightarrow$ evaluate $a(n,l)$ exactly as a numerical value;
$\Longrightarrow$ an explicit description of $\mathcal{E}^J_l(\tau)$ in terms of Hecke basis in $S_k(1)$ (easier than $M_k(4p^2)$);
$\Longrightarrow$ exact evaluation of $L_2(l, f_j, \chi)$ (just a coefficient in the Hecke basis expression).

**Examples** $(p=5, k=12, n=1, \chi=\chi_5, 1 \le r : \text{odd} < 11)$**.**

$$\begin{aligned}
I(l,\nu,r) &= (-1)^{\frac{r+1}{2}}\pi^{r-1}2^{r-1}\nu^{-\frac{r}{2}}\frac{\Gamma(12-r)\Gamma(r)}{\Gamma(11+r)}(4\nu-l^2)^{r-\frac{1}{2}}C^r_{11-r}\left(\frac{l}{2\sqrt{\nu}}\right),\\
L_{-N}(r,\chi_0) &= (-1)^{\frac{r-1}{2}}\frac{2^{r-1}\pi^r}{\Gamma(r)N^{r-\frac{1}{2}}}H(r,N),\\
H(r,N) &= L(1-r,\chi_K)\sum_{a|f}\mu(a)\chi_K(a)\chi(a)a^{r-1}\sigma_{2r-1}(f/a) \quad (-N = D_K f^2),\\
b(n,s) &= \frac{p^2(k-1)}{4\pi^2}\zeta(2s)\frac{p^{2s}-1}{p^{2s}}\cdot a(n,s).
\end{aligned}$$

Here, I introduced $b(n,s)$ following Stopple, in order to compare my calculations with his result. Since, $S_{12}(1) = \mathbf{C}\Delta$, one has $\mathcal{E}^J_r = a(1,r)\Delta$ and

$$b(1,r) = \frac{5^2\cdot 11}{4\pi^2}\zeta(2r)\frac{5^{2r}-1}{5^{2r}}\cdot a(1,r) = \frac{5^2\cdot 11!}{4^{12}\pi^{13}}\cdot\frac{L_2(r,\Delta,\chi_5)}{<\Delta,\Delta>_1}.$$

On the other hand,

$$\begin{aligned}
b(1,r) &= 2d_r\sum_{l=1}^{9}\chi_5(l)C^r_{11-r}\left(\frac{l}{10}\right)H(r,5(100-l^2))\\
&\quad + 5^{r-1}d_r\left\{-2C^r_{11-r}(0)H(r,20) + 4C^r_{11-r}\left(\frac{1}{2}\right)H(r,15)\right\} + \frac{11}{4\pi^2}\zeta(2r)\frac{5^{2r}-1}{5^{2r-2}},\\
d_r &= -11\cdot 2^{2r-4}\pi^{2r-2}5^{-3r+3}\frac{\Gamma(12-r)}{\Gamma(11+r)}.
\end{aligned}$$

We have the following data using Mathematica:

| $(l,-5(100-l^2),D_K,f)$ | $\chi_5(l)$ | $H(1,5(100-l^2))$ | $-252H(3,5(100-l^2))$ | $H(5,5(100-l^2))$ |
|---|---|---|---|---|
| $(1,-495,-55,3)$ | $1$ | $20$ | $25502400$ | $341984\cdot 19765$ |
| $(2,-480,-120,2)$ | $-1$ | $12$ | $21122640$ | $11133604\cdot 513$ |
| $(3,-455,-455,1)$ | $-1$ | $20$ | $21288960$ | $4644833280$ |
| $(4,-420,-420,1)$ | $1$ | $8$ | $14706720$ | $3125844488$ |
| $(6,-320,-20,4)$ | $1$ | $14$ | $7990920$ | $3522\cdot 262657$ |
| $(7,-255,-255,1)$ | $-1$ | $12$ | $4838400$ | $341655552$ |
| $(8,-180,-20,3)$ | $-1$ | $6$ | $1776600$ | $3522\cdot 19603$ |
| $(9,-95,-95,1)$ | $1$ | $8$ | $423360$ | $4033248$ |

$$\begin{aligned}
&H(1,20)=2,\ H(1,15)=2,\\
&H(3,20)=\frac{7560}{-252},\ H(3,15)=\frac{4032}{-252},\\
&H(5,20)=3522,\ H(5,15)=992,\\
&\zeta(2)=\frac{\pi^2}{6},\ \zeta(6)=\frac{\pi^6}{3^3\cdot 5\cdot 7},\ \zeta(10)=\frac{\pi^{10}}{3^5\cdot 5\cdot 7\cdot 11},\\
&C_{10}^1\left(\frac{x}{2}\right)=-1+15x^2-35x^4+28x^6-9x^8+x^{10},\\
&C_8^3\left(\frac{x}{2}\right)=15-210x^2+420x^4-252x^6+45x^8,\\
&C_6^5\left(\frac{x}{2}\right)=-35+420x^2-630x^4+210x^6.
\end{aligned}$$

Accordingly, we get the following values:

$$\begin{aligned}
b(1,1)&=\frac{20901888}{390625}=\frac{2^{12}\cdot 3^6\cdot 7}{5^8},\\
b(1,3)&=\frac{735694848}{3173828125}\pi^4=\frac{2^{12}\cdot 3^2\cdot 7\cdot 2851}{5^{12}\cdot 13}\pi^4,\\
b(1,5)&=\frac{148596228096}{69427490234375}\pi^8=\frac{2^{15}\cdot 3\cdot 1511599}{5^{17}\cdot 7\cdot 13}\pi^8.
\end{aligned}$$

These coincide with the values obtained by Stopple.

# 4 Variation

Our approach suggests some natural variants. Although, I have not worked out yet, I hope to consider the following generalization.

**matrix index Jacobi Eisenstein series.** $2\sigma+k>4$, $(\tau,z)\in H_1\times \mathbf{C}^{2,1}$. Let $T$ be a positive definite integral binary quadratic form,

$$E_{k,T}(\tau,z,s)=\frac{v^s}{2}\sum_{c,d\in\mathbf{Z},\ (c,d)=1}\ \sum_{\lambda\in\mathbf{Z}^2}\frac{e(T[\lambda]\frac{a\tau+b}{c\tau+d}+\frac{2^t\lambda Tz}{c\tau+d}-\frac{cT[z]}{c\tau+d})}{(c\tau+d)^k|c\tau+d|^{2s}}.$$

Restriction $z=0$: Note the followings:

$$\begin{aligned}
E_{k,T}(\gamma\tau,0,s)&=(c\tau+d)^kE_{k,T}(\tau,0,s)\ \forall\gamma\in\Gamma_1,\\
< E_{k,T}(\cdot,0,s),f>&=\frac{\Gamma(s+k-1)}{(4\pi)^{k-1}}\sum_{n=1}^{\infty}\frac{r_T(n)b(n)}{n^{s+k-1}},\\
r_T(n)&=\sharp\{\lambda\in\mathbf{Z}^2;T[\lambda]=n\}.
\end{aligned}$$

In other words, $E_{k,T}(\tau, 0, s)$ is a non-holomorphic kernel of the Rankin convolution series of Hecke eigen cusp form and a binary theta series.

Taking the holomorphic projection and working out the calculation as before, we may compute some special values of the Rankin convolution. Such a special value appears in Böcherer-Mizumoto's formula of the $T$-th Fourier coefficient of Klingen's Eisenstein series of degree 2. As special case, we can consider the holomorphic projection of the restriction of **Hermitian Jacobi Eisenstein series**, which is discussed in Kohama's master thesis.

**higher dimension.** $2\sigma + k > n + l + 1$, $(\tau, z) \in H_n \times \mathbf{C}^{l,n}$. Let $T$ be a positive-definite $l \times l$ half-integral symmetric matrix,

$$\begin{aligned} &E_{k,T}^{(n)}(\tau, z, s) \\ &\quad = (\det \Im v)^s \sum_{\substack{\gamma=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty^{(n)} \backslash Sp_n(\mathbf{Z}) \\ \lambda \in \mathbf{Z}^{l,n}}} \frac{e(\mathrm{tr}\{T[\lambda] \cdot \gamma\tau + 2^t\lambda T z \cdot J(\gamma,\tau)^{-1} - T[z] \cdot J(\gamma,\tau)^{-1} c\})}{\det J(\gamma,\tau)^k |\det J(\gamma,\tau)|^{2s}}, \end{aligned}$$

$J(\gamma, \tau) = c\tau + d$, $\tau = u + iv \in H_n (v > O)$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = (a\tau + b)(c\tau + d)^{-1}$.
Restriction $z = 0$ B. Heim showed that

$$E_{k,T}^{(n)}(\gamma\tau, 0, s) = \det J(\gamma, \tau)^k E_{k,T}^{(n)}(\tau,\ 0,\ s) \quad (\forall \gamma \in Sp_n(\mathbf{Z})),$$

$$< E_{k,T}^{(n)}(*, 0, s), f > \fallingdotseq \sum_{\lambda \in \mathbf{Z}^{l,n}/GL_n(\mathbf{Z})} \frac{\overline{b(T[\lambda])}}{\det(T[\lambda])^{s+k-\frac{n+1}{2}}}$$

for a Siegel cusp form $f(\tau) = \sum_{A>O} b(A) e(\mathrm{tr}(A\tau))$ of degree $n$ and weight $k$. In other words, $E_{k,T}^{(n)}(\tau, 0, s)$ is a non-holomorphic kernel of the theta transform (the standard zeta function). In order to work out our approach in this higher genus case, some problems arise:

(1) computing the Fourier expansion of the real analytic Jacobi-Eisenstein series of higher genus;

(2) calculating integral transforms of the confluent hypergeometric functions of generel degree.

On (2), what kind of special functions of matrix argument occur, which will be an analogue of the Gauss hypergeometric function. Then, what kind of special polynomials appear as analogue of the Gegenbauer poynomials. To compare such a special polynomial and the polynomials arising from Ibukiyama's differential operators seems interesting.

We can generalize our construction to the twisted symmetric square $L$-functions associated to elements in $S_k(N, \psi)$. Also, it seems to be interesting to consider **Hilbert Jacobi Eisenstein series**. See Takase and Mizumoto for Hilbert modular analogue of Zagier's kernel function.

## References


[1] T. Arakawa, Real analytic Eisenstein series for the Jacobi group, Abh. Math. Sem. Univ. Hamburg **60** (1990), 131–148.

[2] T. Arakawa, Selberg zeta functions and Jacobi forms, In: Zeta functions in geometry (Tokyo, 1990), 181–218, Adv. Stud. Pure Math., 21, Kinokuniya, Tokyo, 1992.

[3] H. Cohen, Sums involving the values at negative integers of $L$-functions of quadratic characters, Math. Ann. **217** (1975), 271–285.

[4] K. Doi, H. Hida and H. Ishii, Discriminant of Hecke fields and twisted adjoint $L$-values for GL(2), Invent. Math. **134** (1998), 547–577.

[5] M. Eichler and D. Zagier, The theory of Jacobi forms, Progress in Mathematics, 55, Birkhauser Boston, Inc., Boston, MA, 1985.

[6] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Tables of integral transforms. Vol. I, Based, in part, on notes left by Harry Bateman, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.

[7] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Tables of integral transforms. Vol. II, Based, in part, on notes left by Harry Bateman, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.

[8] K. Goto, A twisted adjoint $L$-value of an elliptic modular form, J. Number Theory **73** (1998), 34–46.

[9] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products, Translated from the fourth Russian edition. Sixth edition. Translation edited and with a preface by Alan Jeffrey, Academic Press, Inc., Boston, MA, 1994.

[10] P. Guerzhoy, Jacobi-Eisenstein series and $p$-adic interpolation of symmetric squares of cusp forms, Ann. Inst. Fourier (Grenoble) **45** (2003), 451–464.

[11] P. Guerzhoy, Jacobi formes and a $p$-adic $L$-function in two variables, Fundam. Prikl. Mat. **6** (2000), 1007–1021.

[12] P. Guerzhoy, Symmetric squares of cusp forms and Jacobi–Eisenstein series, Moscow University Mathematical Bulletin **46** (1991), 39–41.

[13] B. Heim, Analytic Jacobi Eisenstein series and the Shimura method, J. Math. Kyoto Univ. **43** (2003), 451–464.

[14] Y. Hiraoka, Numerical calculation of twisted adjoint $L$-values attached to modular forms, Experiment. Math. **9** (2000), 67–73.

[15] H. Katsurada, Special Values of the Standard Zeta Functions, In: Galois Theory and Modular Forms, 337–356, Developments in Mathematics, 11, Kluwer Acad. Publ., Boston, MA, 2004.

[16] H. Katsurada, Special values of the standard zeta functions for elliptic modular forms, Experiment. Math. **14** (2005), 27–45.

[17] S. Kohama, master thesis, 2011, The Tokushima university.

[18] W. Kohnen and J. Sengupta, On the average of central values of symmetric square $L$-functions in weight aspect, Nagoya Math. J. **167** (2002), 95–100.

[19] W. Kohnen and J. Sengupta, Nonvanishing of symmetric square $L$-functions of cusp forms inside the critical strip, Proc. Amer. Math. Soc. **128** (2000), 1641–1646.

[20] N. Lebedev, Special functions and their applications, Revised edition, translated from the Russian and edited by Richard A. Silverman. Unabridged and corrected republication, Dover Publications, Inc., New York, 1972.

[21] T. Miyake, Modular forms, Translated from the Japanese by Yoshitaka Maeda, Springer-Verlag, Berlin, 1989.

[22] S. Mizumoto, On the symmetric-square zeta functions attached to Hilbert modular forms, Automorphic forms and number theory (Sendai, 1983), 175–183, Adv. Stud. Pure Math., 7, North-Holland, Amsterdam, 1985.

[23] S. Mizumoto, On the second $L$-functions attached to Hilbert modular forms, Math. Ann. **269** (1984), 191–216.

[24] A. Hashim and M. Ram Murty, On Zagier's cusp form and the Ramanujan $\tau$ function, Proc. Indian Acad. Sci. Math. Sci. **104** (1994), 93–98.

[25] A. Panchishkin, Symmetric squares of Hecke series and their values at lattice points, Mat. Sb. (N.S.) **108 (150)** (1979), 393–417, 478.

[26] H. Rademacher, On the Phragmen-Lindelof theorem and some applications, Math. Z. **72** (1959/1960), 192–204.

[27] G. Shimura, On the holomorphy of certain Dirichlet series, Proc. London Math. Soc. (3) **31** (1975), 79–98.

[28] C. Small, Arithmetic of finite fields, Monographs and Textbooks in Pure and Applied Mathematics, 148. Marcel Dekker, Inc., New York, 1991.

[29] J. Stopple, Special values of twisted symmetric square $L$-functions and the trace formula, Compositio Math. **104** (1996), 65–76.

[30] J. Sturm, Projections of $C^\infty$ automorphic forms, Bull. Amer. Math. Soc. (N.S.) **2** (1980), 435–439.

[31] J. Sturm, Special values of zeta functions, and Eisenstein series of half integral weight, Amer. J. Math. **102** (1980), 219–240.

[32] J. Sturm, Evaluation of the symmetric square at the near center point, Amer. J. Math. **111** (1989), 585–598.

[33] K. Takase, On the trace formula of the Hecke operators and the special values of the second $L$-functions attached to the Hilbert modular forms, Manuscripta Math. **55** (1986), 137–170.

[34] D. Zagier, Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields, In: Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), 105–169. Lecture Notes in Math., 627, Springer, Berlin, 1977.

[35] D. Zagier, Introduction to modular forms, In: From number theory to physics (Les Houches, 1989), 238–291, Springer, Berlin, 1992.