Noether's problem and unramified Brauer groups (joint work with M. Kang and B. E. Kunyavskii)

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In this talk, I explain the content of the joint paper [HKK] with Ming-chang Kang and Boris E. Kunyavskii.

1 Introduction

Let k be any field and G be a finite group. Let G act on the rational function field $k(x_g : g \in G)$ by k-automorphisms so that $g \cdot x_h = x_{gh}$ for any $g, h \in G$. Denote by $k(G)$ the fixed field $k(x_g : g \in G)^G$. Noether's problem asks whether $k(G)$ is rational (= purely transcendental) over k. It is related to the inverse Galois problem, to the existence of generic G-Galois extensions over k, and to the existence of versal G-torsors over k-rational field extensions [Sw, Sa1, GMS, 33.1, p. 86]. Noether's problem for abelian groups was studied by Swan, Voskresenskii, Endo, Miyata and Lenstra, etc. The reader is referred to Swan's paper for a survey of this problem [Sw].

On the other hand, just a handful of results about Noether's problem are obtained when the groups are not abelian. It is the case even when G is a p-group.

Before stating the results of Noether's problem for non-abelian p-groups, we recall some relevant definitions.

Definition 1.1. Let $k \subset K$ be an extension of fields. K is rational over k (for short, k-rational) if K is purely transcendental over k. K is stably k-rational if $K(y_1, \ldots, y_m)$ is rational over k for some $y_1, \ldots, y_m$ such that $y_1, \ldots, y_m$ are algebraically independent over K. When k is an infinite field, K is said to be retract k-rational if there is a k-algebra A contained in K such that (i) K is the quotient field of A, (ii) there exist a non-zero polynomial $f \in k[X_1, \ldots, X_n]$ (where $k[X_1, \ldots, X_n]$ is the polynomial ring) and k-algebra homomorphisms $\varphi\colon A \to k[X_1, \ldots, X_n][1/f]$ and $\psi\colon k[X_1, \ldots, X_n][1/f] \to A$ satisfying $\psi \circ \varphi = 1_A$. (See [Sa2, Ka] for details.) It is not difficult to see that "k-rational" $\Rightarrow$ "stably k-rational" $\Rightarrow$ "retract k-rational".

Definition 1.2. Let $k \subset K$ be an extension of fields. The notion of the unramified Brauer group of K over k, denoted by $\mathrm{Br}_{v,k}(K)$ was introduced by Saltman [Sa3]. By definition, $\mathrm{Br}_{v,k}(K) = \bigcap_R \mathrm{Image}\{\mathrm{Br}(R) \to \mathrm{Br}(K)\}$ where $\mathrm{Br}(R) \to \mathrm{Br}(K)$ is the natural map of Brauer groups and R runs over all the discrete valuation rings R such that $k \subset R \subset K$ and K is the quotient field of R.

Lemma 1.3 (Saltman [Sa3, Sa4])**.** *If k is an infinite field and K is retract k-rational, then the natural map $\mathrm{Br}(k) \to \mathrm{Br}_{v,k}(K)$ is an isomorphism. In particular, if k is an algebraically closed field and K is retract k-rational, then $\mathrm{Br}_{v,k}(K) = 0$.*

Theorem 1.4 (Bogomolov, Saltman [Bo, Sa5, Theorem 12])**.** *Let G be a finite group, k be an algebraically closed field with $\gcd\{|G|, \mathrm{char}\, k\} = 1$. Let μ denote the multiplicative* *subgroup of all roots of unity in k. Then $\mathrm{Br}_{v,k}(k(G))$ is isomorphic to the group $B_0(G)$ defined by*

$$B_0(G) = \bigcap_A \mathrm{Ker}\{\mathrm{res} : H^2(G, \mu) \to H^2(A, \mu)\}$$

where A runs over all the bicyclic subgroups of G (a group A is called bicyclic if A is either a cyclic group or a direct product of two cyclic groups).

Note that $B_0(G)$ is a subgroup of $H^2(G,\mu)$ (where $\gcd\{|G|, \mathrm{char}\, k\} = 1$). Since $H^2(G, \mu) \simeq H_2(G)$, which is the Schur multiplier of G (see [Kar]), we will call $B_0(G)$ the Bogomolov multiplier of G, following the convention in [Ku]. Because of Theorem 1.4 we will not distinguish $B_0(G)$ and $\mathrm{Br}_{v,k}(k(G))$ when k is algebraically closed and $\gcd\{|G|, \mathrm{char}\, k\} = 1$. In this situation, $B_0(G)$ is canonically isomorphic to $\bigcap_A \mathrm{Ker}\{\mathrm{res} : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\}$, i.e. we may replace the coefficient μ by $\mathbb{Q}/\mathbb{Z}$ in Theorem 1.4.

Using the unramified Brauer groups, Saltman and Bogomolov are able to establish counter-examples to Noether's problem for non-abelian p-groups.

Theorem 1.5. *Let p be any prime number, k be any algebraically closed field with* $\mathrm{char}\, k \neq p$.
(1) (Saltman [Sa3]) *There is a group G of order p^9 such that $B_0(G) \neq 0$. In particular, $k(G)$ is not retract k-rational. Thus $k(G)$ is not k-rational.*
(2) (Bogomolov [Bo]) *There is a group G of order p^6 such that $B_0(G) \neq 0$. Thus $k(G)$ is not k-rational.*

For p-groups of small order, we have the following result.

Theorem 1.6 (Chu and Kang [CK])**.** *Let p be any prime number, G is a p-group of order $\leq p^4$ and of exponent e. If k is a field satisfying either* (i) $\mathrm{char}\, k = p$, *or* (ii) *k contains a primitive e-th root of unity, then $k(G)$ is k-rational.*

Because of the above Theorems 1.5 and 1.6, we may wonder what happens to non-abelian p-groups of order p^5.

Theorem 1.7 (Chu, Hu, Kang and Prokhorov [CHKP])**.** *Let G be a group of order* 32 *and of exponent e. If k is a field satisfying either* (i) $\mathrm{char}\, k = 2$, *or* (ii) *k contains a primitive e-th root of unity, then $k(G)$ is k-rational. In particular, $B_0(G) = 0$.*

Working on p-groups, Bogomolov developed a lot of techniques and interesting results. Here is one of his results.

Theorem 1.8. (1) [Bo, Lemma 4.11] *If G is a p-group with $B_0(G) \neq 0$ and $G/[G,G] \simeq C_p \times C_p$, then $p \geq 5$ and $|G| > p^7$.*
(2) [Bo, Lemma 5.6; BMP, Corollary 2.11] *If G is a p-group of order $\leq p^5$, then $B_0(G) = 0$.*

Because of part (2) of the above theorem, Bogomolov proposed to classify all the groups G with $|G| = p^6$ satisfying $B_0(G) \neq 0$ [Bo, page 479].

It came as a surprise that Moravec's recent paper [Mo1] disproved the above Theorem 1.8.

Theorem 1.9 (Moravec [Mo1, Section 5])**.** *If G is a group of order* 243*, then $B_0(G) \neq 0$ if and only if $G = G(243, i)$ with $28 \leq i \leq 30$, where $G(243, i)$ is the i-th group among groups of order* 243 *in the database of GAP.*

Moravec proves Theorem 1.9 by using computer calculations. No theoretic proof is given. A file of the GAP functions and commands for computing $B_0(G)$ can be found at

Moravec's website `www.fmf.uni-lj.si/~moravec/b0g.g`. Recently, using this computer package, Moravec was able to classify all groups G of order 5^5 and 7^5 such that $B_0(G) \neq 0$.

Before stating the main result of this paper, we recall the classification of p-groups of order $\le p^6$ and introduce the notion of isoclinism.

A list of groups of order 2^5 (resp. 3^5, 5^5, 7^5) can be found in the database of GAP. However the classification of groups of order p^5 dated back to Bagnera (1898), Bender (1927), R. James (1980), etc. [Ba, Be, Ja], although some minor errors might occur in the classification results finished before the computer-aided time. For example, in Bender's classification of groups of order 3^5, one group is missing, i.e. the group $\Delta_{10}(2111)a_2$ which was pointed by [Ja, page 613]. A beautiful formula for the total number of the groups of order p^5, for $p \ge 3$, was found by Bagnera [Ba] as

$$2p + 61 + \gcd\{4, p-1\} + 2\gcd\{3, p-1\}.$$

Note that the above formula is correct only when $p \ge 5$ (see the second paragraph of Section 4).

On the other hand, groups of order 2^n ($n \le 6$) were classified by M. Hall and Senior [HaS]. There are 267 groups of order 2^6 in total. Groups of order 2^7 were classified by R. James, Newman and O'Brien [JNOB].

Definition 1.10. Two p-groups G_1 and G_2 are called isoclinic if there exist group isomorphisms $\theta\colon G_1/Z(G_1) \to G_2/Z(G_2)$ and $\phi\colon [G_1, G_1] \to [G_2, G_2]$ such that $\phi([g,h]) = [g', h']$ for any $g, h \in G_1$ with $g' \in \theta(gZ(G_1))$, $h' \in \theta(hZ(G_1))$ (note that $Z(G)$ and $[G,G]$ denote the center and the commutator subgroup of the group G respectively).

For a prime number p and a fixed integer n, let $G_n(p)$ be the set of all non-isomorphic groups of order p^n. In $G_n(p)$ consider an equivalence relation: two groups G_1 and G_2 are equivalent if and only if they are isoclinic. Each equivalence class of $G_n(p)$ is called an isoclinism family.

If p is an odd prime number, then there are precisely 10 isoclinism families for groups of order p^5; each family is denoted by Φ_i, $1 \le i \le 10$ [Ja, pages 619–621]. As for groups of order 64, there are 27 isoclinism families [JNOB, page 147].

The main result is the following theorem.

Theorem 1.11. *Let p be any odd prime number, G be a group of order p^5. Then $B_0(G) \neq 0$ if and only if G belongs to the isoclinism family Φ_{10}. Each group G in the family Φ_{10} satisfies the condition $G/[G,G] \simeq C_p \times C_p$. There are precisely 3 groups in this family if $p = 3$. For $p \ge 5$, the total number of non-isomorphic groups in this family is*

$$1 + \gcd\{4, p-1\} + \gcd\{3, p-1\}.$$

Note that, for $p = 3$, the isoclinism family Φ_{10} consists of the groups $\Phi_{10}(2111)a_r$ (where $r = 0, 1$) and $\Phi_{10}(5)$ [Ja, page 621], which are just the groups $G(3^5, i)$ with $28 \le i \le 30$ in the GAP code numbers. This confirms the computation of Moravec [Mo1]. Similarly, when $p = 5$, the isoclinism family Φ_{10} consists of the groups $G(5^5, i)$ with $33 \le i \le 38$; when $p = 7$, the isoclinism family consists of the groups $G(7^5, i)$ with $37 \le i \le 42$. They agree with Moravec's computer results.

We use the computer package provided by Moravec to study groups of order 11^5. We find that, for a group G of order 11^5, $B_0(G) \neq 0$ if and only if $G \simeq G(11^5, i)$ with $39 \le i \le 42$, also confirming the above Theorem 1.11.

It may be interesting to record the computing time to determine $B_0(G)$ for all p-groups of order p^5 with $p = 3, 5, 7, 11$. When $p = 3, 5, 7$, it requires only 20 seconds, one hour and two days respectively. When $p = 11$, it requires more than one month by parallel computing at four cores.

As a corollary of Theorem 1.11, we record the following result.

Theorem 1.12. *Let n be a positive integer and k be a field with $\gcd\{|G|, \operatorname{char} k\} = 1$. If $2^6 \mid n$ or $p^5 \mid n$ for some odd prime number p, then there is a group G of order n such that $B_0(G) \neq 0$. In particular, $k(G)$ is not stably k-rational; when k is an infinite field, $k(G)$ is not even retract k-rational.*

For completeness, we record the result for groups of order 2^6. Recall that there are 267 non-isomorphic groups of order 2^6 and 27 isoclinism families in total [JNOB].

Theorem 1.13 (Chu, Hu, Kang and Kunyavskii [CHKK]). *Let G be a group of order 2^6.*
(1) $B_0(G) \neq 0$ if and only if G belongs to the 16th isoclinism family, i.e. $G = G(2^6, i)$ where $149 \le i \le 151$, $170 \le i \le 172$, $177 \le i \le 178$, or $i = 182$.
(2) If $B_0(G) = 0$ and k is an algebraically closed field with $\operatorname{char} k \neq 2$, then $k(G)$ is rational over k except possibly for groups G belonging to the 13rd isoclinism family, i.e. $G = G(2^6, i)$ with $241 \le i \le 245$.

Finally we mention a recent result which supplements Moravec's result in Theorem 1.9.

Theorem 1.14 (Chu, Hoshi, Hu and Kang [CHHK]). *Let G be a group of order 3^5 and of exponent e. If k is a field containing a primitive e-th root of unity and $B_0(G) = 0$, then $k(G)$ is rational over k except possibly for groups $G \in \Phi_7$, i.e. $G = G(3^5, i)$ with $56 \le i \le 60$.*

We explain briefly the idea of the proof of Theorem 1.11. Let G be a group of order p^5 where p is an odd prime number. For the proof of $B_0(G) = 0$ when G belongs to the isoclinism family Φ_6, we use the 7-term cohomology exact sequence in [DHW]. We remark that in [HKK] we prove not only $B_0(G) = 0$, but also $k(G)$ is retract k-rational or the $k(G)$'s are k-isomorphic for the groups G belonging to the same isoclinism family. Moravec has another proof showing that $B_0(G) = 0$ when G is a group of order p^5 not belonging to the isoclinism family Φ_{10} [Mo2].

On the other hand, to show that $B_0(G) \neq 0$, we find suitable generators and relations for G. It turns out that $B_0(G) \neq 0$ if some relations are satisfied. All the groups in the isoclinism family Φ_{10} satisfy these relations. The proof relies on the 5-term exact sequence of Hochschild and Serre [HS]

$$0 \to H^1(G/N, \mathbb{Q}/\mathbb{Z}) \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^1(N, \mathbb{Q}/\mathbb{Z})^G \to H^2(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\psi} H^2(G, \mathbb{Q}/\mathbb{Z})$$

where ψ is the inflation map. The crux of showing $B_0(G) \neq 0$ is to prove that the image of ψ is non-zero and is contained in $B_0(G)$.

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