The Adams operation on stable presentable symmetric monoidal ∞-categories

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1 Adams operation of Grothendieck groups

For regular noetherian separated scheme X of finite dimensional, Grothendieck [6] proved that the Grothendieck group $K_0(X)$ of the exact category of vector bundles over schemes X admits the Adams operations $\psi^k : K_0(X) \to K_0(X)$ $(k = 1, 2, \ldots)$ and the decomposition to the eigen-spaces after tensoring with $\mathbb{Q}$:

$$\bigoplus_{r \geq 0} K_0(X)^{(r)}_{\mathbb{Q}} \cong K_0(X)_{\mathbb{Q}},$$

where $K_0(X)^{(r)}$ denotes the intersection of the kernel of the endmorphisms $\psi^k - k^r \cdot \mathrm{id}$ $(k = 1, 2, \ldots)$ and $-_{\mathbb{Q}}$ means $- \otimes_{\mathbb{Z}} \mathbb{Q}$. After, Hiller [7], Krazer [11], Soulé [15] and Riou [14] proved that the higher K-groups $K_n(X)$ $(n \geq 0)$ of separated schemes X admit the Adams operations $\psi^k : K_n(X) \to K_n(X)$ $(k = 1, 2, \ldots)$.Let A be a commutative ring. Then the Adams operation $\Psi^k : K_0(A) \to K_0(A)$ is defined by

$$\psi^k([M]) = \chi'([\mathrm{Kos}^k(M)_\bullet]) \quad (k = 1, 2, \ldots)$$

for any finitely generated projective A-module M, where χ' is the secondary Euler characteristic functor and $\mathrm{Kos}^k(M)_\bullet$ is the Koszul complex of M. After Grothendieck [6], Hiller [7], Krazer [11], Soulé [15], Grayson [5] and Riou [14] independently proved that the algebraic higher K-groups $K_n(X)$ $(n \geq 0)$ of separated schemes X admit the Adams operations $\psi^k : K_n(X) \to K_n(X)$ $(k = 1, 2, \ldots)$. In particular, Grayson [5] gave combinatorial constructions Koszul complexes and the secondary Euler characteristic on the K-theory spaces of the exact category of projective A-modules.

2 Preliminaries of stable ∞-categories and symmetric monoidal ∞-categories

2.1 The definition of ∞-categories

An ∞-category is a marked simplicial set which is a fibrant object with respect to Cartesian model structure on the category of marked simplicial sets. We recall the definition of Cartesian model structure and marked simplicial sets.

Let Set_Δ denote the category of simplicial sets. Let $\Delta^n \in \mathrm{Set}_\Delta$ be the standard n-simplex and $\Lambda^n_i \subset \Delta^n$ be the sub-simplicial set obtained by deleting the interior and the face opposite for the i-th vertex.

An *inner fibration* $f : X \to S$ of simplicial sets is a morphism of simplicial sets which has the right lifting property with respect to all inclusion $\Lambda^n_i \to \Delta^n$ for any $n \geq 0$ and $0 < i < n$.

Definition 2.1 ([12, Definition 2.4.1.3]). Let $p : X \to S$ be an inner fibration of simplicial sets. An edge $f : x \to y$ in X is *p-Cartesian* if the induced map

$$X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$$

is a trivial Kan fibration.

Definition 2.2 ([13, Definition 2.4.2.1]). A map $p : X \to S$ of simplicial sets is a *Cartesian fibration* if p is an inner fibration and for every edge $f : x \to y$ in S and every vertex $\tilde{y}$ of X with $p(\tilde{y}) = y$, there exists a p-Cartesian edge $\tilde{f} : \tilde{x} \to \tilde{y}$ such that $p(\tilde{f}) = f$. We say that p is a *coCartesian fibration* if $p : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is a Cartesian fibration.

A marked simplicial set is a pair $(X, \mathcal{E})$, where X is a simplicial set and $\mathcal{E}$ is a set of edges of X_1 which contains the set of all degenerate edges $s_0(X_0)$. Here $s_0 : X_0 \to X_1$ is the degeneracy map. The set $\mathcal{E}$ is called *marked edges*. A morphism $f : (X, \mathcal{E}) \to (Y, \mathcal{E}')$ of marked simplicial sets is a map $f : X \to Y$ satisfying $f(\mathcal{E}) \subset \mathcal{E}'$. Let Set_Δ^+ denotes the category of marked simplicial sets. We write $S^\sharp = (S, S_1)$ and $S^\flat = (S, s_0(S_0))$. Let $p : X \to S$ be a Cartesian fibration of simplicial sets. Then $X^\natural$ denotes the marked simplicial set $(X, \mathcal{E})$, where $\mathcal{E}$ is the set of p-Cartesian edges of X.

Let S be a simplicial set. Let X, Y be two marked simplicial set over $S^\sharp$ and $q : Y \to S$ be a Cartesian fibration. Let $\mathrm{Map}^\flat_S(X, Y)$ denote the underlying simplicial set Y^X and $\mathrm{Map}^\sharp_S(X, Y) \subset \mathrm{Map}^\flat_S(X, Y)$ denote the largest Kan complex. We define Cartesian equivalences:

Definition 2.3 ([12, p. 155]). Let S be a simplicial set and $p : X \to Y$ a morphism in $(\mathrm{Set}_\Delta^+)_{/S^\sharp}$. Then p is called a *Cartesian equivalence* if for every Cartesian fibration $Z \to S$, the induced map

$$\mathrm{Map}^\sharp_S(Y, Z^\natural) \to \mathrm{Map}^\sharp_S(X, Z^\natural)$$

is a homotopy equivalence of Kan complexes.

We use the following Cartesian model structures [12, Proposition 3.1.3.7].

Theorem 2.4 ([12, Proposition 3.1.3.7, p. 157]). *Let S be a simplicial set. There exists a left proper combinatorial model structure on $(\mathrm{Set}_\Delta^+)_{/S^\sharp}$ which described as follows*:

(C) *Cofibrations are monomorphisms.*

(W) *Weak equivalences in $(\mathrm{Set}_\Delta^+)_{/S^\sharp}$ are Cartesian equivalences.*

(F) *Fibrations are those morphisms which have the right lifting property with respect to every morphism satisfying both* (C) *and* (W).

The above model structure on $(\mathrm{Set}_\Delta^+)_{/S^\sharp}$ is called *Cartesian model structure*. An *∞-category* is a fibrant object of the Cartesian model category $(\mathrm{Set}_\Delta^+)_{/\Delta^{0,\sharp}}$. This model structure of $(\mathrm{Set}_\Delta^+)_{/S^\sharp}$ is simplicial and the simplicial model category $(\mathrm{Set}_\Delta^+)_{/S^\sharp}$ admits mapping objects given by $\mathrm{Map}^\sharp_S(X, Y)$ (see [12, Corollary 3.1.4.4, p. 162]). Hence the Cartesian model category $(\mathrm{Set}_\Delta^+)_{/S^\sharp}$ has a symmetric monoidal structure determined by mapping object.

Definition 2.5. An ∞-category $\mathcal{C}$ is *presentable* if $\mathcal{C}$ satisfies the followings:

(i) There exists a regular cardinal κ such that $\mathcal{C}$ is (Cartesian) equivalent to a κ-Ind-category of some small ∞-category. These ∞-categories are said to be *accessible*.

(ii) The ∞-category $\mathcal{C}$ admits small colimits.

Note that any idempotent complete small ∞-category is accessible [12, Corollary 5.4.3.6]. Furthermore any ∞-category admits an idempotent completion [12, Proposition 5.1.4.2]. We have the Adjoint functor theorem on ∞-categories:

Proposition 2.1 ([12, Corollary 5.5.2.9])**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories. Fix a regular cardinal κ.*
(1) *The functor F admits a right adjoint if and only if it preserves all κ-small colimits.*
(2) *The functor F admits a left adjoint if and only if it preserves all κ-small limits and κ-compact objects, and it is κ-continuous (We say that these functors are preserving small limits and κ-accessible).*

2.2 Stable ∞-categories and group complete ∞-categories

We recall the definition of the simplicial nerve of the category of simplicial categories:

Definition 2.6 ([12, Definition 1.1.5.3])**.** Let Cat_Δ be the category of simplicial categories. For the standard simplex Δ^*, the simplicial category $\mathfrak{C}[\Delta^*]$ is defined by

$$\mathfrak{C}[\Delta^n] = \begin{cases} \emptyset & (j < i), \\ \{I \subset [n] \mid 0,\, n \in I\} & (i \leq j) \end{cases}$$

for each $n \geq 0$. For general simplicial sets S, $\mathfrak{C}[S]$ is defined as the colimit $\varinjlim_{\Delta^* \to S} \mathfrak{C}[\Delta^*]$. Then $\mathfrak{C}[S]$ is the simplicial category.

We obtain a functor $\mathfrak{C} : \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$. The functor $N_\Delta : \mathrm{Cat}_\Delta \to \mathrm{Set}_\Delta$ is a left adjoint of $\mathfrak{C}$ defined by

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n,\, N_\Delta(\mathcal{C})) = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n],\, \mathcal{C})$$

for each $n \geq 0$. The simplicial set $N_\Delta(\mathcal{C})$ is called the *simplicial nerve* of $\mathcal{C}$. Joyal [9] showed the existence of a Quillen equivalence

$$N_\Delta : \mathrm{Cat}_\Delta \rightleftarrows \mathrm{Set}_\Delta : \mathfrak{C}.$$

Let Cat_∞ denotes the simplicial nerve of the simplicial category of ∞-categories. The ∞-category is called the ∞-category of ∞-categories. The ∞-category Cat_∞ has a symmetric monoidal structure determined by the symmetric monoidal structure on the Cartesian model category $(\mathrm{Set}_\Delta)_{/\Delta^{0},\sharp}$ (see [12, Section 3.1.3, p. 154]).

A zero object of $\mathcal{C}$ is an object which is both initial and final. Let $\mathcal{C}$ be an ∞-category which has an zero object 0. A diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z \end{array}$$

is said to be a *fiber sequence* if it is a pull-back square, and a *cofiber sequence* if it is a push-forward square. If the diagram is a fiber sequence, then X is said to be a *fiber* of g, and it is a cofiber sequence if Z is a *cofiber* of f.

Definition 2.7 ([13, Definition 1.1.1.9])**.** Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is a stable ∞-category if $\mathcal{C}$ satisfying the following properties:

(i) There exists a zero object $0 \in \mathcal{C}$ such that it is both initial and final.

(ii) Every morphism in $\mathcal{C}$ admits a fiber and a cofiber.

(iii) A diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ is a fiber sequence if and only if a cofiber sequence.

Example 2.8. A *pointed* ∞*-category* is an ∞-category with a zero object. Let $\mathrm{Cat}_\infty^{\mathrm{ex}}$ denotes the presentable ∞-category of stable ∞-categories and $\mathrm{Cat}_{\infty,*}$ denotes the presentable ∞-category of pointed ∞-categories. Since the ∞-category of stable ∞-categories admits all small limits and the embedding functor $\mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathrm{Cat}_{\infty,*}$ is accessible, it admits a left adjoint by Proposition 2.1. Let Stab denote the left adjoint functor. The functor Stab is called the *stabilization*. Let $\mathcal{S}$ be the ∞-category determined by the simplicial nerve of the simplicial category of Kan complexes. This ∞-category is said to be the ∞-category of *spaces*. Then Sp denote the stabilization of the pointed ∞-category $\mathcal{S}_*$, and Sp is called the stable ∞-category of *spectra*.

3 ∞-operads and symmetric monoidal ∞-categories

In this section, following [13], we explain the definitions of ∞-operads and symmetric monoidal ∞-categories. In the last part of this section, we recall the definition of the k-times symmetric product functor ($k \geq 0$).

3.1 ∞-operads and symmetric monoidal ∞-categories

Definition 3.1 ([13, Notation 2.0.0.2])**.** Write $\langle n\rangle = \{*, 1, 2, \ldots, n\}$. Let Fin_* be the category of finite pointed sets $\langle n\rangle$ with the base point $*$ and morphisms in Fin_* are maps of pointed sets.

A map $f : \langle m\rangle \to \langle n\rangle$ in Fin_* is said to be *innert* if $f^{-1}(i)$ has exactly one element for each $1 \leq i \leq n$. A map $f : \langle m\rangle \to \langle n\rangle$ is said to be *active* if $f^{-1}(*) = \{*\}$. For every pair of integers $1 \leq i \leq n$, let $\rho^i : \langle n\rangle \to \langle 1\rangle$ denote the morphism given by the formula

$$\rho^i(j) = \begin{cases} 1 & (i = j) \\ * & (i \neq j). \end{cases}$$

Definition 3.2 ([13, Definition 2.1.1.10])**.** Let $\mathcal{O}^\otimes$ be a simplicial set. An ∞-operad $p : \mathcal{O}^\otimes \to N_\Delta(\mathrm{Fin}_*)$ is a coCartesian fibration satisfying the following conditions:

(i) For any innert morphism $f : \langle m\rangle \to \langle n\rangle$ and $C \in \mathcal{O}^\otimes_{\langle n\rangle}$, there exists a p-coCartesian edge $\bar{f} : C \to C'$ such that $\bar{f}$ is a lifting of f.

(ii) Let $C \in \mathcal{O}^\otimes_{\langle m\rangle}$ and $C' \in \mathcal{O}^\otimes_{\langle n\rangle}$ be two objects, $f : \langle m\rangle \to \langle n\rangle$ a morphism in Fin_*, and let $\mathrm{Map}^f_{\mathcal{O}^\otimes}(C, C')$ the connected components of the pull-back of f by $\mathrm{Map}_{\mathcal{O}^\otimes}(C, C') \to \mathrm{Map}_{N_\Delta(\mathrm{Fin}_*)}(\langle m\rangle, \langle n\rangle)$. Choose p-coCartesian morphisms $C' \to C'_i$ lying over the inert morphisms ρ^i for $1 \leq i \leq n$. Then the induced map

$$\mathrm{Map}^f_{\mathcal{O}^\otimes}(C, C') \to \prod_{1\leq i\leq n} \mathrm{Map}^{\rho^i\circ f}_{\mathcal{O}^\otimes}(C, C'_i)$$

is a homotopy equivalence.

(iii) For every finite collection of objects $C_1, \ldots, C_n \in \mathcal{O}^\otimes_{\langle 1\rangle}$, there exists an object $C \in \mathcal{O}^\otimes_{\langle n\rangle}$ and a collection of p-coCartesian morphisms $C \to C_i$ covering ρ^i.

Definition 3.3 ([13, Definition 2.1.2.3]). Let $p : \mathcal{O}^{\otimes} \to N_{\Delta}(\mathrm{Fin}_*)$ be an ∞-operad and $f : X \to Y$ a morphism in $\mathcal{O}^{\otimes}$. We say that f is *innert* if $p(f)$ is innert and f is p-coCartesian. We say that f is *active* if $p(f)$ is active.

An $\mathcal{O}$-monoidal ∞-category is a Cartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$. If $\mathcal{O}^{\otimes} = N_{\Delta}(\mathrm{Fin}_*)$, then we say that $p : \mathcal{C}^{\otimes} \to N_{\Delta}(\mathrm{Fin}_*)$ is a *symmetric monoidal ∞-category.*

Example 3.4. The stable ∞-category of spectra Sp has a canonical symmetric monoidal structure determined by smash products on the category of pointed Kan complexes.

Definition 3.5 ([13, Definition 2.1.2.7]). Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ be ∞-operads. A map $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ in $(\mathrm{Set}^+_{\Delta})_{/N_{\Delta}(\mathrm{Fin}_*)^{\sharp}}$ is an *∞-operad map* if f carries inert morphisms in $\mathcal{O}^{\otimes}$ to inert morphisms in $\mathcal{O}'^{\otimes}$.

Let $\mathrm{Alg}_{\mathcal{O}}(\mathcal{O}')$ denotes the full subcategory of $\mathrm{Fun}_{N_{\Delta}(\mathrm{Fin}_*)}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ spanned by ∞-operad maps.

Example 3.6. Let $p : \mathcal{C}^{\otimes} \to N_{\Delta}(\mathrm{Fin}_*)$ be a symmetric monoidal ∞-category. Write $\mathrm{CAlg}(\mathcal{C}) = \mathrm{Alg}_{N_{\Delta}(\mathrm{Fin}_*)}(\mathcal{C})$. Objects of $\mathrm{CAlg}(\mathcal{C})$ are called *commutative algebra objects* of $\mathcal{C}$. In particular, objects of CAlg(Sp) is called $\mathbb{E}_{\infty}$-rings.

4 The ∞-category of vector bundles over an E_{∞}-rings

Let R be an $\mathbb{E}_{\infty}$-ring and $\mathrm{CAlg}_{/R}$ the ∞-category of $\mathbb{E}_{\infty}$-rings over R. An object of $\mathrm{CAlg}_{/R}$ is called an R-algebra. Note that $\mathrm{CAlg}_{/R}$ is pointed ∞-category with the base point $\mathrm{id}_R : R \to R$. The stabilization $\mathrm{Stab}(\mathrm{CAlg}_{/R})$ is a stable presentable ∞-category, and it is called R-modules [13, Corollary 8.3.4.14, p. 900]. Let Mod_R denotes the stable presentable ∞-category of R-modules. Let $\mathrm{Mod}_R^{\otimes}$ denotes stable he stable presentable ∞-category of R-modules with symmetric monoidal structure $\otimes_R$ with the unit object R. We assume that R is compact: that is the functor $\mathrm{Map}(R, -) : \mathrm{Mod}_R \to \mathrm{Sp}$ commutes with all small colimits. Let $\mathrm{PMod}_R^{\otimes}$ denotes the smallest full stable subcategory generated by R.

An object of $\mathrm{PMod}_R^{\otimes}$ is given by taking iterated finite times finite colimits and direct summands of objects $R[i]$ $(i \in \mathbb{Z})$. In this paper, we call an object of $\mathrm{PMod}_R^{\otimes}$ a *perfect R-modules.* Then the functor

$$\mathrm{PMod} : \mathrm{CAlg} \ni R \mapsto \mathrm{PMod}_R^{\otimes} \in \mathrm{Cat}_{\infty}^{\mathrm{Perf}}$$

Let Vect_R^n denotes the maximal Kan complex generated by free R-modules of rank $n \geq 1$. The Kan complex Vect_R^n is a sub-symmetric monoidal ∞-category of $\mathrm{PMod}_R^{\otimes}$. The geometric realization of Vect_R^n is equivalent to the ∞-groupoid $\mathrm{BGL}_n(R)$. Let $\mathrm{BGL}(R)$ is the colimit of $\mathrm{BGL}_n(R)$ and $\mathrm{BGL}_n^+(R)$ is the group completion of the monoidal structure of $\mathrm{BGL}(R)$. Then BGL^+ is an E_{∞}-ring by Gepner–Snaith [3].

Let K is the connective K-theory functor defined by Barwick [1]. Then we have the composition functor $K \circ \mathrm{PMod} : \mathrm{CAlg} \to \mathrm{Sp}$, and the following comparison theorem:

Theorem 4.1. *The K-theory functors $K \circ \mathrm{PMod}$ and BGL^+ are equivalent as E_{∞}-rings.*

Furthermore, we have an Adams operation Ψ^k by a similar argument of Riou's work [14]:

Theorem 4.2 ([14]). *Let $\mathcal{SH}(\mathbb{Z})_{-}Q$ be the rationalized stable $\mathbb{A}^1$-homotopy category of* $\mathrm{Spec}\,\mathbb{Z}$ *defined by Voevodsky [17].*

Then we have an isomorphism

$$\mathrm{End}_{\mathcal{SH}(\mathbb{Z})_{\mathbb{Q}}}(\mathrm{BGL}^+_{\mathbb{Q}}) \cong \mathbb{Q}^{\mathbb{Z}}.$$

Under the isomorphism, we can define the Adams operation Ψ^k by $\Psi^k = (k^n)_{n\in\mathbb{Z}} \in \mathbb{Q}^{\mathbb{Z}}$ for $k \in \mathbb{Z} - \{0\}$ (See [14]). Then we have $\Psi^k \circ \Psi^l = \Psi^{kl} = \Psi^l \circ \Psi^k$ for any $k, l \in \mathbb{Z} - \{0\}$ and $\Psi^1 = \mathrm{Id}_{\mathrm{BGL}}$.

Note that Ψ^k is bijective for any $k \in \mathbb{Z} - \{0\}$. We can regard that Ψ^k is an action of the multiplicative group $\mathbb{G}_m$ by the following sense:

$$\mathrm{Aut}_{\mathcal{SH}(\mathbb{Z})_{\mathbb{Q}}}(\mathrm{BGL}^+_{\mathbb{Q}}) = \mathbb{G}_m(\mathrm{End}_{\mathcal{SH}(\mathbb{Z})_{\mathbb{Q}}}(\mathrm{BGL}^+_{\mathbb{Q}})) = \mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}\left(\mathbb{Q}[t, t^{-1}], \mathrm{End}_{\mathcal{SH}(\mathbb{Z})_{\mathbb{Q}}}(\mathrm{BGL}^+_{\mathbb{Q}})\right).$$

Let A be a commutative ring. Then giving an action of $\mathbb{G}_m$ on $X = \mathrm{Spec}\, A$ is equivalent to giving a $\mathcal{Z}$-grading on A. :

Lemma 4.1. *Let A be an E_∞-ring flat over a base E_∞-ring R. Then an action ϕ of $\mathbb{G}_{m,R}$ on* Spec A *gives a grading $A \simeq \bigoplus_{n\in\mathbb{Z}} A^{(n)}$, where $A^{(n)}$ is homotopy equalizer*

$$A^{(n)} \to A \underset{k^n\cdot\mathrm{Id}_A}{\overset{\phi^*}{\rightrightarrows}} A[t, t^{-1}].$$

□

We define the rationalized Adams operation Ψ^k ($k \in \mathbb{Q} - \{0\}$) on the K-theory functor $K_{\mathbb{Q}} \circ \mathrm{PMod}$ by being an action of $\mathbb{G}_m$. Then we get the followings:

Theorem 4.3. *The rationalized Adams operation Ψ^k ($k \in \mathbb{Q} - \{0\}$) on $K_{\mathbb{Q}} \circ$* PMod *is coincides with the Adams operation which gives the eigen-space decomposition with eigenvalue k^n ($n \in \mathbb{Z}$).* □

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