Spectral average of central values of automorphic L-functions for holomorphic cusp forms on $SO_0(m,2)$

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1 Introduction

1.1

Let $\mathrm{PSL}_2(\mathbb{Z})$ be the projective unimodular group which acts on the upper half plane $\mathfrak{H} = \{\tau = x+iy \mid x \in \mathbb{R},\ y > 0\}$ by the linear fractional transformations. For an integer $l \geqslant 2$, let S_l denote the space of all the holomorphic functions $\phi : \mathfrak{H} \to \mathbb{C}$ which satisfy the modular transformation property

$$\phi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2l}\,\phi(\tau) \text{ for all } \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PSL}_2(\mathbb{Z})$$

and are cuspidal at $i\infty$. Thus, $\phi \in S_l$ has the Fourier expansion

$$\phi(\tau) = \sum_{n=1}^{\infty} a_\phi(n)\,q^n, \quad q = \exp(2\pi i\tau)$$

with $a_\phi(n)$ the n-th Fourier coefficient of ϕ. For each $n \in \mathbb{N} := \{1,2,\dots\}$, the n-th Hecke operator $T(n)$, acting on the space S_l by

$$[T(n)\phi](\tau) = n^{2l-1} \sum_{ad=n,0\leqslant b<d} \phi\left(\frac{a\tau+b}{d}\right) d^{-2l}$$

is self-adjoint with respect to the Petersson inner-product on S_l

$$\langle \phi_1, \phi_2\rangle := \int_{\mathrm{PSL}_2(\mathbb{Z})\backslash\mathfrak{H}} \phi_1(\tau)\,\overline{\phi_2(\tau)} y^{2l-2}\,\mathrm{d}x\,\mathrm{d}y.$$

Moreover, the operators $T(n)$ commute with each other. Thus, there exists a basis $\mathcal{B}_l$ of S_l consisting of joint eigenfunctions of all the Hecke operators $T(n)$ such that each $\phi \in \mathcal{B}_l$ is normalized in the sense that $a_\phi(1) = 1$.

For

$$\phi(\tau) = \sum_{n=1}^{\infty} a_\phi(n)\,q^n \in \mathcal{B}_l,$$

define its rescaled Fourier coefficients by

$$\mathfrak{a}_\phi(n) \overset{\text{def}}{=} \frac{(4\pi n)^{-(l-1/2)}\,\Gamma(2l)^{1/2}}{\|\phi\|}\,a_\phi(n), \quad (n = 1,2,\dots).$$

Like the orthogonality relations for Dirichlet characters, the rescaled Fourier coefficients $\mathfrak{a}_\phi(n)$ satisfy the *asymptotic orthogonality relation*, or Petersson's formula:

For any $m, n \in \mathbb{N}$,

$$\frac{1}{2l} \sum_{\phi \in \mathcal{B}_l} \mathfrak{a}_\phi(n)\, \overline{\mathfrak{a}_\phi(m)} = \delta_{m,n} + O\left(\frac{(mn)^{1/4+\epsilon}}{\sqrt{l}}\right) \quad (l \to \infty).$$

Here we should remark that from the well-known dimension formula of S_l,

$$\sharp \mathcal{B}_l = \frac{l}{6} + o(1) \quad (l \to +\infty).$$

Besides the aspect of orthogonality, the rescaled Fourier coefficients resemble to the Dirichlet characters in the aspect of boundedness. While any Dirichlet character is obviously bounded taking its values in the unit circle, the rescaled Fourier coefficients $\mathfrak{a}_\phi(n)$ are bounded only asymptotically in the sense that for any $\epsilon > 0$,

$$|\mathfrak{a}_\phi(n)| \ll (l\, n)^\epsilon, \quad \phi \in \mathcal{B}_l,\, n \in \mathbb{N}.$$

Despite its simple appearance, this is a deep result. For the n-aspect, we need Deligne's estimate $|a_\phi(n)| = O(n^{(2l-1)/2+\epsilon})$, and for the l-aspect, we use the relation

$$\|\phi\|^2 \, (4\pi)^{-(2l-1)}\, \Gamma(2l)^{-1} \asymp \mathrm{Res}_{s=2l} L(\phi \times \bar{\phi}, s)$$

proved by the Rankin-Selberg method and then imvoke Iwaniec's estimate $\mathrm{Res}_{s=2l} L(\phi \times \bar{\phi}, s) \asymp l^\epsilon$.

1.2

Let us recall the L-series $L(\phi, s)$ of $\phi \in \mathcal{B}_l$. In the classical normalization, it is given by the Euler product

$$L(\phi, s) := \sum_{n=1}^{\infty} \frac{a_\phi(n)}{n^s} = \prod_{p:\text{primes}} (1 - a_\phi(p)\, p^{-s} + p^{-2s+2l-1})^{-1}, \quad \mathrm{Re}(s) \gg 0.$$

Then, the completed L-function

$$\Lambda(\phi, s) := \Gamma_{\mathbb{C}}(s)\, L(\phi, s)$$

has a holomorphic continuation to $\mathbb{C}$ satisfying the functional equation

$$\Lambda(\phi, 2l - s) = (-1)^l\, \Lambda(\phi, s).$$

In particular, the central value $L(\phi, l)$ is zero unless l is even, in which cases its positivity is known. Suppose l is even. The central value $L(\phi, l)$ is of some interest. From Petersson's formula, by taking summation in n and by setting $m = 1$, we heuristically obtain the asymptotic formula

$$\spadesuit : \ \frac{1}{2l} \sum_{\phi \in \mathcal{B}_l} L\,(\phi, l)\ |\mathfrak{a}_\phi(1)|^2 \sim 1, \quad (l \to +\infty)$$

for their average (= the first moment), which is *true* actually (*cf.* [1]). Since $\mathfrak{a}_\phi(1) \asymp l^\epsilon$ $(\forall \epsilon > 0)$ and $\sharp \mathcal{B}_l \asymp l$, this is consistent with the Lindelöf hypothesis

$$(\forall \epsilon > 0) \quad L(\phi, l) = O(l^\epsilon), \quad \phi \in \mathcal{B}_l,\, l \in 2\mathbb{N}$$

in the weight aspect. At this point, to study the asymptotic for higher moments

$$\frac{1}{2l} \sum_{\phi \in \mathcal{B}_l} |L(\phi, l)|^n |\mathfrak{a}_\phi(1)|^2, \quad (n = 2, 3, \dots)$$

seems a natural direction for us to go. However, noting $\mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{SO}_0(2, 1)$, we see another possible way:
Regard $\mathrm{PSL}(2, \mathbb{R})$ as the first layer of groups $\mathrm{SO}_0(2, m)$ ($m = 1, 2, \dots$), go up the layer and ask the question:

♡ Is there analogous formula for the first moment of higher degree Euler products for holomorphic cusp forms on $\mathrm{SO}_0(2, m)$?

In this note, we consider this problem and announce a $\mathrm{SO}(2, m)$ counterpart of the formula ♠ (enunciated in Theorem 6). We should note that in a recent preprint [2] a similar question is raised for the spinor L-values of Siegel modular cusp forms on $Sp_2(\mathbb{Z})$; the authors proved an asymptotic formula for a weighted average of L-values *in the convergent range* of the Euler product, with the weight factor being constructed from the Bessel period of the cusp forms. When viewed through the accidental isomorphism $PGSp(2) \cong SO(5)$, our result should yield an asymptotic formula for the weighted average of *the central spinor L-values* with the same weight factor as in [2].

Let us explain the structure of this article briefly. We heavily rely on the theory of completed L-functions for orthogonal groups developed by Murase and Sugano; section 2 is a review of necessary facts from their theory. In section 3, we introduce notation of holomorphic cusp forms and their L-functions on the type IV tube domain, recalling the integral representation of the L-function due to Andrianov and Murase-Sugano. In section 4, we state our main result, whose proof is sketched in the final section.

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2 L-functions for orthogonal groups (Review of Murase-Sugano's work)

The basic references here are [3] and [4].

Let $M \cong \mathbb{Z}^n$ be a lattice, i.e., a free $\mathbb{Z}$-module with $\mathbb{Z}$-bilinear form $(X, Y) \in \mathbb{Z}$ such that the associated quadratic form $Q[X] := (X, X)$ on the $\mathbb{Q}$-vector space $M_\mathbb{Q}$ is non-degenerate. We suppose that

(i) M is even (i.e., $Q[X] \in 2\mathbb{Z}$ for all $X \in M$).

(ii) M is maximal (i.e., if L is an even lattice containing M, then $L = M$).

Let O_M be the orthogonal group scheme of M, i.e,

$$O_M(R) = \{g \in \mathrm{GL}(M_R) |\, Q[g\,X] = Q[X] \text{ for all } X \in M\,\}$$

for any commutative ring R, where the scalar extension $M \otimes_\mathbb{Z} R$ is denoted by M_R.

2.1 Local factors

Let p be a prime and ν the Witt index of the scalar extension $M_{\mathbb{Q}_p}$. Then, $O_M(\mathbb{Z}_p)$ is a maximal compact subgroup of $O_M(\mathbb{Q}_p)$. Let $\mathbf{K}^*_{M,p}$ be the kernel of the reduction homomorphism $O_M(\mathbb{Z}_p) \to \mathrm{GL}(M^\vee_{\mathbb{Z}_p}/M_{\mathbb{Z}_p})$; E_p denotes the quotient group $O_M(\mathbb{Z}_p)/\mathbf{K}^*_{M,p}$. We call $\mathbf{K}^*_{M,p}$ the *discriminant subgroup* of $O_M(\mathbb{Z}_p)$. Let $\mathcal{H}_{M,p}$ be the Hecke algebra for the pair $(O_M(\mathbb{Q}_p), \mathbf{K}^*_{M,p})$ in the usual sense and define $\mathcal{H}^+_{M,p} = \{\phi \in \mathcal{H} \mid \phi(uku^{-1}) = \phi(k) \text{ for all } u \in \mathbf{K}^*_{M,p}\}$. By the theory of Satake isomorphism ([4]), the characters of the $\mathbb{C}$-algebra $\mathcal{H}^+_{M,p}$ are parametrized by the set of Satake parameters (λ, χ), where $\lambda = (\lambda_j)_{1 \leqslant j \leqslant \nu}$ is a ν-tuple of unramified quasicharacters of $\mathbb{Q}_p^\times$ and χ a class of finite dimensional irreducible representations of E_p. For a character $\Xi : \mathcal{H}^+_{M,p} \to \mathbb{C}$ with Satake parameter (λ, χ), its local standard L-factor is defined by

$$L_p(\Xi; s) = \prod_{j=1}^{\nu} \{(1 - \lambda_j(p)\, p^{-s})(1 - \lambda_j(p)^{-1}\, p^{-s})\}^{-1} A_{\chi,p}(s),$$

where $A_{\chi,p}(s)$ is the modification factor given by [4, Formula (1.18)]. We only note that if $M_{\mathbb{Z}_p} = M^\vee_{\mathbb{Z}_p}$ then E_p is trivial and the factor $A_{\chi,p}(s)$ is 1; thus, $L_p(\Xi; s)$ agrees with the local standard L-factor pertaining to the embedding of the dual group

$${}^L SO_M = \begin{cases} \mathrm{Sp}(\nu; \mathbb{C}) & (\mathrm{rk} M = 2\nu + 1) \\ \mathrm{SO}(2\nu) & (\mathrm{rk} M = 2\nu) \end{cases} \hookrightarrow \mathrm{GL}(2\nu, \mathbb{C}).$$

2.2 Euler products

Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and $\mathbb{A}_\mathbf{f}$ the finite adele ring of $\mathbb{Q}$. A general point of $O_M(\mathbb{R})$ (resp. $O_M(\mathbb{A}_\mathbf{f})$) is often denoted by g_∞ (resp. $g_\mathbf{f}$). When a point g of the adele group $O_M(\mathbb{A})$ is given, the symbols g_∞ and $g_\mathbf{f}$ are used to denote its archimedean component and its finite component, respectively; thus $g = g_\infty\, g_\mathbf{f}$, $g_\infty \in O_M(\mathbb{R})$ and $g_\mathbf{f} \in O_M(\mathbb{A}_\mathbf{f})$.

Let

$$F : O_M(\mathbb{Q}) \backslash O_M(\mathbb{A}) / \prod_{p:\text{primes}} \mathbf{K}^*_{M,p} \to \mathbb{C}$$

be an L^2-automorphic form such that, for any prime p, there exists a character Ξ_p^F of $\mathcal{H}^+_{M,p}$ such that

$$\int_{O_M(\mathbb{Q}_p)} F(hg_p)\, \varphi(g_p)\, \mathrm{d}g_p = \Xi_p^F(\varphi)\, F(h) \quad \text{for all } \varphi \in \mathcal{H}^+_{M,p}.$$

Define

$$L(F, s) := \prod_{p:\text{primes}} L(\Xi_p^F, s), \quad \mathrm{Re}(s) \gg 0.$$

Actually, the infinite product converges absolutely on the half plane $\mathrm{Re}(s) > \mathrm{rk}(M)/2$.

2.2.1 The definite case

Suppose M is positive definite. Thus,

$$\sharp[O_M(\mathbb{Q}) \backslash O_M(\mathbb{A}_\mathbf{f}) / \prod_{p:\text{primes}} \mathbf{K}^*_{M,p}] < +\infty.$$

Let f be a function on this double coset space satisfying a joint eigenequation for the action of $\mathcal{H}^+_{M,p}$ for all p. The Euler product $L(f,s)$ is defined as above. Let us define the gamma factor by

$$\Gamma_M(s) = \prod_{j=1}^{[n/2]} \Gamma_{\mathbb{C}}(s-j+n/2) \begin{cases} \mathfrak{d}(M)^{s/2} & n: \text{even}, \\ (2^{-1}\mathfrak{d}(M))^{s/2} & n: \text{odd}, \end{cases}$$

where $n = \mathrm{rk}(M)$.

Theorem 1 ([3], [4])**.** *The completed L-function*

$$\Lambda(f,s) = \Gamma_M(s)\, L(f,s)$$

has a meromorphic continuation to $\mathbb{C}$ satisfying the functional equation $\Lambda(f, 1-s) = \Lambda(f,s)$. It is holomorphic away from the possible simple poles at

$$s = \frac{n}{2} - j \quad (0 \leqslant j \leqslant n-1).$$

Moreover, $\Lambda(f,s)$ has a pole at $s = n/2$ if and only if f is a constant function.

3 Cusp forms on $O(2,m)$ and their L-functions

We refer to [6] and [5].

Let m be an integer with $m \geqslant 3$ and $L \cong \mathbb{Z}^{m+2}$ a lattice of signature $(2-, m+)$ satisfying

(i) L is even and is maximal.

(ii) L admits the orthogonal splitting

$$L = \mathbb{Z}\,\varepsilon_1 \oplus L_1 \oplus \mathbb{Z}\,\varepsilon_1', \quad L_1 = \mathbb{Z}\,\varepsilon_0 \oplus L_0 \oplus \mathbb{Z}\,\varepsilon_0'$$

with ε_j, ε_j' isotropic vectors such that $(\varepsilon_j, \varepsilon_j') = 1$. Thus, L_0 is positive definite.

Let

$$\tilde{\mathcal{D}} = \{\mathfrak{z} \in L_{1,\mathbb{C}} |\, Q[\mathrm{Im}(\mathfrak{z})] < 0\,\}.$$

For $(\mathfrak{z}, g) \in \tilde{\mathcal{D}} \times O_L(\mathbb{R})$, define $g\langle \mathfrak{z} \rangle \in \tilde{\mathcal{D}}$ and $J(g,\mathfrak{z}) \in \mathbb{C}^*$ by the relation

$$g\,P(\mathfrak{z}) = J(g,\mathfrak{z})\, P(g\langle\mathfrak{z}\rangle),$$

where

$$P(\mathfrak{z}) = (-2^{-1}\,Q[\mathfrak{z}])\,\varepsilon_1 + \mathfrak{z} + \varepsilon_1' \in L_{\mathbb{C}}.$$

Choose a vector $\eta_0^- \in L_{1,\mathbb{R}}$ such that $Q[\eta_0^-] = -1$ and set $\mathfrak{z}_0 = \sqrt{2}\eta_0^-/i$, which belongs to $\tilde{\mathcal{D}}$. Let $\mathcal{D}$ be the connected component of $\tilde{\mathcal{D}}$ containing $\mathfrak{z}_0$. Then, the identity component G of $O_L(\mathbb{R})$ acts on $\mathcal{D}$ transitively by the mapping $(g,\mathfrak{z}) \mapsto g\langle\mathfrak{z}\rangle$, and $J(g,\mathfrak{z})$ satisfies the condition of automorphy factor :

$$J(gg', \mathfrak{z}) = J(g, g'\langle\mathfrak{z}\rangle)\, J(g', \mathfrak{z}).$$

Let $\mathbf{K}_\infty$ be the stabilizer of $\mathfrak{z}_0$ in G; then, it is a maximal compact subgroup of G and

$$G/\mathbf{K}_\infty \cong \mathcal{D}, \quad g\mathbf{K}_\infty \mapsto g\langle\mathfrak{z}_0\rangle.$$

We fix a G-invariant Kähler structure on $\mathcal{D}$ by the 2-form $2^{-1}\sqrt{-1}\,\partial\bar{\partial}\,Q[\mathrm{Im}(\mathfrak{z})]$. Let $\mathrm{d}\mu_{\mathcal{D}}$ be the G-invariant measure on $\mathcal{D}$ associated with the Kähler volume form, which in turn yields a Haar measure $\mathrm{d}g_\infty$ on $O_L(\mathbb{R})$ so that the quotient of $\mathrm{d}g_\infty$ by the probability Haar measure on $\mathbf{K}_\infty$ corresponds to $\mathrm{d}\mu_{\mathcal{D}}$ on $\mathcal{D} \cong G/\mathbf{K}_\infty$. We endow the adele group $O_L(\mathbb{A}) = O_L(\mathbb{R}) \times O_L(\mathbb{A}_\mathbf{f})$ with the product measure $\mathrm{d}g = \mathrm{d}g_\infty \otimes \mathrm{d}g_\mathbf{f}$, where $\mathrm{d}g_\mathbf{f}$ is the Haar measure on the finite adele group $O_L(\mathbb{A}_\mathbf{f})$ such that $\mathrm{vol}(\mathbf{K}^*_{L,\mathbf{f}}) = 1$, where $\mathbf{K}^*_{L,\mathbf{f}} = \prod_p \mathbf{K}^*_{L,p}$.

3.1 Cusp forms on the adele group

Let $l \in \mathbb{N}$. A function $F : O_L(\mathbb{A}) \to \mathbb{C}$ is called a holomorphic cusp form of weight l if it satisfies the conditions:

(i) $F(\gamma g k_{\mathbf{f}}) = F(g)$ for all $(\gamma, g, k_{\mathbf{f}}) \in O_L(\mathbb{Q}) \times O_L(\mathbb{A}) \times \mathbf{K}^*_{L,\mathbf{f}}$.

(ii) For any $g_{\mathbf{f}} \in O_L(\mathbb{A}_{\mathbf{f}})$, the function

$$G \ni g_\infty \to F(g_\infty g_{\mathbf{f}})\, J(g_\infty, \mathfrak{z}_0)^l$$

factors through a holomorphic function on the domain $G/\mathbf{K}_\infty \cong \mathcal{D}$.

(iii) $F(g)$ is bounded on $O_L(\mathbb{A})$.

Let $\mathfrak{S}_l$ be the space of all the holomorphic cusp forms of weight l. We define the inner-product of $\mathfrak{S}_l$ by

$$\langle F_1 | F_2 \rangle := \int_{O_L(\mathbb{Q}) \backslash O_L(\mathbb{A})} F_1(g)\overline{F_2(g)}\, \mathrm{d}g.$$

Then, it is known that $\mathfrak{S}_l$ is a finite dimensional and

$$\dim_{\mathbb{C}} \mathfrak{S}_l \asymp l^m, \quad (l \to +\infty)$$

from the Hirzebruch-Mumford proportionality theorem.

3.2 Fourier expansion

Given $g_{\mathbf{f}} \in O_L(\mathbb{A}_{\mathbf{f}})$, there exists a $\mathbb{Z}$-lattice $L_1(g_{\mathbf{f}}) \subset L_{1,\mathbb{Q}}$ such that for any $F \in \mathfrak{S}_l$,

$$F(g_{\mathbf{f}}\, g_\infty) = \sum_{\eta \in L_1(g_{\mathbf{f}}) \cap \sqrt{-1}\mathcal{D}} a_F(g_{\mathbf{f}};\, \eta)\, \mathcal{W}_l^\eta(g_\infty), \quad g_\infty \in G,$$

where $a_F(g_{\mathbf{f}};\, \eta) \in \mathbb{C}$ are the Fourier coefficients and

$$\mathcal{W}_l^\eta(g_\infty) = J(g_\infty, \mathfrak{z}_0)^{-l} \exp(2\pi i(\eta, g_\infty\langle \mathfrak{z}_0 \rangle))$$

is the archimedean Whittaker function.

3.3 Averaged Fourier coefficients

From now on, we *fix* $\xi \in L_{1,\mathbb{Q}}$ satisfying the following conditions.

(1) (signature condition) $\xi \in \sqrt{-1}\mathcal{D}$, in particular $Q[\xi] < 0$.

(2) (primitivity) ξ is a primitive vector in the dual lattice $L_1^\vee$.

(3) (maximality) $L_1^\xi := L_1 \cap \xi^\perp$ is maximal in $L_{1,\mathbb{Q}}^\xi$.

Set

$$O_{L_1} = \mathrm{Stab}_{O_L}(\varepsilon_1, \varepsilon_1'), \quad O_L^\xi = \mathrm{Stab}_{O_L}(\xi), \quad O_{L_1}^\xi = O_L^\xi \cap O_{L_1}.$$

Since L_1^ξ is positive definite, the orthogonal group $O_{L_1}^\xi$ is anisotropic.

Let

$$\mathbf{K}^*_{L_1^\xi,\mathbf{f}} = \prod_{p:\text{primes}} \mathbf{K}^*_{L_1^\xi,p},$$

and

$$f : O_{L_1}^\xi(\mathbb{Q}) \backslash O_{L_1}^\xi(\mathbb{A}_{\mathbf{f}}) / \mathbf{K}^*_{L_1^\xi,\mathbf{f}} \to \mathbb{C}$$

be a joint eigenfunction of the Hecke algebras $\mathcal{H}^+_{L_1^\xi,p}$ for all p.

3.3.1

For $F \in \mathfrak{S}_l$ with adelic Fourier coefficients $a_F(g_{\mathbf{f}}; \eta)$, we call the following quantity the *Fourier-Bessel coefficient* associated with (ξ, f) (which was introduced in [6] without being called so).

$$a_F^f(\xi) = \mu_\xi{}^{-1} \sum_{j=1}^{h} \frac{f(u_j)}{e_\xi(j)}\, a_F(u_j;\, \xi),$$

where $u_j \in O_{L_1}^{\xi}(\mathbb{A}_{\mathbf{f}})$ $(1 \leqslant j \leqslant h)$ is a complete set of representatives for the double coset space $O_{L_1}^{\xi}(\mathbb{Q})\backslash O_{L_1}^{\xi}(\mathbb{A}_{\mathbf{f}})/\mathbf{K}^*_{L_1^{\xi},\mathbf{f}}$ and we set

$$e_\xi(j) = \sharp[O_{L_1}^{\xi}(\mathbb{Q}) \cap u_j\, \mathbf{K}^*_{L_1^{\xi},\mathbf{f}}\, u_j^{-1}], \quad \mu_\xi = \sum_{j=1}^{h} (1/e_\xi(j)).$$

3.3.2

Another way to define $a_F^f(\xi)$ is as follows. For $g_\infty \in G$, consider the integral

$$\int_{O_{L_1}^{\xi}(\mathbb{Q})\backslash O_{L_1}^{\xi}(\mathbb{A}_{\mathbf{f}})} f(h)\, \mathrm{d}h \int_{N_L(\mathbb{Q})\backslash N_L(\mathbb{A})} F(n\, h\, g_\infty b_\infty^{\xi}) \psi_\xi(n)\, \mathrm{d}n \tag{3.1}$$

where $P_L = \mathrm{Stab}_{O_L}(\mathbb{Q}\varepsilon_1)$ the maximal parabolic subgroup of O_L with the unipotent radical N_L and ψ_ξ is the character of $N_L(\mathbb{A})$ determined by ξ. Then, (3.1), regarded as a function in g_∞, is proportional to $\mathcal{W}_l^{\xi}(g_\infty)$. The proportionality constant is nothing but $a_F^f(\xi)$.

3.4 L-functions

Let (ξ, f) be as in 3.3. Let P^ξ be the $\mathbb{Q}$-parabolic subgroup of O_L^{ξ} which stabilizes the isotropic line $\mathbb{Q}\varepsilon_1$ and N^ξ the unipotent radical of P^ξ. For $(t, h) \in \mathrm{GL}(1) \times O_{L_1}^{\xi}$, let $\mathsf{m}(t;\, h)$ be the element of O_L^{ξ} such that

$$\mathsf{m}(t;\, h)\, \varepsilon_1 = t\, \varepsilon_1, \quad \mathsf{m}(t;\, h)\, \varepsilon_1' = t^{-1}\, \varepsilon_1', \quad \mathsf{m}(t;\, h)|L_1^{\xi} = h.$$

Then, $M^\xi = \{\mathsf{m}(t;\, h) |\, t \in \mathrm{GL}(1),\, h \in O_{L_1}^{\xi}\}$ is a Levi subgroup of P^ξ. Fix an element $b_\infty^{\xi} \in O_{L_1}(\mathbb{R})^0$ such that $b_\infty^{\xi}\, \eta_0^- = |Q[\xi]|^{-1/2}\, \xi$ and define $\mathbf{K}_\infty^{\xi} = [b_\infty^{\xi} \mathbf{K}_\infty b_\infty^{\xi}{}^{-1}] \cap O_L^{\xi}(\mathbb{R})$. Then, we have the Iwasawa decomposition $O_L^{\xi}(\mathbb{A}) = P^\xi(\mathbb{A})\, \mathbf{K}^*_{L^\xi,\mathbf{f}}\, \mathbf{K}_\infty^{\xi}$. For $s \in \mathbb{C}$, let us define a function $\mathsf{f}^{(s)}$ on $O_L^{\xi}(\mathbb{A})$ by

$$\mathsf{f}^{(s)}(\mathsf{m}(t;\, h)\, n\, k) = |t|_{\mathbb{A}}^{s+\rho} f(h), \quad \mathsf{m}(t;\, h)\, n \in P^\xi(\mathbb{A}), \quad k \in \mathbf{K}^*_{L^\xi,\mathbf{f}}\, \mathbf{K}_\infty^{\xi}$$

and consider the corresponding Eisenstein series on $O_L^{\xi}(\mathbb{A})$:

$$E_{P^\xi}^{O_L^{\xi}}(f, s;\, h) = \sum_{\gamma \in P^\xi(\mathbb{Q})\backslash O_L^{\xi}(\mathbb{Q})} \mathsf{f}^{(s)}(\gamma\, h), \quad h \in O_L^{\xi}(\mathbb{A}), \quad \mathrm{Re}(s) > (m-1)/2.$$

As a function in s, this has a meromorphic continuation to the whole complex plane in such a way that, at its regular point s, the function $E_{P^\xi}^{O_L^{\xi}}(f, s;\, h)$ in h is an automorphic form on

$O_L^\xi(\mathbb{A})$. A precise functional equation relating $E_{P^\xi}^{O_L^\xi}(f,s;\,h)$ and $E_{P^\xi}^{O_L^\xi}(f,-s;\,h)$ is proved in [4].

Let $F \in \mathfrak{S}_l$ be a joint eigenfunction of the Hecke algebras $\mathcal{H}_{L,p}^+$ for all primes p. As we explained in 2.1, we have the Euler product $L(F,s)$. The complete L-function $\Lambda(F,s)$ is defined by

$$\Lambda(F,s) := \Gamma_L(l,s)\, L(F,s).$$

with the gamma factor

$$\Gamma_L(l,s) := \Gamma_{\mathbb{C}}(s - m/2 + l) \prod_{j=1}^{[m/2]} \Gamma_{\mathbb{C}}(s + m/2 - j) \times \begin{cases} \mathfrak{d}(L)^{s/2} & m : \text{even}, \\ (2^{-1}\mathfrak{d}(L))^{s/2} & m : \text{odd}. \end{cases}$$

Theorem 2 ([6], [5])**.** *Let $F \in \mathfrak{S}_l$ and $E_{P^\xi}^{O_L^\xi}(f,s;h)$ be the Eisenstein series on $O_L^\xi(\mathbb{A})$ induced from f. Fix a Haar measure* dh *on $O_L^\xi(\mathbb{A})$. Then, for s with sufficiently large* Re(s)*,*

$$\begin{aligned} Z_F^f(s) &:= \int_{O_L^\xi(\mathbb{Q})\backslash O_L^\xi(\mathbb{A})} E_{P^\xi}^{O_L^\xi}(f, s-1/2; h)\, F(h\, b_\infty^\xi)\, \mathrm{d}h \\ &= C_l^\xi\, a_F^f(\xi)\, \frac{\Lambda(F,s)}{\Lambda(f, s+1/2)} \times \begin{cases} 1 & m: \text{odd}, \\ \hat{\zeta}(2s) & m: \text{even}. \end{cases} \end{aligned}$$

where C_l^ξ is a positive constant depending on the choice of a Haar measure dh *on $O_L^\xi(\mathbb{A})$. If $a_F^f(\xi) \neq 0$ for some (ξ, f), then $\Lambda(F,s)$ has a meromorphic continuation to $\mathbb{C}$ satisfying the functional equation $\Lambda(F, 1-s) = \Lambda(F,s)$ with possible poles only at $s = m/2 - j$ $(0 \leqslant j \leqslant m-1)$. In particular, $L(F,s)$ is regular at $s = 1/2$.*

4 Results

We keep the notation introduced in the previous section.

4.1 Polynomial bounds

For any $\epsilon > 0$ and for any interval $I \subset \mathbb{R}$, set

$$\mathcal{T}_{\epsilon,I} = \{s \in \mathbb{C} \mid \mathrm{Re}(s) \in I,\ |\mathrm{Im}(s)| \geqslant \epsilon\}.$$

Recall that a meromorphic function $\phi(s)$ on $\mathbb{C}$ holomorphic away from the real axis is said to be bounded in vertical strips of finite width if $|\phi(s)|$ is bounded on the set $\mathcal{T}_{\epsilon,I}$ for any compact interval I and for any $\epsilon > 0$.

Proposition 3 ([7])**.** *Let ξ and f be as in 3.3, i.e., a simultaneous eigenform on $O_{L_1}^\xi(\mathbb{Q})\backslash O_{L_1}^\xi(\mathbb{A}_\mathbf{f})/\mathbf{K}^*_{L_1^\xi,\mathbf{f}}$ of the Hecke algebras $\mathcal{H}^+_{L_1^\xi,p}$ for all primes p.*

(1) *The completed L-function $\Lambda(f,s)$ is bounded in vertical strips of finite width.*

(2) *For any $h \in O_{L_1}^\xi(\mathbb{A})$, the normalized Eisenstein series $\Lambda^*(f,-s)\, E_{P^\xi}^{O_L^\xi}(f,s;\,h)$ is bounded in vertical strips of finite width.*

Proposition 4 ([7])**.** *Let f be as in Proposition 3. For any compact interval I and for any $\epsilon > 0$, there exists $N > 0$ such that the following estimation holds.*

$$|L(f,s)| \ll |\mathrm{Im}(s)|^N, \quad s \in T_{\epsilon,I}.$$

4.2 Rescaled Fourier-Bessel coefficients

Let (ξ, f) be as in 3.3. For $F \in \mathfrak{S}_l$, let us introduce the rescaled Fourier-Bessel coefficient for (ξ, f) by

$$\mathfrak{a}_F^f(\xi) := \frac{(4\pi\sqrt{-2Q[\xi]^2})^{l-\rho-1/2}\,\Gamma(2l-\rho)^{1/2}\,a_F^f(\xi)}{\|f\|\,\|F\|},$$

where $\rho = (m-1)/2$ and

$$\|f\|^2 = \mu_\xi^{-1}\sum_{j=1}^{h}\frac{|f(u_j)|^2}{e_\xi(j)}, \quad \|F\|^2 = \int_{O_L(\mathbb{Q})\backslash O_L(\mathbb{A})} |F(g)|^2\,\mathrm{d}g.$$

The following is regarded as an analogue of the asymptotic orthogonality relation for Fourier coefficients recalled in the introduction.

Theorem 5 ([8])**.** *Let $\mathcal{B}_l$ be an orthonormal basis consisting of Hecke eigenforms in $\mathfrak{S}_l$. Then,*

$$\lim_{l\to+\infty}\frac{1}{2\,l^m}\sum_{F\in\mathcal{B}_l}|\mathfrak{a}_F^f(\xi)|^2 = (2^{-1}\mathfrak{d}(L))^{-1/2}\left(\frac{\pi}{4}\right)^{-\rho}.$$

4.3 The Limit formula

We consider the limiting behavior as $l \to +\infty$ for the sum of the weighted central L-value $L(F, 1/2)\,|\mathfrak{a}_F^f(\xi)|^2$ over $F \in \mathcal{B}_l$ and prove the following asymptotic formula, which is our main theorem.

Theorem 6 ([9])**.** *Let $\mathcal{B}_l$ be an orthonormal basis consisting of Hecke eigenforms in $\mathfrak{S}_l$. For any $\kappa > 0$, we have*

$$\frac{\mathbf{\Gamma}(l)}{4l^m}\sum_{F\in\mathcal{B}_l} L(F,1/2)\,|\mathfrak{a}_F^f(\xi)|^2 = c_L(\xi,f) + O(l^{-\kappa}), \quad (l\to+\infty),$$

where

$$\mathbf{\Gamma}(l) = \frac{l^m\,\Gamma(l-\rho)^2\,\Gamma(l-\rho-1/2)}{\Gamma(l)\,\Gamma(l-\rho/2)\,\Gamma(l-\rho/2+1/2)},$$

$$c_L(\xi,f) = (2^{-1}\mathfrak{d}(L))^{-1/2}\left(\frac{\pi}{4}\right)^{-\rho}\times\begin{cases}\mathrm{CT}_{s=1}L(f,s), & m:\ \textit{odd},\\ L'(f,1) - d_L(\xi)\,L(f,1), & m:\ \textit{even},\end{cases}$$

$$d_L(\xi) = \frac{-1}{2}\log(2^{-1}\mathfrak{d}(L_1^\xi)) + \left(\frac{m}{2}-1\right)\log(2\pi) - \sum_{j=1}^{m/2-1}\frac{\Gamma'}{\Gamma}\left(\frac{m+1}{2}-j\right)$$

with $\mathfrak{d}(L)$ the Gram determinant of a $\mathbb{Z}$-basis of L and $L(f,s)$ the standard L-function of f defined by Murase and Sugano.

In the theorem, $\mathrm{CT}_{s=1}L(f,s)$ means the constant term of the Laurent expansion at $s = 1$. Since $\mathbf{\Gamma}(l) = 1 + O(l^{-1})$, we obtain the following corollary.

Corollary 7.

$$\lim_{l\to+\infty}\frac{1}{4l^m}\sum_{F\in\mathcal{B}_l} L(F,1/2)\,|\mathfrak{a}_F^f(\xi)|^2 = c_L(\xi,f).$$

We remark that $L(f,s)$ has a possible simple pole at $s=1$ when m is odd but is holomorphic at $s=1$ when m is even.

Corollary 8. *Suppose that m is odd and that there exists (ξ, f) as above such that $\mathrm{CT}_{s=1} L(f,s) \neq 0$. Then, there exist infinitely many linearly independent holomorphic Hecke eigenforms F belonging to some $\mathfrak{S}_l$ such that $L(F,1/2) \neq 0$ and $a_F^f(\xi) \neq 0$.*

Questions and remarks. (1) Is the value $L(F,1/2)$ positive (or non-negative) for $F \in \mathfrak{S}_l$? (*cf.* If O_M is the split orthogonal group of odd size and π a *generic* cuspidal automorphic representation of O_M, the non-negativity of $L(\pi,1/2)$ is known by [Lapid and Rallis, Ann. Math. **157** (2003), 891–917].)
(2) Is the value $L(f,1)$ non zero if m is odd ? If $m=3, 5$, then $O_{L_1}^{\xi} = O(2), O(4)$ and $L(f,s)$ is (essentially) the Hecke's L for a grossencharacter or the Jacquet-Langlands' L for an automorphic form on an inner form of GL(2), respectively. Thus, $L(f,1) \neq 0$ is known for both. (*cf.* For *generic* cusp forms π, $L(\pi,1) \neq 0$ is widely proved by Shahidi (see [Gelbart and Lapid, Amer. J. Math **128** (2006), 619–638] also).)

5 The proof of Theorem 6

For details, we refer to [9], whose contents are briefly reviewed in this section.

5.1 Ingredients of the proof of Theorem 6

There are three major ingredients:

- An integral representation of the standard L-functions by Murase and Sugano, which was already recalled in 3.4.
- Construction of an adelic Poincaré series (see 5.5); as its seed, we use the "Shintani functions" (at the archimedean place, explained in 5.2) and the "flat extension" of f (at the finite places, explained in 5.3).
- To compute (f,ξ)-th Fourier-Bessel coefficient of the Poincaré series thus constructed, in two ways (as explained in 5.6 and 5.7); this will give us a kind of summation formula (5.1) which equats the weighted average of L-values with a certain geometric expression arising from the double cosets $P^{\xi}(\mathbb{Q})\,\gamma\, P_L(\mathbb{Q})$.

Besides these, we need the results in 4.1 and 4.2 for the proof. From now on, we fix ξ as in 3.3 and set $\Delta = |Q[\xi]|$.

5.2 The archimedean component (the Shintani function)

Define $\Phi_l^{\xi}(s) : O_L(\mathbb{R}) \to \mathbb{C}$ by setting

$$\Phi_l^{\xi}(s;\, g_\infty) := J(g_\infty, \mathfrak{z}_0)^{-l}\, 2^{-(s+\rho)/2} \left(\frac{(\xi, g_\infty\langle \mathfrak{z}_0 \rangle)}{i\, \Delta^{1/2}} \right)^{s+\rho-l}$$

for $g \in O_L(\mathbb{R})^+$ and $\Phi_l^{\xi}(s;\, g_\infty) = 0$ for $g_\infty \in O_L(\mathbb{R}) - O_L(\mathbb{R})^+$, where $O_L(\mathbb{R})^+$ is the index two subgroup of $O_L(\mathbb{R})$ which stabilizes the connected component $\mathcal{D}$. The function $\Phi_l^{\xi}(s;\, g_\infty)$ corresponds to an $O_L^{\xi}(\mathbb{R}) \times O_L(\mathbb{R})$-intertwining operator

$$T : \mathrm{Ind}_{P^{\xi}(\mathbb{R})}^{O_L^{\xi}(\mathbb{R})}(|\,|^s) \boxtimes D_l \longrightarrow C^\infty(O_L(\mathbb{R}))$$

where $\mathrm{Ind}_{P^\xi(\mathbb{R})}^{O_L^\xi(\mathbb{R})}(|\,|^s)$ is the principal series representation of $O_L^\xi(\mathbb{R}) \cong O(m,1)$ and D_l is the holomorphic discrete series representation of weight l of $O_L(\mathbb{R}) \cong O(m,2)$.

5.3 The non archimedean component

Define $\Phi_{\mathbf{f}}^{f,\xi}(s) : O_L(\mathbb{A}_{\mathbf{f}}) \to \mathbb{C}$ by

$$\Phi_{\mathbf{f}}^{f,\xi}(s;\, g_{\mathbf{f}}) = \begin{cases} \mathsf{f}^{(s)}(h_{\mathbf{f}}), & \text{if } g_{\mathbf{f}} = h_{\mathbf{f}}\, k_{\mathbf{f}} \in O_L^\xi(\mathbb{A}_{\mathbf{f}})\,\mathbf{K}_{L,\mathbf{f}}^*, \\ 0, & \text{if } g_{\mathbf{f}} \notin O_L^\xi(\mathbb{A}_{\mathbf{f}})\,\mathbf{K}_{L,\mathbf{f}}^*, \end{cases}$$

where $\mathsf{f}^{(s)}$ is as in 3.4. The well-definedness is guaranteed by the relation $O_L^\xi(\mathbb{A}_{\mathbf{f}}) \cap \mathbf{K}_{L,\mathbf{f}}^* = \mathbf{K}_{L^\xi,\mathbf{f}}^*$ ([4, (2.11)]).

5.4 An adelic function

For $g \in O_L(\mathbb{A})$ and $\mathrm{Re}(z) > 0$, define

$$\widehat{\mathbf{\Phi}}_{l,\mathbb{A}}^{f,\xi}(\beta, z; g) = \frac{1}{2\pi i}\int_{(\sigma)} \frac{\beta(s)}{z-s}\, D_*(s)\, \Lambda^*(f,-s)\, \Phi_l^\xi(s;\, g_\infty)\, \Phi_{\mathbf{f}}^{f,\xi}(s;\, g_{\mathbf{f}})\, \mathrm{d}s,$$

where (σ) denotes the contour $\mathrm{Re}(s) = \sigma$ such that $0 < \sigma < \mathrm{Re}(z)$,

$$\Lambda^*(f,s) = \Lambda(f,s)\, \times \begin{cases} 1 & m : \text{odd}, \\ \hat{\zeta}(2s) & m : \text{even}, \end{cases}$$

$$D_*(s) = \prod_{\substack{0 \leqslant j \leqslant m-1 \\ j \neq \rho}} (s - \rho + j)$$

and $\beta(s)$ is an entire function on $\mathbb{C}$ satisfying that for any compact interval $I \subset \mathbb{R}$ and for any $N > 0$, the estimation $|\beta(s)| \ll (1 + |\mathrm{Im}(s)|)^{-N}$ holds on the stripe $\mathrm{Re}(s) \in I$.

5.5 An adelic Poincaré series

Let β be as in 5.4. For $\mathrm{Re}(z) \in (\rho, l - 3\rho - 1)$, define $\hat{\mathbb{F}}_l^{\xi,f}(\beta, z) : O_L(\mathbb{Q})\backslash O_L(\mathbb{A}) \to \mathbb{C}$ by

$$\hat{\mathbb{F}}_l^{\xi,f}(\beta, z;\, g) = \sum_{\gamma \in P^\xi(\mathbb{Q})\backslash O_L(\mathbb{Q})} \widehat{\mathbf{\Phi}}_{l,\mathbb{A}}^{f,\xi}(\beta, z; \gamma g), \quad g \in O_L(\mathbb{A}).$$

Proposition 9. (i) *The series converges absolutely if* $\mathrm{Re}(z) \in (\rho, l - 3\rho - 1)$.
(ii) *There exists a positive integer* $l_0\, (> 3\rho + 1)$ *such that* $\hat{\mathbb{F}}_l^{\xi,f}(\beta, z) \in \mathfrak{S}_l$ *if* $l \geqslant l_0$ *and* $\mathrm{Re}(z) \in (\rho, l - 3\rho - 1)$. *For any* $g \in O_L(\mathbb{A})$, *the holomorphic function* $z \mapsto \hat{\mathbb{F}}_l^{\xi,f}(\beta, z;\, g)$ *defined on the stripe* $\mathrm{Re}(z) \in (\rho, l - 3\rho - 1)$ *has a holomorphic continuation to the whole complex plane satisfying the functional equation*

$$\hat{\mathbb{F}}_l^{\xi,f}(\beta, z;\, g) + \hat{\mathbb{F}}_l^{\xi,f}(\beta, -z;\, g) = \frac{-\pi^{m/2}\,\Gamma(l - m/2)}{\Gamma(l)}\, \beta(z)\, D_*(z)\, C_l^\xi \sum_{F \in \mathcal{B}_l} \Lambda^*(f,-z)\, Z_{\bar{F}}^f(z)\, F(g)$$

with the same constant C_l^ξ *occurred in Proposition 2.*
(iii) *The value at* $z = 0$ *of* $\hat{\mathbb{F}}_l^{\xi,f}(\beta, z)$ *with* $l \geqslant l_0$ *is equals to*

$$\beta(0) \times \frac{-\pi^{m/2}\,\Gamma(l - m/2)}{2\,\Gamma(l)}\, C_l^\xi \sum_{F \in \mathcal{B}_l} L(\bar{F}, 1/2)\, a_{\bar{F}}^f(\xi)\, F(g).$$

(iv) *For any compact set* $\mathcal{U} \subset O_L(\mathbb{A})$ *and for any* $\epsilon > 0$,

$$|\hat{\mathbb{F}}_l^{\xi,f}(\beta, z;\, g)| \ll \exp(|z|^{1+\epsilon}), \quad z \in \mathbb{C},\, g \in \mathcal{U}.$$

In particular, the entire function $z \mapsto \hat{\mathbb{F}}_l^{\xi,f}(\beta, z;\, g)$ *is of order* 1.

5.6 Fourier-Bessel coefficients of the Poincare series (the spectral side)

Since the integration domain of (3.1) is compact, it follows from Proposition 9 that the Fourier-Bessel coefficient $a^{\bar{f}}_{\hat{\mathbb{F}}_l^{f,\xi}(\beta,z)}(\xi)$ is holomorphic in $z \in \mathbb{C}$. Its value at $z = 0$ is computed as follows.

Proposition 10. *For any* $l \geqslant l_0$,

$$\mathrm{CT}_{z=0}\, a^{\bar{f}}_{\hat{\mathbb{F}}_l^{f,\xi}(\beta,z)}(\xi) = \beta(0) \times \frac{-\pi^m\, \Gamma(l - m/2)}{2\,\Gamma(l)}\, C_l^{\xi} \sum_{F \in \mathcal{B}_l} L(\bar{F}, 1/2)\, |a_f^{\bar{F}}(\xi)|^2$$

5.7 Fourier-Bessel coefficients of the Poincaré series (the geometric side)

Let w_0, $\bar{\mathsf{n}}(\varepsilon)$, $\bar{\mathsf{n}}(\xi) \in O_L(\mathbb{Q})$ be the elements defined by

$$\begin{aligned} &w_0\, \varepsilon_1 = -\varepsilon_1', && w_0(X) = X, && w_0\, \varepsilon_1' = -\varepsilon_1, \\ &\bar{\mathsf{n}}(\varepsilon)\, \varepsilon_1 = \varepsilon_1 + \varepsilon, && \bar{\mathsf{n}}(\varepsilon)\, X = X - (X, \varepsilon)\, e_1', && \bar{\mathsf{n}}(\varepsilon)\, \varepsilon_1' = \varepsilon_1', \\ &\bar{\mathsf{n}}(\xi)\, \varepsilon_1 = \varepsilon_1 + \xi - 2^{-1}Q[\xi]\varepsilon_1', && \bar{\mathsf{n}}(\xi)\, X = X - (X, \xi)\, \varepsilon_1', && \bar{\mathsf{n}}(\xi)\, \varepsilon_1' = \varepsilon_1', \end{aligned}$$

where $X \in L_{1,\mathbb{Q}}$ is arbitrary and $\varepsilon = Q[\xi]\, \varepsilon_0$.

Lemma 11. (i) *The double coset space* $P^{\xi}(\mathbb{Q})\backslash O_L(\mathbb{Q})/P_L(\mathbb{Q})$ *has 4 elements represented by the rational points* 1, w_0, $\bar{\mathsf{n}}(\varepsilon)$ *and* $\bar{\mathsf{n}}(\xi)$.
(ii) *For each* $\nu_0 \in \{1, w_0, \bar{\mathsf{n}}(\varepsilon), \bar{\mathsf{n}}(\xi)\}$, *set*

$$\mathfrak{X}(\nu_0) = \begin{cases} \{\mathsf{m}(1;\, \delta) \,|\, \delta \in O_{L_1}^{\xi}(\mathbb{Q})\backslash O_{L_1}(\mathbb{Q})\,\} & (\nu_0 \in \{1, w_0\}), \\ \{\mathsf{m}(\tau;\, \delta) \,|\, \tau \in \mathbb{Q}^{\times},\, \delta \in P_{L_1}(\mathbb{Q})\backslash O_{L_1}(\mathbb{Q})\,\} & (\nu_0 = \bar{\mathsf{n}}(\varepsilon)), \\ \{\mathsf{m}(\tau;\, \delta) \,|\, \tau \in \mathbb{Q}^{\times},\, \delta \in O_{L_1}^{\xi}(\mathbb{Q})\backslash O_{L_1}(\mathbb{Q})\,\} & (\nu_0 = \bar{\mathsf{n}}(\xi)), \end{cases}$$

where P_{L_1} *denotes the maximal parabolic subgroup of* O_{L_1} *stabilizing the isotropic line* $\mathbb{Q}\,\varepsilon_0$. *Then*

$$P^{\xi}(\mathbb{Q})\backslash[P^{\xi}(\mathbb{Q})\nu_0 P_L(\mathbb{Q})] = \bigsqcup_{\mu \in \mathfrak{X}(\nu_0)} \nu_0\mu \cdot [N_{L,\mu}(\mathbb{Q})\backslash N_L(\mathbb{Q})]$$

with $N_{L,\mu}(\mathbb{Q}) = N_L(\mathbb{Q}) \cap (\nu_0\mu)^{-1} P^{\xi}(\mathbb{Q})\, (\nu_0\mu)$.

By this lemma and from the definition of the Fourier-Bessel coefficients given in 3.3.2, $a^{\bar{f}}_{\hat{\mathbb{F}}_l^{f,\xi}(\beta,z)}(\xi)$ with $\mathrm{Re}(z) \in (\rho, l - 3\rho - 1)$ is written as a sum of 4 terms:

$$(\hat{\mathbb{I}}_1 + \hat{\mathbb{I}}_{w_0} + \hat{\mathbb{I}}_{\bar{\mathsf{n}}(\varepsilon)} + \hat{\mathbb{I}}_{\bar{\mathsf{n}}(\xi)})(l; \beta, z)$$

where $\hat{\mathbb{I}}_{\nu_0}(l; \beta, z)$ is

$$\exp(2\sqrt{2\Delta}\pi) \int_{\mathsf{G}_1^{\xi}(\mathbb{Q})\backslash \mathsf{G}_1^{\xi}(\mathbb{A})} \bar{f}(h_0)\, \mathrm{d}h_0 \\ \times \sum_{\mu \in \mathfrak{X}(\nu_0)} \int_{\mathsf{N}_{\mu}(\mathbb{Q})\backslash \mathsf{N}(\mathbb{A})} \hat{\boldsymbol{\Phi}}_l^{f,\xi}(\beta, z;\, \nu_0\, \mu\, n\, \mathsf{m}(r;\, h_0)\, b_{\infty})\, \psi_{\xi}(n)^{-1}\, \mathrm{d}n.$$

Moreover, we write the integral $\hat{\mathbb{I}}_{w_0}(l;\,\beta,z)$ as the sum $\hat{\mathbb{I}}^{\rm sing}_{w_0}(l;\,\beta,z)+\hat{\mathbb{I}}^{\rm reg}_{w_0}(l;\,\beta,z)$ according to the splitting $\mathfrak{X}(w_0)=\{1\}\cup[\mathfrak{X}(w_0)-\{1\}]$.

From definitions made so far, we have

$$a^{\bar f}_{\hat{\mathbb{F}}_l^{f,\xi}(\beta,z)}(\xi)=\hat{\mathbb{I}}_1(l;\,\beta,z)+\hat{\mathbb{I}}^{\rm sing}_{w_0}(l;\,\beta,z) \qquad (5.1)$$
$$+\hat{\mathbb{I}}^{\rm reg}_{w_0}(l;\,\beta,z)+\hat{\mathbb{I}}_{\bar{\mathfrak{n}}(\varepsilon)}(l;\,\beta,z)+\hat{\mathbb{I}}_{\bar{\mathfrak{n}}(\xi)}(l;\,\beta,z)$$

for any $l\geqslant l_0$ and $\mathrm{Re}(z)\in(\rho,l-3\rho-1)$.

5.7.1 The main term

On the strip $\mathrm{Re}(z)\in(\rho,l-3\rho-1)$, set

$$Y(l;\,\beta,z)=\hat{\mathbb{I}}_1(l;\beta,z)+\hat{\mathbb{I}}^{\rm sing}_{w_0}(l;\beta,z).$$

Proposition 12. *Let $l>3\rho+1$ be an integer.*
(1) *Let $z\in\mathbb{C}$ be such that $\mathrm{Re}(z)\in(\rho,l-3\rho-1)$. For any $\sigma\in(\rho,\mathrm{Re}(z))$,*

$$Y(l;\,\beta,z)=\left\{\frac{1}{2\pi i}\int_{(\sigma)}\frac{\beta(s)}{s-z}\,(\phi(s)+\phi(-s))\,\mathrm{d}s\right\}\|f\|^2$$

with

$$\phi(s)=\frac{(\sqrt{8\Delta}\,\pi)^{-s-\rho+l}}{\Gamma(-s-\rho+l)}\,D_*(s)\,\Lambda^*(f,-s).$$

(2) *The function $z\mapsto Y(l;\beta,z)$ on $\mathrm{Re}(z)\in(\rho,l-3\rho-1)$ has a holomorphic continuation to the whole complex plane satisfying the functional equation*

$$Y(l;\beta,z)+Y(l;\beta,-z)=\beta(z)\,\|f\|^2\,\{\phi(z)+\phi(-z)\}.$$

(3) *The entire function $Y(l;\beta,z)$ is of order* 1.

5.8 An error term estimation

From now on, we assume our test function $\beta(s)$ introduced in 5.4 satisfies that $\beta(0)=1$ and that there exists a constant $a>\pi$ such that

$$|\beta(s)|\ll\exp(-a|\mathrm{Im}(s)|)$$

holds for any compact interval $I\subset\mathbb{R}$; for example, $\beta(s)=\exp(T\,s^2)$ with $T>0$ is a possible choice. Let $l_0\,(>3\rho+1)$ be a sufficiently large integer as in Proposition 9. Our aim in this section is to show that, in the right-hand side of (5.1), the last three terms combined have a holomorphic continuation to $\mathbb{C}$ and its value at $z=0$ is asymptotically negligible in the limit $l\to\infty$ compared with the first two terms combined. For our purpose, let us introduce the function

$$\mathbf{R}(l;\,\beta,z)=\frac{-\Gamma(l-\rho)}{(\sqrt{8\Delta}\pi)^{l-\rho}}\left\{a^{\bar f}_{\hat{\mathbb{F}}_l^{f,\xi}(\beta,z)}(\xi)-\hat{\mathbb{I}}_1(l;\,\beta,z)-\hat{\mathbb{I}}^{\rm sing}_{w_0}(l;\,\beta,z)\right\} \qquad (5.2)$$

for $l\geqslant l_0$ and $\mathrm{Re}(z)\in(\rho,l-3\rho-1)$. For convenience, we also set

$$\mathbf{\Gamma}(l,s)=\frac{\pi^{m/2}\,\Gamma(l-\rho)\,\Gamma(l-\rho-1/2)}{2\,(\sqrt{8\Delta}\pi)^{l-\rho}\,\Gamma(l)}\,C_l^{\xi}\,\frac{\Gamma_L(l,s+1/2)}{\Gamma_{L_1^{\xi}}(1-s)},$$

where $\Gamma_L(l,s)$ is the common gamma factor for $\Lambda(F,s)$ (see 3.4), $\Gamma_{L_1^{\xi}}(s)$ the gamma factor for $\Lambda(f,s)$ (see 2.2.1) and C_l^{ξ} the constant occurred in Theorem 2.

Proposition 13. (1) *The function* $z \mapsto \mathbf{R}(l;\,\beta,z)$ *on* $\mathrm{Re}(z) \in (\rho, l-3\rho-1)$ *has a holomorphic extension to the whole complex plane. The entire function* $\mathbf{R}(l;\,\beta,z)$ *is of order* 1.
(2) *The sum* $\mathbf{R}(l;\,\beta,z)+\mathbf{R}(l;\,\beta,-z)$ *equals*

$$\mathbf{\Gamma}(l,z)\,\Gamma_{L_1^\xi}(1-z)\sum_{F\in\mathcal{B}_l}|a_F^f(\xi)|^2\,L(F,z+1/2)+\|f\|^2$$
$$\times\left(\frac{(\sqrt{8\Delta}\pi)^{-z}\,\Gamma(l-\rho)}{\Gamma(l-\rho-z)}\,\Lambda(f,1+z)\,\hat{\zeta}(1+2z)^{\epsilon}+\frac{(\sqrt{8\Delta}\pi)^{z}\,\Gamma(l-\rho)}{\Gamma(l-\rho+z)}\,\Lambda(f,1-z)\,\hat{\zeta}(1-2z)^{\epsilon}\right),$$

where ϵ *denotes* 0 *or* 1 *according to* m *is odd or even, respectively.*

Proof. This follows from Proposition 9 and Proposition 12. □

Lemma 14. *We have*

$$\mathbf{R}(l;\,\beta,0)$$
$$=\left\{\mathbf{\Gamma}(l,0)\sum_{F\in\mathcal{B}_l}|a_F^f(\xi)|^2\,L(F,1/2)-\frac{\mathrm{CT}_{s=0}(\Lambda(f,1-s)\,\hat{\zeta}(1-2s)^{\epsilon})}{\Gamma_{L_1^\xi}(1)}\right\}\Gamma_{L_1^\xi}(1)\,D_*(0)\,\beta(0)$$

and

$$\mathbf{\Gamma}(l,0)\sim\left(\frac{\pi}{4}\right)^{\rho}(2^{-1}\mathfrak{d}(L))^{1/2}\,4^{-1}\,(4\pi\sqrt{2\Delta})^{2\rho-2l+1}\,\Gamma(2l-\rho)\,l^{-m},\quad l\to+\infty.$$

Proof. The first formula is inferred from Proposition 9 (iv) and Proposition 12. By a direct computation, we have

$$\mathbf{\Gamma}(l,0)=\frac{\Gamma(l-\rho)^2\,\Gamma(l-\rho-1/2)}{\Gamma(l)\,\Gamma(l-\rho/2)\,\Gamma(l-\rho/2+1/2)}\left(\frac{\pi}{4}\right)^{\rho}(2^{-1}\mathfrak{d}(L))^{1/2}\,4^{-1}\,(4\pi\sqrt{2\Delta})^{2\rho-2l+1}.$$

Since $\Gamma(l+a)/\Gamma(l+b)\sim l^{a-b}$ as $l\to+\infty$ for $a,\,b\in\mathbb{R}$, the first factor on the right hand side asymptotically equals l^{-m} as $l\to+\infty$. □

The next proposition is the cornerstone of our argument; from (5.1) and (5.2), its proof is reduced to the individual estimation of the three terms $\hat{\mathbb{I}}^{\mathrm{reg}}_{w_0}(l;\,\beta,z)$, $\hat{\mathbb{I}}_{\bar{\mathfrak{n}}(\varepsilon)}$ and $\hat{\mathbb{I}}_{\bar{\mathfrak{n}}(\xi)}(l;\beta,z)$, which is done by a rather complicated computation of archimedean integrals involving the Bessel functions.

Proposition 15. *Let* $\sigma\in(\rho,\rho+2)$. *Then, for any* $q\in\mathbb{N}$,

$$|\mathbf{R}(l;\,\beta,z)|\ll l^{-q}\tag{5.3}$$

for any $z\in\sigma+i\mathbb{R}$ *and for any sufficiently large* l *with the implied constant independent of* (z,l).

Lemma 16. *Let* $\sigma>(m+1)/2$. *Then, there exists* $\kappa_1\in\mathbb{R}$ *such that*

$$|\mathbf{\Gamma}(l,z)\,\Gamma_{L_1^\xi}(1-z)\sum_{F\in\mathcal{B}_l}|a_F^f(\xi)|^2\,L(F,z+1/2)|\ll l^{\kappa_1}$$

for any $z\in\sigma+i\mathbb{R}$ *and for any sufficiently large* l *with the implied constant independent of* (z,l).

Proof. Since the argument $z+1/2$ with $z \in \sigma + i\mathbb{R}$ lies in the convergent range of the Euler product, the L-functions $|L(F, z+1/2)|$ are uniformly bounded in j and $z \in \sigma + i\mathbb{R}$. From Stirling's formula, there is a polynomial bound

$$\frac{\mathbf{\Gamma}(l,z)\,\Gamma_{L_1^\xi}(1-z)}{\mathbf{\Gamma}(l,0)\,\Gamma_{L_1^\xi}(1)} = O(l^{\kappa_1}), \quad z \in \sigma + i\mathbb{R}, \quad l \geqslant l_0$$

with some exponent κ_1. From Theorem 5, we obtain

$$\sum_{F \in \mathcal{B}_l} \mathbf{\Gamma}(l,0)\, |a_F^f(\xi)|^2 = O(1), \quad l \to \infty.$$

Combining in all, we are done. □

Lemma 17. *Let $A < B$ and $C > 0$. Let $\varphi(l;z)$ ($l \in \mathbb{N}$) be a family of entire functions in $z \in \mathbb{C}$ such that*

- *$|\varphi(l;z)| \leqslant C\, l^a$ for any $z \in A + i\mathbb{R}$ and for any sufficiently large l.*
- *$|\varphi(l;z)| \leqslant C\, l^b$ for any $z \in B + i\mathbb{R}$ and for any sufficiently large l.*
- *For each l, $\varphi(l;z)$ is of order 1.*

Then,

$$|\varphi(l;\, z)| \leqslant C\, l^{\mathrm{Re}\kappa(z)}$$

on the strip $\mathrm{Re}(z) \in [A,B]$ for any sufficiently large l, where $\kappa(z)$ is the linear function in z such that $\kappa(A) = a$, $\kappa(B) = b$.

Proof. This follows from the Phragmen-Lindelöf convexity principle applied to the function $l^{-\kappa(z)}\,\varphi(l;\, z)$. □

Corollary 18. *For any $q \in \mathbb{N}$, we have*

$$|\mathbf{R}(l;\, \beta, 0)| \ll l^{-q}$$

for sufficiently large l.

Proof. Fix $\sigma \in (\rho, \rho+2)$. From Proposition 13 (2) and Lemmas 15 and 16, together with a polynomial bound of $\Gamma(l-\rho)/\Gamma(l-\rho \pm z) = O(l^\sigma)$ on the line $\mathrm{Re}(z) = \sigma$, we have the estimation on the line $\mathrm{Re}(z) = -\sigma$

$$|\mathbf{R}(l;\, \beta, z)| \ll l^N, \quad \mathrm{Re}(z) = -\sigma,\, l \geqslant l_0 \tag{5.4}$$

with some N. Let $q \in \mathbb{N}$. Thus, applying Lemma 17, from the estimations (5.3) and (5.4), we have the interpolating estimate

$$|\mathbf{R}(l;\, \beta, z)| \ll l^{(N-q)/2}$$

for any $z \in i\mathbb{R}$ and $l \geqslant l_0$ with the implied constant independent on (z, l). This completes the proof. □

Lemma 19.

$$\Gamma_{L_1^\xi}(1)^{-1}\,\mathrm{CT}_{s=0}(\hat{L}(f,1-s)\,\hat{\zeta}(1-2s)^{\epsilon}) = \begin{cases} \mathrm{CT}_{s=1}\,L(f,1), & m\equiv 1 \pmod 2,\\ L'(f,1)-d_L(\xi)\,L(f,1), & m\equiv 0 \pmod 2,\end{cases}$$

where

$$d_L(\xi)=\Gamma'_{L_1^\xi}(1)/\Gamma_{L_1^\xi}(1)=\frac{-1}{2}\log(2^{-1}\mathfrak{d}(L_1^\xi))+\frac{m-2}{2}\log(2\pi)-\sum_{j=1}^{m/2-1}\psi\left(\frac{m+1}{2}-j\right).$$

Proof. This is shown by a direct computation. □

Now, Theorem 6 follows immediately from Lemma 14, Corollary 18 and Lemma 19.

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