Archimedean zeta integrals for the exterior square L-functions on GL_4

Taku Ishii (Seikei University)

Introduction

Let $\pi \cong \otimes'_v \pi_v$ be an automorphic cuspidal representation of $\mathrm{GL}_n(\mathbf{A}_{\mathbf{Q}})$. Let S be a finite set of places of $\mathbf{Q}$ including archimedean place such that π_p ($p \notin S$) is isomorphic to unramified principal series with Satake parameter $\mathrm{diag}(\alpha_{1,p}, \ldots, \alpha_{n,p}) \in \mathrm{GL}_n(\mathbf{C})$. The local L-factors $L(s, \pi_p)$ and $L(s, \pi_p, \wedge^2)$ for the standard and the exterior L-functions are defined by

$$L(s, \pi_p) = \prod_{1 \le i \le n} (1 - \alpha_{i,v} p^{-s})^{-1}, \quad L(s, \pi_p, \wedge^2) = \prod_{1 \le i < j \le n} (1 - \alpha_{i,v} \alpha_{j,p} p^{-s})^{-1},$$

respectively. Let

$$L^S(s, \pi) = \prod_{p \notin S} L(s, \pi_p), \quad L^S(s, \pi, \wedge^2) = \prod_{p \notin S} L(s, \pi_p, \wedge^2)$$

be the partial L-functions. Jacquet and Shalika [6] found an integral representation of the exterior square L-functions and proved an analytic continuation of $L^S(s, \pi, \wedge^2)$. Another integral representation was given by Bump and Friedberg [1]. This zeta integral contains two complex variables and makes us possible to study the standard and the exterior square L-functions simultaneously. In the papers [6] and [1], unramified computations are done, however, study of the local zeta integrals at archimedean and ramified places are not enough. Then their results are limited to the partial L-functions.

In this article we compute the archimedean local zeta integrals in [1] for GL_4. When π_∞ is the class one principal series, Stade [12] carried out archimedean calculus. Our aim here is to extend Stade's result to non-spherical cases. Using our explicit formulas of the principal series Whittaker functions on GL_4 given in [3], we show that the archimedean zeta integral coincides with the product of two archimedean L-factors. As a consequence we can prove the analytic continuation and functional equations for the completed L-functions.

Contrary to the zeta integrals method, the Langlands-Shahidi method gives more satisfactory results. Kim [7] proved the analytic continuations and the functional equation for the completed exterior square L-functions. Miller and Schmid [8], [9] bring new approach for archimedean theory, and they also obtained global results for the completed L-functions.

1 Zeta integrals

In this section we recall the zeta integral introduced by Bump and Friedberg [1]. We note that they worked on GL_n, however, we only look at GL_4. In this note the base field is $\mathbf{Q}$, and we denote by $\mathbf{A}$ the adele ring of $\mathbf{Q}$.

1.1 global Whittaker functions

Let π be a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbf{A})$ with the unitary central character ω. We denote by N the maximal unipotent subgroup of GL_4 consisting of upper triangular unipotent matrices. We fix a nontrivial additive character $\psi = \prod_v \psi_v : \mathbf{A}/\mathbf{Q} \to \mathbf{C}^{(1)}$ and extend it to a character $\psi_{\mathrm{N}} = \prod_v \psi_{\mathrm{N},v}$ of $\mathrm{N}(\mathbf{A})$ by $\psi_N(n) = \psi(n_{12})\psi(n_{23})\psi(n_{34})$ with $n = (n_{ij}) \in \mathrm{N}(\mathbf{A})$. For a cusp form $\varphi \in \pi$, we define the global Whittaker function W_φ attached to φ by

$$W_\varphi(g) = \int_{\mathrm{N}(\mathbf{Q})\backslash\mathrm{N}(\mathbf{A})} \varphi(ng)\psi_{\mathrm{N}}(n^{-1})\, dn, \quad (g \in \mathrm{GL}_4(\mathbf{A})),$$

which satisfies $W_\varphi(ng) = \psi_{\mathrm{N}}(n)W_\varphi(g)$ for all $(n,g) \in \mathrm{N}(\mathbf{A})\times\mathrm{GL}_4(\mathbf{A})$. The space $\mathcal{W}(\pi,\psi) = \{W_\varphi \mid \varphi \in \pi\}$, on which $\mathrm{GL}_4(\mathbf{A})$ acts by right translation, is called Whittaker model of π. Since the cusp form φ is recovered from Whittaker function W_φ through Fourier expansion ([11])

$$\varphi(g) = \sum_{\gamma\in \mathrm{N}_3(\mathbf{Q})\backslash\mathrm{GL}_3(\mathbf{Q})} W_\varphi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right)$$

(N_3 is the maximal unipotent subgroup of GL_3 consisting of upper triangular unipotent matrices), we have $W_\varphi \neq 0$ for $\varphi \neq 0$.

The notion of Whittaker model also makes sense over a local field. Let

$$\begin{aligned}&\mathcal{W}(\psi_v)\\ &\quad = \{W : \mathrm{GL}_4(\mathbf{Q}_v) \to \mathbf{C} \text{ smooth} \mid W(ng) = \psi_{\mathrm{N},v}(n)W(g),\ \forall(n,g) \in \mathrm{N}(\mathbf{Q}_v) \times \mathrm{GL}_4(\mathbf{Q}_v)\}.\end{aligned}$$

For a smooth irreducible admissible representation π_v of $\mathrm{GL}_4(\mathbf{Q}_v)$, we call the image of π_v in $\mathcal{W}(\psi_v)$ the Whittaker model of π_v, and denote it by $\mathcal{W}(\pi_v,\psi_v)$. For a cuspidal automorphic representation $\pi \cong \otimes'_v\pi_v$, it is known that each π_v has a unique Whittaker model. Furthermore, if a cusp form φ is decomposable, that is, $\varphi \to \otimes_v\xi_v$ under the isomorphism $\pi \cong \otimes'_v\pi_v$, then the global Whittaker function can be factorized as

$$W_\varphi(g) = \prod_v W_{\xi_v}(g_v), \quad g = (g_v) \in \mathrm{GL}_4(\mathbf{A}),$$

where the local Whittaker function W_{ξ_v} is the image of $\xi_v \in \pi_v$ under $\pi_v \hookrightarrow \mathcal{W}(\psi_v)$.

1.2 Eisenstein series

We construct the Eisenstein series on $\mathrm{GL}_2(\mathbf{A})$. Let Φ be a Schwartz-Bruhat function on $\mathbf{A}^2$ and $\eta : \mathbf{A}^\times/\mathbf{Q}^\times \to \mathbf{C}^{(1)}$ a unitary idele class character. We set

$$f(s,g,\Phi,\eta) = |\det g|_{\mathbf{A}}^s \int_{\mathbf{A}^\times} \Phi((0,1)zg)\, |z|_{\mathbf{A}}^{2s}\, \eta(z)\, dz, \quad s \in \mathbf{C},\ g \in \mathrm{GL}_2(\mathbf{A}).$$

This converges for $\mathrm{Re}(s) > 1/2$ and satisfies

$$f(s, \begin{pmatrix}a & b\\ 0 & d\end{pmatrix} g, \Phi, \eta) = \left|\frac{a}{d}\right|^s \eta^{-1}(d) f(s,g,\Phi,\eta).$$

We define the Eisenstein series $E(s,g,\Phi,\eta)$ on $\mathrm{GL}_2(\mathbf{A})$ by

$$E(s,g,\Phi,\eta) = \sum_{\gamma\in \mathrm{B}_2(\mathbf{Q})\backslash\mathrm{GL}_2(\mathbf{Q})} f(s,\gamma g,\Phi,\eta),$$

which converges absolutely for $\mathrm{Re}(s) > 1$. Here B_2 is the standard Borel subgroup of GL_2 consisting of upper triangular matrices. Since this Eisenstein series can be written as the Mellin transform of theta function, the Poisson summation leads the following properties.

Proposition 1.1. *The Eisenstein series $E(s, g, \Phi, \eta)$ has a meromorphic continuation to the whole s-plane, and satisfies the functional equation*

$$E(s, g, \Phi, \eta) = E(1 - s, {}^t g^{-1}, \widehat{\Phi}, \eta^{-1}),$$

where $\widehat{\Phi}$ is the Fourier transform of Φ: $\widehat{\Phi}(x_1, x_2) = \int_{\mathbf{A}^\times} \Phi(y_1, y_2)\psi(x_1y_1 + x_2y_2)\, dy_1dy_2$. If η is not of form $|\cdot|^{\sqrt{-1}\sigma}$ ($\sigma \in \mathbf{R}$), then $E(s, g, \Phi, \eta)$ is entire. If $\eta = |\cdot|^{\sqrt{-1}\sigma}$ for some $\sigma \in \mathbf{R}$, then $E(s, g, \Phi, \eta)$ has possible simple poles at $s = -\sqrt{-1}\sigma$ and $s = 1 - \sqrt{-1}\sigma$.

1.3 global zeta integrals

Let $\mathcal{Z}$ be the center of GL_4. We define an embedding $J : \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$ by

$$\left(g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \mapsto J(g_1, g_2) = \begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}.$$

For a cusp form φ on GL_4 and $s_1, s_2 \in \mathbf{C}$, we define the global zeta integral by

$$\begin{aligned} Z(s_1, s_2, \varphi, \Phi) = \int_{\mathcal{Z}(\mathbf{A})(\mathrm{GL}_2(\mathbf{Q})\times\mathrm{GL}_2(\mathbf{Q}))\backslash\mathrm{GL}_2(\mathbf{A})\times\mathrm{GL}_2(\mathbf{A})} & \varphi(J(g_1, g_2)) \\ \times E(s_2, g_2, \Phi, \omega) \left|\frac{\det g_1}{\det g_2}\right|^{s_1-1/2} & dg_1dg_2. \end{aligned}$$

Then the substitution $(g_1, g_2) \to ({}^t g_1^{-1}, {}^t g_2^{-1})$ implies

$$Z(s_1, s_2, \varphi, \Phi) = Z(1 - s_1, 1 - s_2, \widetilde{\varphi}, \widehat{\Phi}), \tag{1.1}$$

where $\widetilde{\varphi}(g) = \varphi({}^t g^{-1})$, and $\widetilde{\varphi} \in \widetilde{\pi}$. Here $\widetilde{\pi}$ is the contragredient representation of π and has the central character ω^{-1}. Using the Fourier expansion of φ, we can reach the basic identity ([1, Theorem 2]):

$$\begin{aligned} Z(s_1, s_2, \varphi, \Phi) &= \int_{\mathcal{Z}(\mathbf{A})(\mathrm{N}_2(\mathbf{A})\times\mathrm{N}_2(\mathbf{A}))\backslash\mathrm{GL}_2(\mathbf{A})\times\mathrm{GL}_2(\mathbf{A})} W_\varphi(J(g_1, g_2)) \\ &\qquad \times f(s_2, g_2, \Phi, \omega) \left|\frac{\det g_1}{\det g_2}\right|^{s_1-1/2} dg_1dg_2 \\ &= \int_{\mathrm{N}_2(\mathbf{A})\backslash\mathrm{GL}_2(\mathbf{A})} \int_{\mathrm{N}_2(\mathbf{A})\backslash\mathrm{GL}_2(\mathbf{A})} W_\varphi(J(g_1, g_2)) \\ &\qquad \times \Phi((0, 1)g_2)\, |\det g_1|^{s_1-1/2} |\det g_2|^{-s_1+s_2+1/2}\, dg_1dg_2. \end{aligned}$$

Therefore, if φ is decomposable, then we have

$$Z(s_1, s_2, \varphi, \Phi) = \prod_v Z_v(s_1, s_2, W_v, \Phi_v).$$

Here $Z_v(s_1, s_2, W_v, \Phi_v)$ is the local zeta integral given by

$$Z_v(s_1, s_2, W_v, \Phi_v) = \int_{\mathrm{N}_2(\mathbf{Q}_v)\backslash \mathrm{GL}_2(\mathbf{Q}_v)} \int_{\mathrm{N}_2(\mathbf{Q}_v)\backslash \mathrm{GL}_2(\mathbf{Q}_v)} W_v(J(g_1, g_2)) \times \Phi_v((0,1)g_2)\,|\det g_1|^{s_1-1/2}|\det g_2|^{-s_1+s_2+1/2}\,dg_1 dg_2,$$

where $\mathrm{N}_2(\mathbf{R}) = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbf{R} \right\}$.

1.4 Unramified computation

Let π_p be the unramified principal series representation of $\mathrm{GL}_4(\mathbf{Q}_p)$. Bump and Friedberg performed the unramified computation:

Proposition 1.2 ([1, Theorem 3]). *Let $p < \infty$ be an unramified place. For an unramified Whittaker function W_p^o and $\Phi_p^o = \mathrm{ch}_{\mathbf{Z}_p \oplus \mathbf{Z}_p}$ (characteristic function of $\mathbf{Z}_p \bigoplus \mathbf{Z}_p$), we have*

$$Z_p(s_1, s_2, W_p^o, \Phi_p^o) = L(s_1, \pi_p)L(s_2, \pi_p, \wedge^2).$$

2 Representation theory of $\mathrm{GL}_4(\mathbf{R})$

2.1 Lie groups and algebras

Let $G = \mathrm{GL}_4(\mathbf{R})$ and fix a maximal compact subgroup $K = O(4)$ of G. Let $N = \mathrm{N}(\mathbf{R})$ and

$$A = \{\mathrm{diag}(a_1, a_2, a_3, a_4) \mid a_i > 0 \text{ for } 1 \le i \le 4\},$$

Then we have the Iwasawa decomposition $G = NAK$. For our later use, we introduce new coordinates on A by

$$y = a[y_1, y_2, y_3, y_4] = \mathrm{diag}(y_1y_2y_3y_4, y_2y_3y_4, y_3y_4, y_4),$$

with $y_i > 0$ $(1 \le i \le 4)$.

We denote by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{n}$ and $\mathfrak{a}$ the Lie algebras of G, K, N and A, respectively. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form: $\mathfrak{p} = \{X \in \mathfrak{g} = \mathfrak{gl}(n, \mathbf{R}) \mid X = {}^tX\}$. We denote by $\mathfrak{p}^0 = \{X \in \mathfrak{p} \mid \mathrm{tr}(X) = 0\}$. Let E_{ij} be the matrix unit of size 4 with 1 at the (i,j)-th entry and 0 at the other entries. For $1 \le i, j \le n$ we set

$$K_{ij} = E_{ij} - E_{ji}, \quad X_{ij} = \begin{cases} E_{ij} + E_{ji} & \text{if } i \ne j, \\ 2E_{ii} - (1/2)E_4 & \text{if } i = j, \end{cases}$$

where E_4 is the unit matrix of size 4. Then we have

$$\mathfrak{k} = \bigoplus_{1 \le i < j \le n} \mathbf{R}K_{ij}, \quad \mathfrak{p}^0 = \bigoplus_{1 \le i \le j \le n} \mathbf{R}X_{ij}.$$

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_{\mathbf{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $\mathfrak{l}$. Let f be a smooth function on G. We denote by R the right regular action of G, and also denote by R the action of $\mathfrak{l}$ determined by the differential of R:

$$R(X)f(g) = \frac{d}{dt}\Big|_{t=0} f(g\exp(tX)), \quad X \in \mathfrak{l},\ g \in G.$$

This action of $\mathfrak{l}$ can be extended the that of the universal enveloping algebra $U(\mathfrak{l}_{\mathbf{C}})$ of $\mathfrak{l}_{\mathbf{C}}$.

2.2 Representations of K

We introduce finite dimensional representations (τ_i, V_i) $(i = 0, 1, 2)$ of K and their basis as follows:

- τ_0 : trivial representation on $V_0 = \mathbf{C} = \mathbf{C}v_0$;
- τ_1 : the standard representations on $V_1 = \mathbf{C}^4 = \bigoplus_{1 \le i \le 4} \mathbf{C}v_i$;
- τ_2 : the exterior representation of τ_1 on $V_2 = \wedge \mathbf{C}^4 = \bigoplus_{1 \le i < j \le 4} \mathbf{C}v_{ij}$,

where $v_0 = 1$, v_i $(1 \le i \le 4)$ is the i-th standard basis of $\mathbf{C}^4$, and $v_{ij} = v_i \wedge v_j$. The $\mathfrak{k}$-actions on V_1 and V_2 are given by

$$\begin{aligned} d\tau_1(K_{ij})v_p &= \delta_{jp}v_i - \delta_{ip}v_j, \\ d\tau_2(K_{ij})v_{pq} &= \delta_{jp}v_{iq} + \delta_{jq}v_{pi} - \delta_{ip}v_{jq} - \delta_{iq}v_{pj}. \end{aligned}$$

We note that τ_2 is direct sum of two (3-dimensional) irreducible representations.

For $\delta \in \mathbf{Z}/2\mathbf{Z}$, let $\tau_{(i,\delta)}$ be a representation of K on V_i given by

$$\tau_{(i,\delta)}(k) = \det(k)^{\delta}\tau_i(k), \quad k \in K.$$

2.3 Principal series representations

Let M be the centralizer of A in K:

$$M = \{m = \mathrm{diag}(m_1, m_2, m_3, m_4) \mid m_i \in \{\pm 1\} \ (1 \le i \le 4)\}.$$

For a subset I of $\{1, 2, 3, 4\}$ we define a representation σ_I of M by

$$\sigma_I(\mathrm{diag}(m_1, m_2, m_3, m_4)) = \prod_{1 \le i \le 4} m_i^{\delta_i},$$

where $\delta_i \equiv \delta_{i,I}$ $(1 \le i \le 4)$ is given by

$$\delta_i \equiv \delta_{i,I} = \begin{cases} 0 & \text{if } i \notin I, \\ 1 & \text{if } i \in I. \end{cases}$$

A linear form $\nu \in \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$ is identified with a tuple of complex numbers $(\nu_1, \nu_2, \nu_3, \nu_4)$ by $\nu(E_{ii}) = \nu_i$. We define a character e^{ν} of A by

$$e^{\nu}(a) = \prod_{1 \le i \le 4} a_i^{\nu_i}, \quad a = \mathrm{diag}(a_1, a_2, a_3, a_4) \in A.$$

Let ρ be the half sum of the standard positive roots of $(\mathfrak{g}, \mathfrak{a})$. Then we have $e^{\rho}(a) = a_1^3 a_2^2 a_3$. Under the data above, we call the induced representation

$$\pi_{I,\nu} = \mathrm{Ind}_{MAN}^{G}(\sigma_I \otimes e^{\nu+\rho} \otimes 1_N)$$

the principal series representation of G. The representation space $H_{I,\nu}$ is

$$H_{I,\nu} = \{f \in L^2(K) \mid f(mk) = \sigma_I(m)f(k) \text{ for } (m,k) \in M \times K\},$$

on which G acts by

$$(\pi_{I,\nu}(g))f(k) = a(kg)^{\nu+\rho}f(\kappa(kg)).$$

Here $g = n(g)a(g)\kappa(g)$ $(n(g) \in N, a(g) \in A, \kappa(g) \in K)$ is the Iwasawa decomposition of $g \in G$. We call the cardinality h of the set I the helicity of the principal series $\pi_{I,\nu}$, and denote by $\tilde{h} = \min(h, 4-h)$. When ν is in a general position, it is known that $\pi_{I,\nu}$ is irreducible, and we assume that $\pi_{I,\nu}$ is irreducible throughout this paper.

Lemma 2.1. *The minimal K-type of $\pi_{I,\nu}$ is $\tau_{(\tilde{h},\delta)}$ where $\delta = 0$ for $h = 0, 1, 2$, and $\delta = 1$ for $h = 3, 4$.*

2.4 Whittaker functions

Let ψ_∞^c ($c \in \mathbf{R}$) be the unitary character of $\mathbf{R}$ defined by $\psi_\infty^c(x) = \exp(2\pi\sqrt{-1}cx)$. For $c = (c_1, c_2, c_3) \in \mathbf{R}^3$, we define the character ψ_∞^c of N by

$$\psi_\infty^c(n) := \psi^{c_1}(n_{12})\psi^{c_2}(n_{23})\psi^{c_3}(n_{34}) = \exp\{2\pi\sqrt{-1}(c_1n_{12} + c_2n_{23} + c_3n_{34})\},$$

for $n = (n_{ij}) \in N$. A nondegenerate unitary character of N is of the form ψ_∞^c for some $c \in (\mathbf{R}^\times)^3$. We use the convention $\psi_\infty^{(c)} = \psi_\infty^{(c,c,c)}$ for $c \in \mathbf{R}$.

For $c \in (\mathbf{R}^\times)^3$, we introduce the space

$$C^\infty(N\backslash G; \psi_\infty^c) = \{f \in C^\infty(G) \mid f(ng) = \psi_\infty^c(n)f(g),\ (n, g) \in N \times G\},$$

on which G acts by right translation. For the principal series $(\pi_{I,\nu}, H_{I,\nu})$ of G we denote by $H_{I,\nu}^\infty$ the subspace of $H_{I,\nu}$ consisting of smooth functions. We call the space

$$\mathcal{W}(\pi_{I,\nu}, \psi_\infty^c) = \{\Phi(f) \mid f \in H_{I,\nu}^\infty, \Phi \in \mathrm{Hom}_G(H_{I,\nu}^\infty, C^\infty(N\backslash G; \psi_\infty^c))\}$$

Whittaker model of $\pi_{I,\nu}$, and a function in this space Whittaker function for $\pi_{I,\nu}$. According to the results of Shalika, Kostant and Wallach, the dimension of the space of $\mathcal{W}(\pi_{I,\nu}, \psi_\infty^c)$ is one. To describe Whittaker functions as functions on G, we take a K-type (τ, V_τ) of $\pi_{I,\nu}$, and a vector $v \in V_\tau$. For the unique (up to constant) intertwining operator $\Phi \in \mathrm{Hom}_G(H_{I,\nu}^\infty, C^\infty(N\backslash G; \psi_\infty^c))$, the function

$$W(v; g) := \Phi(v) \in \mathcal{W}(\pi_{I,\nu}, \psi_\infty^c)$$

satisfies the relation

$$W(v; ngk) = \psi_\infty^c(n)W(\tau(k)v; g), \quad (n, g, k) \in N \times G \times K.$$

Because of the Iwasawa decomposition, $W(v; g)$ is determined by $W(v; a(g))$, which we call the radial part of $W(v; g)$.

Remark 1. For $W(v; g) \in \mathcal{W}(\pi_{I,\nu}, \psi_\infty^{(1)})$ ($v \in V_\tau$), if we set

$$W^c(v; g) = W(v; \mathrm{diag}(c_1c_2c_3, c_2c_3, c_3, 1)g),$$

then we have $W^c(v; g) \in \mathcal{W}(\pi_{I,\nu}, \psi_\infty^c)$, and

$$W^c(v; a[y_1, y_2, y_3, y_4]) = W(\tau(m_c)v; a[|c_1|y_1, |c_2|y_2, |c_3|y_3, y_4])$$

with $m_c = \mathrm{diag}(\mathrm{sgn}(c_1c_2c_3), \mathrm{sgn}(c_2c_3), \mathrm{sgn}(c_3), 1) \in M$.

2.5 L- and ε- factors

We recall the definition of the archimedean L- and ε- factors via the Langlands parametrizations. For an irreducible admissible representation π_∞ of $\mathrm{GL}_4(\mathbf{R})$, we denote by $L(s, \pi_\infty)$

and $L(s,\pi_\infty,\wedge^2)$ the L-factors of the standard and the exterior square L-functions, respectively. We denote by $\varepsilon(s,\pi_\infty,\psi^1_\infty)$ and $\varepsilon(s,\pi_\infty,\wedge^2,\psi^1_\infty)$ the corresponding ε-factors. When $\pi_\infty=\pi_{I,\nu}$, the archimedean L- and ε- factors are defined as follows:

$$L(s,\pi_\infty)=\prod_{1\le i\le 4}\Gamma_{\mathbf{R}}(s+\nu_i+\delta_i),\qquad \varepsilon(s,\pi_\infty,\psi^1_\infty)=\sqrt{-1}^{\,h};$$
$$L(s,\pi_\infty,\wedge^2)=\prod_{1\le i<j\le 4}\Gamma_{\mathbf{R}}(s+\nu_i+\nu_j+\delta_{ij}),\qquad \varepsilon(s,\pi_\infty,\wedge^2,\psi^1_\infty)=\sqrt{-1}^{\,h(n-h)},$$

where we denote by $\Gamma_{\mathbf{R}}(s)=\pi^{-s/2}\Gamma(s/2)$, and $\delta_{ij}\in\{0,1\}$ is defined by $\delta_{ij}\equiv\delta_i+\delta_j \pmod 2$.

3 Evaluation of archimedean zeta integrals

3.1 Explicit formulas for Whittaker functions

We first review the Mellin-Barnes type integral representations of the radial parts $W(v;y)$ ($y\in A$) of the Whittaker functions at the minimal K-types of $\pi_{I,\nu}$. In [5] (n: general, $h=0$), [3] ($n=4$, $0\le h\le 4$) and [4] (n,h: general), we expressed Whittaker functions on $\mathrm{SL}_n(\mathbf{R})$ for the principal series of helicity h, in terms of Whittaker functions on $\mathrm{SL}_{n-1}(\mathbf{R})$ for the principal series of helicity $h-1$. We denote by

$$I=\{i_1,\dots,i_h\}\ (i_1<\cdots<i_h),\quad I'=\{1,2,3,4\}\backslash I=\{i'_1,\dots,i'_{4-h}\}\ (i'_1<\cdots<i'_{4-h}).$$

Theorem 3.1 ([5], [3], [4]). *Let $\tilde h=\min(h,4-h)$ and $(\tau_{(\tilde h,\delta)},V_{\tilde h})$ be the minimal K-type of the irreducible principal series $\pi_{I,\nu}$ of helicity $h=\sharp(I)$. For each vector $v\in V_{\tilde h}$, there exists the Whittaker function $W^c_\nu(v;g)$ corresponding to v, whose radial part is given by*

$$W^c_\nu(v;y)=\frac{y_1^{3/2}y_2^2y_3^{3/2}y_4^{|\nu|}}{(2\pi\sqrt{-1})^3}\int_{s_1,s_2,s_3}V^c_\nu(v;s_1,s_2,s_3)\prod_{1\le i\le 3}(|c_i|y_i)^{-s_i}ds_i,$$

with the path of integration in each s_i being a vertical line in the complex plane, of sufficiently large real part to keep the poles of $V^c_\nu(v;s_1,s_2,s_3)$ on its left. Here $|\nu|:=\nu_1+\nu_2+\nu_3+\nu_4$ and $V^c_\nu(v;s_1,s_2,s_3)$ can be written as follows.
(1) *When $h=0,4$, we have*

$$\begin{aligned}V^c_\nu(v_0;s_1,s_2,s_3)=\frac{1}{(4\pi\sqrt{-1})^3}\int_{t_1,t_2,u}&\Gamma_{\mathbf{R}}(u+\nu_3)\Gamma_{\mathbf{R}}(u+\nu_4)\\ &\times\Gamma_{\mathbf{R}}(t_1+\nu_2)\Gamma_{\mathbf{R}}(t_1-u)\Gamma_{\mathbf{R}}(t_2-u+\nu_2)\Gamma_{\mathbf{R}}(t_2+\nu_3+\nu_4)\\ &\times\Gamma_{\mathbf{R}}(s_1+\nu_1)\Gamma_{\mathbf{R}}(s_1-t_1)\Gamma_{\mathbf{R}}(s_2-t_1+\nu_1)\Gamma_{\mathbf{R}}(s_2-t_2)\\ &\times\Gamma_{\mathbf{R}}(s_3-t_2+\nu_1)\Gamma_{\mathbf{R}}(s_3+\nu_2+\nu_3+\nu_4)\,du\,dt_1dt_2.\end{aligned}$$

(2) *When $h=1,3$, if we set*

$$(j_1,j_2,j_3,j_4)=\begin{cases}(i_1,i'_1,i'_2,i'_3) & \text{if } h=1;\\ (i'_1,i_1,i_2,i_3) & \text{if } h=3,\end{cases}$$

then we have

$$
\begin{aligned}
V_\nu^c(v_p; s_1, s_2, s_3) = \frac{\sqrt{-1}^{-p} \prod_{i=p}^3 \mathrm{sgn}(c_i)}{(4\pi\sqrt{-1})^3} \int_{t_1,t_2,u} & \Gamma_{\mathbf{R}}(u+\nu_{j_3})\Gamma_{\mathbf{R}}(u+\nu_{j_4}) \\
& \times \Gamma_{\mathbf{R}}(t_1+\nu_{j_2})\Gamma_{\mathbf{R}}(t_1-u)\Gamma_{\mathbf{R}}(t_2-u+\nu_{j_2})\Gamma_{\mathbf{R}}(t_2+\nu_{j_3}+\nu_{j_4}) \\
& \times \Gamma_{\mathbf{R}}(s_1+\nu_{j_1}+\chi_p(1))\Gamma_{\mathbf{R}}(s_1-t_1+1-\chi_p(1)) \\
& \times \Gamma_{\mathbf{R}}(s_2-t_1+\nu_{j_1}+\chi_p(2))\Gamma_{\mathbf{R}}(s_2-t_2+1-\chi_p(2)) \\
& \times \Gamma_{\mathbf{R}}(s_3-t_2+\nu_{j_1}+\chi_p(3))\Gamma_{\mathbf{R}}(s_3+\nu_{j_2}+\nu_{j_3}+\nu_{j_4}+1-\chi_p(3)) \\
& \times du dt_1 dt_2
\end{aligned}
$$

for $1 \le p \le 4$. *Here*

$$
\chi_p(a) = \begin{cases} 1 & \text{if } 1 \le a \le p-1; \\ 0 & \text{if } p \le a \le 3. \end{cases}
$$

(3) *When* $h = 2$, *we have*

$$
\begin{aligned}
& V_\nu^c(v_{pq}; s_1, s_2, s_3) \\
&= \sum_{p \le r \le q-1} \frac{\sqrt{-1}^{-(p+q)} \prod_{i=p}^3 \mathrm{sgn}(c_i) \prod_{j=q}^3 \mathrm{sgn}(c_j)}{(4\pi\sqrt{-1})^3} \int_{t_1,t_2,u} \Gamma_{\mathbf{R}}(u+\nu_{i_1'})\Gamma_{\mathbf{R}}(u+\nu_{i_2'}) \\
&\quad \times \Gamma_{\mathbf{R}}(t_1+\nu_{i_2}+\chi_r(1))\Gamma_{\mathbf{R}}(t_1-u+1-\chi_r(1)) \\
&\quad \times \Gamma_{\mathbf{R}}(t_2-u+\nu_{i_2}+\chi_r(2))\Gamma_{\mathbf{R}}(t_2+\nu_{i_1'}+\nu_{i_2'}+1-\chi_r(2)) \\
&\quad \times \Gamma_{\mathbf{R}}(s_1+\nu_{i_1}+\chi_r^{p,q}(1))\Gamma_{\mathbf{R}}(s_1-t_1+\bar{\chi}_r^{p,q}(1)) \\
&\quad \times \Gamma_{\mathbf{R}}(s_2-t_1+\nu_{i_1}+\chi_r^{p,q}(2))\Gamma_{\mathbf{R}}(s_2-t_2+\bar{\chi}_r^{p,q}(2)) \\
&\quad \times \Gamma_{\mathbf{R}}(s_3-t_2+\nu_{i_1}+\chi_r^{p,q}(3))\Gamma_{\mathbf{R}}(s_3+\nu_{i_2}+\nu_{i_1'}+\nu_{i_2'}+\bar{\chi}_r^{p,q}(3))\, du dt_1 dt_2
\end{aligned}
$$

for $1 \le p < q \le 4$. *Here*

$$
\begin{aligned}
\chi_r(a) &= \begin{cases} 1 & \text{if } 1 \le a \le r-1; \\ 0 & \text{if } r \le a \le 2, \end{cases} \\
\chi_r^{p,q}(a) &= \begin{cases} 1 & \text{if } 1 \le a \le p-1 \text{ or } r+1 \le a \le q-1; \\ 0 & \text{if } p \le a \le r \text{ or } q \le a \le 3, \end{cases} \\
\bar{\chi}_r^{p,q}(a) &= \begin{cases} 1 & \text{if } p \le a \le r-1 \text{ or } q \le a \le 3; \\ 0 & \text{if } 1 \le a \le p-1 \text{ or } r \le a \le q-1. \end{cases}
\end{aligned}
$$

3.2 Contragredient Whittaker functions

Let $\widetilde{\pi}_{I,\nu}$ be the contragredient representation of $\pi_{I,\nu}$. The representation $\widetilde{\pi}_{I,\nu}$ also has Whittaker model, and in fact we have

$$
\mathcal{W}(\widetilde{\pi}_{I,\nu}, \psi_\infty^{(-c_3,-c_2,-c_1)}) = \{\widetilde{W} \mid W \in \mathcal{W}(\pi_{I,\nu}, \psi_\infty^{(c_1,c_2,c_3)})\},
$$

where we set

$$
\widetilde{W}(g) = W(w\,{}^t g^{-1}), \quad w = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.
$$

Using our explicit formulas in Theorem 3.1, we can determine the radial part of $\widetilde{W}$.

Proposition 3.2. *The contragredient representation* $\widetilde{\pi}_{I,\nu}$ *of* $\pi_{I,\nu}$ *is isomorphic to* $\pi_{I,-\nu}$*. Moreover, for* $W(g) = W_\nu^c(v; g)$ *with* $v \in V_{\tilde{h}}$*, we have* $\widetilde{W} \in \mathcal{W}(\pi_{I,-\nu}, \psi_\infty^{(-c_3,-c_2,-c_1)})$ *and the radial part of* $\widetilde{W}$ *are given as*

$$\widetilde{W}(y) = |c_1c_2c_3|^{|\nu|} \cdot C(v) \cdot W_{-\nu}^{(-c_3,-c_2,-c_1)}(v; y).$$

Here the constants $C(v)$ *are*

$$C(v) = \begin{cases} 1 & \text{if } \tilde{h} = 0 \text{ and } v = v_0; \\ -\sqrt{-1}\mathrm{sgn}(c_1c_2c_3) & \text{if } \tilde{h} = 1 \text{ and } v = v_p \ (1 \le p \le 4); \\ 1 & \text{if } \tilde{h} = 2 \text{ and } v = v_{pq} \ (1 \le p < q \le 4). \end{cases}$$

3.3 Calculus of archimedean zeta integrals

For $W \in \mathcal{W}(\pi_{I,\nu}, \psi_\infty^c)$ and $s_1, s_2 \in \mathbf{C}$, we wish to compute the following archimedean zeta integral:

$$Z_\infty(s_1, s_2, W, \Phi_n) = \int_{\mathrm{N}_2(\mathbf{R})\backslash \mathrm{GL}_2(\mathbf{R})} \int_{\mathrm{N}_2(\mathbf{R})\backslash \mathrm{GL}_2(\mathbf{R})} W(J(g_1, g_2))\Phi_n((0,1)g_2) \times |\det g_1|^{s_1-1/2} |\det g_2|^{-s_1+s_2+1/2}\, dg_1 dg_2,$$

where $\Phi_n(x_1, x_2) = (\mathrm{sgn}(n)\sqrt{-1}x_1 + x_2)^{|n|} \exp\{-\pi(x_1^2 + x_2^2)\}$. Using the Iwasawa decomposition of $\mathrm{GL}_2(\mathbf{R})$, we have

$$Z_\infty(s_1, s_2, W, \Phi_n) = 2^{-3}\Gamma_{\mathbf{R}}(2s_2 + |\nu| + |n|) \sum_{0 \le i \le 3} \int_{(\mathbf{R}_+)^3} \int_0^{2\pi}\int_0^{2\pi} W(m_i\, a[y_1, y_2, y_3, 1]\, \kappa_{\theta_1,\theta_2}) \times \exp(\sqrt{-1}n\theta_2)\, y_1^{s_1-3/2} y_2^{s_2-2} y_3^{s_1+s_2-3/2} \prod_{i=1}^{2} \frac{d\theta_i}{2\pi} \prod_{i=1}^{3} \frac{dy_i}{y_i},$$

where

- $\kappa_{\theta_1,\theta_2} = J(\kappa_{\theta_1}, \kappa_{\theta_2})$ with $\kappa_{\theta_i} = \begin{pmatrix} \cos\theta_i & \sin\theta_i \\ -\sin\theta_i & \cos\theta_i \end{pmatrix}$ $(i = 1, 2)$,
- $m_0 = 1_4,\ m_1 = \mathrm{diag}(-1, 1, 1, 1),\ m_2 = \mathrm{diag}(1, -1, 1, 1),\ m_3 = m_1 m_2$.

Here is our main result.

Theorem 3.3. *We use the same notation as in Theorem 3.1. For* $\varepsilon \in \{\pm 1\}$*, we abbreviate* $W_\nu^\varepsilon = W_\nu^{(\varepsilon,\varepsilon,\varepsilon)}$ *and* $V_\nu^\varepsilon = V_\nu^{(\varepsilon,\varepsilon,\varepsilon)}$*. We take a pair* $(W, \Phi) \in \mathcal{W}(\pi_{I,\nu}, \psi_\infty^{(\varepsilon)}) \times \mathcal{S}(\mathbf{R}^2)$ *of Whittaker function and Schwartz function as the following.*

(1) *When* $h = 0$*, we set*

$$W(g) = W_\nu^\varepsilon(v_0; g),\ \Phi = 2\Phi_0.$$

(2) *When* $h = 1$*, we set*

$$W(g) = W_\nu^\varepsilon(v_2 + \sqrt{-1}v_4; g),\ \Phi = -2\sqrt{-1}\Phi_{-1}.$$

(3) *When* $h = 2$*, we set*

$$W(g) = \frac{1}{4\pi\sqrt{-1}}\{R(X_{12})W_\nu^\varepsilon(v_{12}; g) - R(X_{23})W_\nu^\varepsilon(v_{23}; g) + R(X_{34})W_\nu^\varepsilon(v_{34}; g) + R(X_{14})W_\nu^\varepsilon(v_{14}; g)\},$$
$$\Phi = -2\sqrt{-1}\Phi_0.$$

(4) *When $h=3$, we set*

$$W(g) = \frac{1}{4\pi\sqrt{-1}}\{R(X_{23}+\sqrt{-1}X_{34})W_\nu^\varepsilon(v_1;g) - R(X_{12}+\sqrt{-1}X_{14})W_\nu^\varepsilon(v_3;g)\},$$
$$\Phi = 2\sqrt{-1}\Phi_{-1}.$$

(5) *When $h=4$, we set*

$$W(g) = \frac{1}{(4\pi\sqrt{-1})^2}R(X_{12}X_{34}-X_{23}X_{14})W_\nu^\varepsilon(v_0;g),\ \ \Phi = 2\Phi_0.$$

Then we have

$$Z_\infty(s_1,s_2,W,\Phi) = L(s_1,\pi_\infty)L(s_2,\pi_\infty,\wedge^2).$$

Proof. The case of $h=0$ is done by Stade [12]. We illustrate our proof when $h=2$. We first check that $W(g\kappa_{\theta_1,\theta_2}) = W(g)$. For $1 \le p < q \le 4$, the function $W_{pq}(g) := R(X_{pq})W_\nu^\varepsilon(v_{pq},g)$ satisfies the relation

$$W_{pq}(gk) = W(\tau_{(2,0)}(k)v_{pq}; g\cdot kX_{pq}k^{-1})\ (k\in K).$$

Since $\mathfrak{p}_{\mathbf{C}}^0$ can be identified with V_2 via the adjoint action, and

$$\begin{aligned}&(\tau_{(2,0)}(\kappa_{\theta_1,\theta_2})v_{12}, \tau_{(2,0)}(\kappa_{\theta_1,\theta_2})v_{23}, \tau_{(2,0)}(\kappa_{\theta_1,\theta_2})v_{34}, \tau_{(2,0)}(\kappa_{\theta_1,\theta_2})v_{14})\\ &= (v_{12},v_{23},v_{34},v_{14})\begin{pmatrix} c_1c_2 & c_1s_2 & s_1s_2 & s_1c_2\\ -c_1s_2 & c_1c_2 & s_1c_2 & -s_1s_2\\ s_1s_2 & -s_1c_2 & c_1c_2 & -c_1s_2\\ -s_1c_2 & -s_1s_2 & c_1s_2 & c_1c_2\end{pmatrix}\end{aligned}$$

with $(c_i,s_i) = (\cos\theta_i, \sin\theta_i)$, we can see that $W(g\kappa_{\theta_1,\theta_2}) = W(g)$.

Let us compute the radial part $W(y)$. For $1\le i<j\le 4$, we have

$$\begin{aligned}R(X_{ij})W_\nu^\varepsilon(v_{ij};y) &= R(2E_{ij}-K_{ij})W_\nu^\varepsilon(v_{ij};y)\\ &= R(2E_{ij})W_\nu^\varepsilon(v_{ij};y)\\ &= \begin{cases}4\pi\sqrt{-1}\varepsilon y_i W_\nu^\varepsilon(v_{ij};y) & \text{if } j=i+1;\\ 0 & \text{if } j\ge i+2.\end{cases}\end{aligned}$$

Then, combined with $W(m_iy) = W(y)$ $(0\le i\le 3)$, we have

$$W(y) = 4\pi\sqrt{-1}\varepsilon\{y_1W_\nu^\varepsilon(v_{12};y) - y_2W_\nu^\varepsilon(v_{23};y) + y_3W_\nu^\varepsilon(v_{34};y)\},$$

and thus

$$\begin{aligned}Z(s_1,s_2,W,\Phi) = -\varepsilon\sqrt{-1}\Gamma_{\mathbf{R}}(2s_2+|\nu|)\{&V_\nu^\varepsilon(v_{12};s_1+1,s_2,s_1+s_2)\\ &- V_\nu^\varepsilon(v_{23};s_1,s_2+1,s_1+s_2) + V_\nu^\varepsilon(v_{34};s_1,s_2,s_1+s_2+1)\}.\end{aligned}$$

Using our explicit formula in Theorem 3.1 (ii) and Barnes' first lemma [10, §4.2]:

$$\begin{aligned}&\frac{1}{4\pi\sqrt{-1}}\int_z \Gamma_{\mathbf{R}}(z+a)\Gamma_{\mathbf{R}}(z+b)\Gamma_{\mathbf{R}}(-z+c)\Gamma_{\mathbf{R}}(-z+d)\,dz\\ &= \frac{\Gamma_{\mathbf{R}}(a+c)\Gamma_{\mathbf{R}}(a+d)\Gamma_{\mathbf{R}}(b+c)\Gamma_{\mathbf{R}}(b+d)}{\Gamma_{\mathbf{R}}(a+b+c+d)},\end{aligned}$$

we get

$$
\begin{aligned}
&Z(s_1,s_2,W,\Phi)\\
&\quad=\Gamma_{\mathbf{R}}(2s_2+|\nu|)\Gamma_{\mathbf{R}}(s_1+\nu_{i_1}+1)\Gamma_{\mathbf{R}}(s_1+\nu_{i_2}+1)\\
&\qquad\times\Gamma_{\mathbf{R}}(s_1+s_2+\nu_{i_1}+\nu_{i'_1}+\nu_{i'_2}+1)\Gamma_{\mathbf{R}}(s_1+s_2+\nu_{i_2}+\nu_{i'_1}+\nu_{i_2})\\
&\qquad\times\frac{1}{4\pi\sqrt{-1}}\int_u\frac{\Gamma_{\mathbf{R}}(u+\nu_{i'_1})\Gamma_{\mathbf{R}}(u+\nu_{i'_2})\Gamma_{\mathbf{R}}(s_2-u+\nu_{i_1}+1)\Gamma_{\mathbf{R}}(s_2-u+\nu_{i_2}+1)}{\Gamma_{\mathbf{R}}(s_1+s_2-u+\nu_{i_1}+\nu_{i_2}+2)\Gamma_{\mathbf{R}}(s_1+2s_2-u+|\nu|+2)}\\
&\qquad\times\{\Gamma_{\mathbf{R}}(s_1-u+2)\Gamma_{\mathbf{R}}(s_2+\nu_{i_1}+\nu_{i_2})\Gamma_{\mathbf{R}}(s_1+s_2-u+\nu_{i_1}+\nu_{i_2})\Gamma_{\mathbf{R}}(s_2+\nu_{i'_1}+\nu_{i'_2}+2)\\
&\qquad+\Gamma_{\mathbf{R}}(s_1-u)\Gamma_{\mathbf{R}}(s_2+\nu_{i_1}+\nu_{i_2}+2)\Gamma_{\mathbf{R}}(s_1+s_2-u+\nu_{i_1}+\nu_{i_2})\Gamma_{\mathbf{R}}(s_2+\nu_{i'_1}+\nu_{i'_2}+2)\\
&\qquad+\Gamma_{\mathbf{R}}(s_1-u)\Gamma_{\mathbf{R}}(s_2+\nu_{i_1}+\nu_{i_2}+2)\Gamma_{\mathbf{R}}(s_1+s_2-u+\nu_{i_1}+\nu_{i_2}+2)\Gamma_{\mathbf{R}}(s_2+\nu_{i'_1}+\nu_{i'_2})\}\,du.
\end{aligned}
$$

In view of $\Gamma_{\mathbf{R}}(s+2)=(2\pi)^{-1}s\Gamma_{\mathbf{R}}(s)$, the bracket $\{\ \}$ in the integrand above can be written as

$$
\begin{aligned}
&\Gamma_{\mathbf{R}}(s_1-u)\Gamma_{\mathbf{R}}(s_2+\nu_{i_1}+\nu_{i_2})\Gamma_{\mathbf{R}}(s_2+\nu_{i'_1}+\nu_{i'_2})\\
&\quad\times\frac{\Gamma_{\mathbf{R}}(s_1+s_2-u+\nu_{i_1}+\nu_{i_2}+2)\Gamma_{\mathbf{R}}(2s_2+|\nu|+2)}{\Gamma_{\mathbf{R}}(2s_2+|\nu|)}.
\end{aligned}
$$

Therefore we can perform the integration over u by means of Barnes' second lemma [10, §4.2]:

$$
\begin{aligned}
&\frac{1}{4\pi\sqrt{-1}}\int_z\frac{\Gamma_{\mathbf{R}}(z+a)\Gamma_{\mathbf{R}}(z+b)\Gamma_{\mathbf{R}}(z+c)\Gamma_{\mathbf{R}}(-z+d)\Gamma_{\mathbf{R}}(-z+e)}{\Gamma_{\mathbf{R}}(z+a+b+c+d+e)}\,dz\\
&\quad=\frac{\Gamma_{\mathbf{R}}(a+d)\Gamma_{\mathbf{R}}(a+e)\Gamma_{\mathbf{R}}(b+d)\Gamma_{\mathbf{R}}(b+e)\Gamma_{\mathbf{R}}(c+d)\Gamma_{\mathbf{R}}(c+e)}{\Gamma_{\mathbf{R}}(a+b+d+e)\Gamma_{\mathbf{R}}(b+c+d+e)\Gamma_{\mathbf{R}}(c+a+d+e)},
\end{aligned}
$$

to finish the proof of (iii). □

From Theorem 3.3 and Proposition 3.2 we can show the following:

Corollary 3.4. *When $\pi_\infty\cong\pi_{I,\nu}$, there exists a pair $(W,\Phi)\in\mathcal{W}(\pi_\infty,\psi_\infty^{(+1)})\times\mathcal{S}(\mathbf{R}^2)$ such that*

$$Z_\infty(s_1,s_2,W,\Phi)=L(s_1,\pi_\infty)L(s_2,\pi_\infty,\wedge^2)$$

and

$$Z_\infty(s_1,s_2,\widetilde{W},\widehat{\Phi})=\varepsilon(s_1,\pi_\infty,\psi_\infty^1)\varepsilon(s_2,\pi_\infty,\wedge^2,\psi_\infty^1)L(s_1,\widetilde{\pi}_\infty)L(s_2,\widetilde{\pi}_\infty,\wedge^2).$$

3.4 Global functional equation

Theorem 3.5. *Let $\pi\cong\otimes'_v\pi_v$ be a cuspidal automorphic representation of* $\mathrm{GL}_4(\mathbf{A_Q})$ *satisfying the following conditions:*
(1) *For $v=p<\infty$, π_v is isomorphic to the unramified principal series representation of* $\mathrm{GL}_4(\mathbf{Q}_p)$*;*
(2) *For $v=\infty$, π_v is isomorphic to the principal series representation $\pi_{I,\nu}$ of* $\mathrm{GL}_4(\mathbf{R})$.
Then the completed exterior square L-function $L(s,\pi,\wedge^2)=\prod_{v\le\infty}L(s,\pi_v,\wedge^2)$ can be holomorphically continued to the whole s-plane with at most simple poles at $s=0,1$, and satisfies the functional equation

$$L(s,\pi,\wedge^2)=\varepsilon(s,\pi,\wedge^2)L(1-s,\widetilde{\pi},\wedge^2).$$

Proof. The functional equation (1.1) of zeta integral, Proposition 1.2 and Corollary 3.4 implies that

$$L(s_1,\pi)L(s_2,\pi,\wedge^2) = \varepsilon(s_1,\pi)\varepsilon(s_2,\pi,\wedge^2)L(1-s_1,\widetilde{\pi})L(1-s_2,\widetilde{\pi},\wedge^2).$$

Since the functional equation for the standard L-function $L(s,\pi)$ has established in [2], we are done. □

We remark some related works for GL_n.

Remark 2. (i) When π_∞ is isomorphic to the class one principal series representation of $\mathrm{GL}_n(\mathbf{R})$, Stade [13] computed the archimedean zeta integrals to prove the result above.
(ii) Via Langlands-Shahidi method, Kim [7] proved the analytic continuations and the functional equations for the completed exterior square L-functions on GL_n. He proved $L(s,\pi,\wedge^2)$ is holomorphic except that n is even and π is self-dual.
(iii) Recently Miller and Schmidt ([8], [9]) discover a new way for archimedean theory of automorphic L-functions. They introduce "automorphic distribution method" and prove the global functional equation for the exterior square L-function on $\mathrm{GL}_n(\mathbf{A}_\mathbf{Q})$ without any assumptions on π_∞.

References

[1] D. Bump and S. Friedberg, The exterior square automorphic L-functions on GL(n), In: Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989), 47–65, Israel Math. Conf. Proc., 3, Weizmann, Jerusalem, 1990.

[2] R. Godement and H. Jacquet, Zeta functions of simple algebras, Lecture Notes in Mathematics, 260, Springer-Verlag, Berlin-New York, 1972.

[3] T. Hina, T. Ishii and T. Oda, Calculus of principal series Whittaker functions on $SL(4,\mathbf{R})$, preprint.

[4] T. Ishii and T. Oda, Calculus of principal series Whittaker functions on $SL(n,\mathbf{R})$, preprint.

[5] T. Ishii and E. Stade, New formulas for Whittaker functions on $GL(n,\mathbf{R})$, J. Funct. Anal. **244** (2007), 289–314.

[6] H. Jacquet and J. Shalika, Exterior square L-functions, In: Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), 143–226, Perspect. Math., 11, Academic Press, Boston, MA, 1990.

[7] H. H. Kim, Langlands-Shahidi method and poles of automorphic L-functions: application to exterior square L-functions, Canad. J. Math. **51** (1999), 835–849.

[8] S. D. Miller and W. Schmid, The Rankin-Selberg method for automorphic distributions, In: Representation theory and automorphic forms, 111–150, Prog. Math., 255, Birkhäuser Boston, MA, 2008.

[9] S. D. Miller and W. Schmid, The archimedean theory of the Exterior Square L-functions over $\mathbf{Q}$, arXiv:1109.4190.

[10] L. J. Slater, Generalized hypergeometric functions, Cambridge University Press, Cambridge, 1966.

[11] J. Shalika The multiplicity one theorem for GL_n, Ann. Math. (2) **100** (1974), 171–193.

[12] E. Stade, On explicit integral formulas for $GL(n, \mathbf{R})$-Whittaker functions, Duke Math. J. **60** (1990), 313–362.

[13] E. Stade, Mellin transforms of $GL(n, \mathbf{R})$ Whittaker functions, Amer. J. Math. **123** (2001), 121–161.