Mock Theta Functions and Mathieu Moonshine

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1 Introduction

The monstrous moonshine is now well known [7]: the Fourier coefficients of the modular j-function are related with the dimensions of the irreducible representations of the Monster. It has been also studied such a connection between the modular forms and the finite groups (see, *e.g.*, [25, 8, 23]).

Recently observed is a similar phenomenon, *Mathieu moonshine* [15]; therein a *mock* modular form plays a role. The mock theta function was first introduced by Ramanujan in his last letter to Hardy in 1920 [27, 1]. Zwegers demystified the mathematical structure of the mock theta function [30], and the mock theta function is identified with a holomorphic part of the harmonic Maass form with weight-1/2. Studies of mock theta functions have been also applied to quantum invariants of certain 3-manifolds [22, 2, 24].

Purpose of this article is to survey the Mathieu moonshine based on [9, 10, 13]. In Section 2 we study a relationship between the superconformal character and the mock theta function, and explain the Mathieu moonshine. In Section 3 we give the Poincaré series for the Fourier coefficients of the mock theta functions.

2 Mathieu Moonshine

2.1 Superconformal Characters

Our motivation is to study the $K3$ surface by use of the superconformal algebras. The $K3$ surface is described by use of the $\mathcal{N} = 4$ superconformal algebra with $c = 6$, and we use the superconformal characters $\mathrm{ch}^{\widetilde{R}}_{h,\ell}(z;\tau)$ computed in [16, 17], where h and ℓ respectively denote the conformal dimension and the isospin. See Appendix for definition of the Jacobi theta series.

- non-BPS representations ($h > \frac{1}{4}$ and $\ell = \frac{1}{2}$),

$$\mathrm{ch}^{\widetilde{R}}_{h,\ell=\frac{1}{2}}(z;\tau) = q^{h-\frac{3}{8}} \frac{\left[\theta_{11}(z;\tau)\right]^2}{\left[\eta(\tau)\right]^3},$$

- BPS representations ($h = \frac{1}{4}$ and $\ell = 0, \frac{1}{2}$),

$$\mathrm{ch}^{\widetilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) = \frac{\left[\theta_{11}(z;\tau)\right]^2}{\left[\eta(\tau)\right]^3} \mu(z;\tau), \tag{1}$$

$$\mathrm{ch}^{\widetilde{R}}_{h=\frac{1}{4},\ell=\frac{1}{2}}(z;\tau) + 2\ \mathrm{ch}^{\widetilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) = q^{-\frac{1}{8}} \frac{\left[\theta_{11}(z;\tau)\right]^2}{\left[\eta(\tau)\right]^3}.$$

Here $\mu(z;\tau)$ is defined by

$$\mu(z;\tau) = \frac{\mathrm{i}\, \mathrm{e}^{\pi \mathrm{i} z}}{\theta_{11}(z;\tau)} \sum_{n\in\mathbb{Z}} (-1)^n \frac{q^{\frac{1}{2}n(n+1)}\, \mathrm{e}^{2\pi \mathrm{i} n z}}{1 - q^n\, \mathrm{e}^{2\pi \mathrm{i} z}}.$$

2.2 Mock Theta Function

The function $\mu(z;\tau)$ is a mock theta function studied in [30]. Its typical property is seen in the modular S-transformation,

$$\mu(z;\tau) = -\sqrt{\frac{\mathrm{i}}{\tau}}\,\mu\left(\frac{z}{\tau};-\frac{1}{\tau}\right) + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\pi\mathrm{i}\tau x^2}}{\cosh(\pi x)}\,\mathrm{d}x.$$

We define a *completion* of $\mu(z;\tau)$;

$$\widehat{\mu}(z;\tau) = \mu(z;\tau) - \frac{1}{2\sqrt{\mathrm{i}}}\int_{-\overline{\tau}}^{\mathrm{i}\infty} \frac{[\eta(x)]^3}{\sqrt{x+\tau}}\,\mathrm{d}x,$$

which is the harmonic Maass form satisfying

$$\widehat{\mu}\left(\frac{z}{c\tau+d};\gamma(\tau)\right) = \chi(\gamma)\,\sqrt{c\tau+d}\,\widehat{\mu}(\tau),$$
$$(\Im\tau)^{\frac{3}{2}}\,\frac{\partial}{\partial\tau}\sqrt{\Im\tau}\frac{\partial}{\partial\overline{\tau}}\widehat{\mu}(z;\tau) = 0.$$

Here the multiplier system χ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2;\mathbb{Z})$ is given by

$$\chi(\gamma) = \begin{cases} \mathrm{i}^{\frac{3}{2}}\,\mathrm{e}^{-\frac{a+d}{4c}\pi\mathrm{i}+3s(d,c)\pi\mathrm{i}}, & \text{for } c\neq 0, \\ \mathrm{e}^{-\frac{b}{4}\pi\mathrm{i}}, & \text{for } c = 0. \end{cases}$$

where $s(d,c)$ is the Dedekind sum,

$$s(d,c) = \sum_{k \bmod c} \left(\!\left(\frac{k}{c}\right)\!\right)\left(\!\left(\frac{k\,d}{c}\right)\!\right), \qquad ((x)) = \begin{cases} x - \lfloor x\rfloor - \frac{1}{2}, & \text{for } x\in\mathbb{R}-\mathbb{Z}, \\ 0, & \text{for } x\in\mathbb{Z}. \end{cases}$$

We note that

$$\sqrt{\Im\tau}\,\frac{\partial}{\partial\overline{\tau}}\widehat{\mu}(z;\tau) = \frac{\mathrm{i}}{2\sqrt{2}}\,[\eta(-\overline{\tau})]^3\,,$$

and that, in the sense of Zagier [29], the mock theta function $\mu(z;\tau)$ has $[\eta(\tau)]^3$ as a "shadow".

2.3 Character Decomposition

We pay attention to the elliptic genus of the $K3$ surface

$$Z_{K3}(z;\tau) = 2\,\phi_{0,1}(z;\tau) = 8\,\left[\left(\frac{\theta_{10}(z;\tau)}{\theta_{10}(0;\tau)}\right)^2 + \left(\frac{\theta_{00}(z;\tau)}{\theta_{00}(0;\tau)}\right)^2 + \left(\frac{\theta_{01}(z;\tau)}{\theta_{01}(0;\tau)}\right)^2\right].$$

By use of the superconformal characters we decompose $Z_{K3}(z;\tau)$ as

$$Z_{K3}(z;\tau) = 20\,\mathrm{ch}^{\widetilde{R}}_{\frac{1}{4},0}(z;\tau) - 2\,\mathrm{ch}^{\widetilde{R}}_{\frac{1}{4},\frac{1}{2}}(z;\tau) + \sum_{n=1}^{\infty} A(n)\,\mathrm{ch}^{\widetilde{R}}_{n+\frac{1}{4},\frac{1}{2}}(z;\tau), \tag{2}$$

where $A(n)$ denotes the number of the non-BPS representations in the elliptic genus;

n	1	2	3	4	5	6	7	8	9	$\cdots$
$A(n)$	90	462	1540	4554	11592	27830	61686	131100	265650	$\cdots$

The number $A(n)$ can be identified with the Fourier coefficients of the mock theta function [9]. Observation in [15] is that $A(n)$ is decomposed into a sum of dimensions of the irreducible representations of M_{24},

$$\sum_R \mathrm{mult}_R(n)\, \dim R = A(n),$$

and that the multiplicities $\mathrm{mult}_R(n)$ are conjectured to be positive integer. Here R denotes the irreducible representation of M_{24}. Hence by introducing

$$V(n) = \sum_R \mathrm{mult}_R(n)\, R,$$

we have

$$A(n) = \mathrm{Tr}_{V(n)}\, 1.$$

2.4 Twisted Elliptic Genus

As the Mathieu group M_{24} has 26 conjugacy classes, we define for conjugacy class g a variant of the number $A(n)$ of the non-BPS characters by

$$A_g(n) = \mathrm{Tr}_{V(n)}\, g = \sum_R \mathrm{mult}_R(n)\, \chi_R^g.$$

See Table 1 for the character table χ_R^g of M_{24} (*e.g.* [6]). Using $A_g(n)$, we define the twisted elliptic genus $Z_g(z;\tau)$ by the character decomposition as

$$Z_g(z;\tau) = (\chi_g - 4)\, \mathrm{ch}^{\widetilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) - 2\, \mathrm{ch}^{\widetilde{R}}_{h=\frac{1}{4},\ell=\frac{1}{2}}(z;\tau) + \sum_{n=1}^{\infty} A_g(n)\, \mathrm{ch}^{\widetilde{R}}_{h=n+\frac{1}{4},\ell=\frac{1}{2}}(z;\tau).$$

Here $\chi_g \in \mathbb{Z}$ is the Witten index,

$$\chi_g = Z_g(z=0;\tau),$$

i.e.

g	1A	2A	3A	5A	4B	7A	8A	6A	11A	15A	14A	23A	others
χ_g	24	8	6	4	4	3	2	2	2	1	1	1	0

Explicit forms of $Z_g(z;\tau)$ are summarized in Table 2 [4, 20, 19, 13]. See Table 3 for the number $A_g(n)$ of the non-BPS characters. The multiplicities $\mathrm{mult}_R(n)$ are given in Table 4. We have also give several explicit values of the Fourier coefficients $c_g(n,\ell)$ of the twisted elliptic genus in Table 5, where

$$Z_g(z;\tau) = \sum_{n=0}^{\infty} \sum_{\ell\in\mathbb{Z}} c_g(n,\ell)\, q^n\, \mathrm{e}^{2\pi \mathrm{i} \ell z}. \tag{3}$$

Therein we have used

$$c_g(4\,n - \ell^2) = c_g(n,\ell).$$

These coefficients play an important role in the Borcherds product [14].

1A	2A	3A	5A	4B	7A	7B	8A	6A	11A	15A	15B	14A	14B	23A	23B	12B	6B	4C	3B	2B	10A	21A	21B	4A	12A
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
23	7	5	3	3	2	2	1	1	1	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
252	28	9	2	4	0	0	0	1	−1	−1	−1	0	0	−1	−1	0	0	0	0	12	2	0	0	4	1
253	13	10	3	1	1	1	−1	−2	0	0	0	−1	−1	0	0	1	1	1	1	−11	−1	1	1	−3	0
1771	−21	16	1	−5	0	0	−1	0	0	1	1	0	0	0	0	−1	−1	−1	7	11	1	0	0	3	0
3520	64	10	0	0	−1	−1	0	−2	0	0	0	1	1	1	1	0	0	0	−8	0	0	−1	−1	0	0
45	−3	0	0	1	e_7^+	e_7^-	−1	0	1	0	0	$-e_7^+$	$-e_7^-$	−1	−1	1	−1	1	3	5	0	e_7^-	e_7^+	−3	0
$\overline{45}$	−3	0	0	1	e_7^-	e_7^+	−1	0	1	0	0	$-e_7^-$	$-e_7^+$	−1	−1	1	−1	1	3	5	0	e_7^+	e_7^-	−3	0
990	−18	0	0	2	e_7^+	e_7^-	0	0	0	0	0	e_7^+	e_7^-	1	1	1	−1	−2	3	−10	0	e_7^-	e_7^+	6	0
$\overline{990}$	−18	0	0	2	e_7^-	e_7^+	0	0	0	0	0	e_7^-	e_7^+	1	1	1	−1	−2	3	−10	0	e_7^+	e_7^-	6	0
1035	−21	0	0	3	$2e_7^+$	$2e_7^-$	−1	0	1	0	0	0	0	0	0	−1	1	−1	−3	−5	0	$-e_7^-$	$-e_7^+$	3	0
$\overline{1035}$	−21	0	0	3	$2e_7^-$	$2e_7^+$	−1	0	1	0	0	0	0	0	0	−1	1	−1	−3	−5	0	$-e_7^+$	$-e_7^-$	3	0
1035′	27	0	0	−1	−1	−1	1	0	1	0	0	−1	−1	0	0	0	2	3	6	35	0	−1	−1	3	0
231	7	−3	1	−1	0	0	−1	1	0	e_{15}^+	e_{15}^-	0	0	1	1	0	0	3	0	−9	1	0	0	−1	−1
$\overline{231}$	7	−3	1	−1	0	0	−1	1	0	e_{15}^-	e_{15}^+	0	0	1	1	0	0	3	0	−9	1	0	0	−1	−1
770	−14	5	0	−2	0	0	0	1	0	0	0	0	0	e_{23}^+	e_{23}^-	1	1	−2	−7	10	0	0	0	2	−1
$\overline{770}$	−14	5	0	−2	0	0	0	1	0	0	0	0	0	e_{23}^-	e_{23}^+	1	1	−2	−7	10	0	0	0	2	−1
483	35	6	−2	3	0	0	−1	2	−1	1	1	0	0	0	0	0	0	3	0	3	−2	0	0	3	0
1265	49	5	0	1	−2	−2	1	1	0	0	0	0	0	0	0	0	0	−3	8	−15	0	1	1	−7	−1
2024	8	−1	−1	0	1	1	0	−1	0	−1	−1	1	1	0	0	0	0	0	8	24	−1	1	1	8	−1
2277	21	0	−3	1	2	2	−1	0	0	0	0	0	0	0	0	0	2	−3	6	−19	1	−1	−1	−3	0
3312	48	0	−3	0	1	1	0	0	1	0	0	−1	−1	0	0	0	−2	0	−6	16	1	1	1	0	0
5313	49	−15	3	−3	0	0	−1	1	0	0	0	0	0	0	0	0	0	−3	0	9	−1	0	0	1	1
5796	−28	−9	1	4	0	0	0	−1	−1	1	1	0	0	0	0	0	0	0	0	36	1	0	0	−4	−1
5544	−56	9	−1	0	0	0	0	1	0	−1	−1	0	0	1	1	0	0	0	0	24	−1	0	0	−8	1
10395	−21	0	0	−1	0	0	1	0	0	0	0	0	0	−1	−1	0	0	3	0	−45	0	0	0	3	0

Table 1: Character table χ_R^g of the Mathieu group M_{24}. Here we have used $e_p^\pm = \frac{1}{2}\left(\pm\sqrt{-p}-1\right)$. We have used $\overline{\bullet}$ and $\bullet'$ to classify irreducible representation R.

g	$Z_g(z;\tau)$
1A	$2\,\phi_{0,1}(z;\tau)$
2A	$\frac{2}{3}\,\phi_{0,1}(z;\tau)+\frac{4}{3}\,\phi_2^{(2)}(\tau)\,\phi_{-2,1}(z;\tau)$
3A	$\frac{1}{2}\,\phi_{0,1}(z;\tau)+\frac{3}{2}\,\phi_2^{(3)}(\tau)\,\phi_{-2,1}(z;\tau)$
5A	$\frac{1}{3}\,\phi_{0,1}(z;\tau)+\frac{5}{3}\,\phi_2^{(5)}(\tau)\,\phi_{-2,1}(z;\tau)$
7A	$\frac{1}{4}\,\phi_{0,1}(z;\tau)+\frac{7}{4}\,\phi_2^{(7)}(\tau)\,\phi_{-2,1}(z;\tau)$
4B	$\frac{1}{3}\,\phi_{0,1}(z;\tau)+\left(-\frac{1}{3}\,\phi_2^{(2)}(\tau)+2\,\phi_2^{(4)}(\tau)\right)\,\phi_{-2,1}(z;\tau)$
6A	$\frac{1}{6}\,\phi_{0,1}(z;\tau)+\left(-\frac{1}{6}\,\phi_2^{(2)}(\tau)-\frac{1}{2}\,\phi_2^{(3)}(\tau)+\frac{5}{2}\,\phi_2^{(6)}(\tau)\right)\,\phi_{-2,1}(z;\tau)$
8A	$\frac{1}{6}\,\phi_{0,1}(z;\tau)+\left(-\frac{1}{2}\,\phi_2^{(4)}(\tau)+\frac{7}{3}\,\phi_2^{(8)}(\tau)\right)\,\phi_{-2,1}(z;\tau)$
11A	$\frac{1}{6}\,\phi_{0,1}(z;\tau)+\left(\frac{11}{6}\,\phi_2^{(11)}(\tau)-\frac{22}{5}\,[\eta(\tau)\,\eta(11\,\tau)]^2\right)\,\phi_{-2,1}(z;\tau)$
14A	$\frac{1}{12}\,\phi_{0,1}(z;\tau)+\left(-\frac{1}{36}\,\phi_2^{(2)}(\tau)-\frac{7}{12}\,\phi_2^{(7)}(\tau)+\frac{91}{36}\,\phi_2^{(14)}(\tau)-\frac{14}{3}\,\eta(\tau)\,\eta(2\,\tau)\,\eta(7\,\tau)\,\eta(14\,\tau)\right)\,\phi_{-2,1}(z;\tau)$
15A	$\frac{1}{12}\,\phi_{0,1}(z;\tau)+\left(-\frac{1}{16}\,\phi_2^{(3)}(\tau)-\frac{5}{24}\,\phi_2^{(5)}(\tau)+\frac{35}{16}\,\phi_2^{(15)}(\tau)-\frac{15}{4}\,\eta(\tau)\,\eta(3\,\tau)\,\eta(5\,\tau)\,\eta(15\,\tau)\right)\,\phi_{-2,1}(z;\tau)$
23A	$\frac{1}{12}\,\phi_{0,1}(z;\tau)+\left(\frac{23}{12}\,\phi_2^{(23)}(\tau)-\frac{23}{22}\,f_{23,1}(\tau)-\frac{161}{22}\,f_{23,2}(\tau)\right)\,\phi_{-2,1}(z;\tau)$
2B	$2\,\frac{\eta(\tau)^8}{\eta(2\tau)^4}\,\phi_{-2,1}(z;\tau)$
4A	$2\,\frac{\eta(2\tau)^8}{\eta(4\tau)^4}\,\phi_{-2,1}(z;\tau)$
4C	$2\,\frac{\eta(\tau)^4\,\eta(2\tau)^2}{\eta(4\tau)^2}\,\phi_{-2,1}(z;\tau)$
3B	$2\,\frac{\eta(\tau)^6}{\eta(3\tau)^2}\,\phi_{-2,1}(z;\tau)$
6B	$2\,\frac{\eta(\tau)^2\,\eta(2\tau)^2\,\eta(3\tau)^2}{\eta(6\tau)^2}\,\phi_{-2,1}(z;\tau)$
12B	$2\,\frac{\eta(\tau)^4\,\eta(4\tau)\,\eta(6\tau)}{\eta(2\tau)\,\eta(12\tau)}\,\phi_{-2,1}(z;\tau)$
10A	$2\,\frac{\eta(\tau)^3\,\eta(2\tau)\,\eta(5\tau)}{\eta(10\tau)}\,\phi_{-2,1}(z;\tau)$
12A	$2\,\frac{\eta(\tau)^3\,\eta(4\tau)^2\,\eta(6\tau)^3}{\eta(2\tau)\,\eta(3\tau)\,\eta(12\tau)^2}\,\phi_{-2,1}(z;\tau)$
21A	$\left(\frac{7}{3}\,\frac{\eta(\tau)^3\,\eta(7\,\tau)^3}{\eta(3\,\tau)\,\eta(21\,\tau)}-\frac{1}{3}\,\frac{\eta(\tau)^6}{\eta(3\,\tau)^2}\right)\,\phi_{-2,1}(z;\tau)$

Table 2: Twisted elliptic genus $Z_g(z;\tau)$.

$n \setminus g$	1A	2A	3A	5A	4B	7A	8A	6A	11A	15A	14A	23A	12B	6B	4C	3B	2B	10A	21A	4A	12A
1	90	−6	0	0	2	−1	−2	0	2	0	1	−2	2	−2	2	6	10	0	−1	−6	0
2	462	14	−6	2	−2	0	−2	2	0	−1	0	2	0	0	6	0	−18	2	0	−2	−2
3	1540	−28	10	0	−4	0	0	2	0	0	0	−1	2	2	−4	−14	20	0	0	4	−2
4	4554	42	0	−6	2	4	−2	0	0	0	0	0	0	4	−6	12	−38	2	−2	−6	0
5	11592	−56	−18	2	8	0	0	−2	−2	2	0	0	0	0	0	0	72	2	0	−8	−2
6	27830	86	20	0	−2	−2	2	−4	0	0	2	0	0	0	6	−16	−90	0	−2	6	0
7	61686	−138	0	6	−10	2	−2	0	−2	0	2	0	−2	−2	−2	30	118	−2	2	6	0
8	131100	188	−30	0	4	−3	0	2	2	0	−1	0	0	0	−12	0	−180	0	0	−4	2
9	265650	−238	42	−10	10	0	−2	2	0	2	0	0	−2	6	10	−42	258	−2	0	−14	−2
10	521136	336	0	6	−8	0	−4	0	0	0	0	2	−2	2	16	42	−352	−2	0	0	0
11	988770	−478	−60	0	−14	6	2	−4	2	0	−2	0	0	0	−6	0	450	0	0	18	0
12	1830248	616	62	8	8	0	0	−2	2	2	0	0	2	−6	−16	−70	−600	0	0	−8	−2
13	3303630	−786	0	0	22	−6	2	0	0	0	−2	2	0	−4	6	84	830	0	0	−18	0
14	5844762	1050	−90	−18	−6	0	2	6	0	0	0	2	0	0	18	0	−1062	−2	0	10	−2
15	10139734	−1386	118	4	−26	−4	−2	6	0	−2	0	0	2	2	−10	−110	1334	4	2	22	−2
16	17301060	1764	0	0	12	0	0	0	−4	0	0	0	2	6	−28	126	−1740	0	0	−12	0
17	29051484	−2212	−156	14	28	0	−4	−4	0	−1	0	0	0	0	12	0	2268	−2	0	−36	0
18	48106430	2814	170	0	−18	8	−2	−6	−2	0	0	−2	2	−6	38	−166	−2850	0	2	14	2
19	78599556	−3612	0	−24	−36	0	0	0	2	0	0	0	−2	−6	−20	210	3540	0	0	36	0
20	126894174	4510	−228	14	14	−6	−2	4	0	2	2	0	0	0	−42	0	−4482	−2	0	−18	0
21	202537080	−5544	270	0	48	4	4	6	−2	0	0	0	−2	6	16	−282	5640	0	−2	−40	2
22	319927608	6936	0	18	−16	−7	4	0	0	0	−1	0	0	4	48	300	−6968	2	−1	24	0
23	500376870	−8666	−360	0	−58	0	−2	−8	4	0	0	2	0	0	−18	0	8550	0	0	54	0
24	775492564	10612	400	−36	28	0	0	−8	0	0	0	0	0	−8	−60	−392	−10556	4	0	−28	−4
25	1191453912	−12936	0	12	64	12	−4	0	0	0	0	0	2	−10	32	462	13064	4	0	−72	0
26	1815754710	15862	−510	0	−34	0	−6	10	0	0	0	−1	0	0	78	0	−15930	0	0	22	−2
27	2745870180	−19420	600	30	−76	−10	4	8	−2	0	−2	0	0	8	−36	−600	19268	−2	2	84	0
28	4122417420	23532	0	0	36	2	0	0	0	0	−2	0	0	12	−84	660	−23460	0	2	−36	0
29	6146311620	−28348	−762	−50	100	−6	4	−10	−2	−2	2	0	0	0	36	0	28548	−2	0	−92	−2
30	9104078592	34272	828	22	−40	0	4	−12	4	−2	0	0	0	−8	96	−840	−34352	−2	0	48	0
31	13401053820	−41412	0	0	−116	0	−4	0	0	0	0	−2	−2	−10	−44	966	41180	0	0	108	0
32	19609321554	49618	−1062	34	50	18	2	10	−2	−2	2	0	0	0	−126	0	−49518	2	0	−46	2
33	28530824630	−59178	1220	0	126	0	−6	12	0	0	0	2	−4	12	62	−1204	59430	0	0	−138	0
34	41286761478	70758	0	−72	−66	−10	−6	0	6	0	2	0	0	12	150	1332	−70890	0	2	54	0
35	59435554926	−84530	−1518	26	−154	6	2	−14	0	2	2	0	0	0	−66	0	84222	2	0	158	2
36	85137361430	100310	1670	0	70	−12	−2	−10	0	0	0	0	−2	−18	−170	−1666	−100170	0	0	−74	−2

Table 3: The number of the non-BPS characters $A_g(n)$.

$n \setminus R$	1	23	252	253	1771	3520	$\frac{45}{45}$	$\frac{990}{990}$	$\frac{1035}{1035}$	$1035'$	$\frac{231}{231}$	$\frac{770}{770}$	483	265	2024	2277	3312	5313	5796	5544	10395
1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0
6	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
7	0	0	0	0	2	0	0	0	0	0	0	0	0	0	2	0	0	2	2	2	2
8	0	0	0	0	0	2	0	1	1	0	0	0	0	2	0	2	2	4	2	2	6
9	0	0	0	0	2	4	0	0	2	2	0	2	2	0	2	2	4	4	8	8	10
10	0	0	0	2	4	8	0	2	2	2	2	0	2	4	4	6	6	12	10	10	24
11	0	0	0	0	8	12	0	4	4	6	0	4	0	2	10	8	14	22	26	24	40
12	0	2	2	4	12	30	0	8	8	4	2	6	4	12	12	18	26	40	40	38	80
13	0	0	4	2	26	44	2	14	14	18	2	10	6	16	30	28	44	70	84	80	136
14	0	0	4	6	38	86	0	24	24	22	8	16	14	34	46	58	80	128	132	126	254
15	0	0	12	8	78	144	2	40	44	46	8	38	18	46	86	88	138	218	246	238	424
16	0	2	18	22	122	252	2	72	72	68	18	50	36	100	140	170	232	378	400	382	742
17	0	2	30	26	212	410	8	116	124	130	25	94	54	140	246	262	392	630	704	670	1222
18	0	6	50	58	342	704	6	194	202	192	50	148	100	256	388	454	654	1044	1120	1074	2058
19	0	4	80	72	582	1116	18	318	332	346	68	252	150	394	664	722	1062	1702	1880	1800	3320
20	0	14	128	138	904	1836	20	516	536	520	126	390	254	676	1036	1196	1716	2764	2980	2846	5408
21	2	20	214	200	1476	2902	40	814	860	872	182	652	396	1020	1684	1862	2742	4384	4828	4622	8572
22	2	32	328	346	2302	4616	55	1298	1348	1336	314	988	640	1686	2630	3000	4324	6950	7532	7204	13620
23	2	40	512	496	3638	7166	98	2020	2118	2144	460	1590	972	2546	4162	4624	6768	10856	11898	11376	21204
24	0	80	798	824	5584	11192	132	3140	3278	3236	744	2426	1544	4050	6376	7248	10500	16834	18294	17504	32976
25	8	108	1232	1208	8654	17084	234	4814	5038	5084	1106	3764	2336	6108	9892	11042	16112	25840	28288	27056	50524
26	6	174	1860	1904	13090	26148	322	7348	7670	7626	1742	5677	3602	9444	14968	16940	24566	39428	42894	41022	77176
27	12	252	2836	2802	19914	39436	514	11092	11618	11666	2560	8688	5394	14100	22744	25462	37148	59564	65114	62294	116494
28	16	398	4238	4310	29772	59330	742	16686	17418	17356	3922	12912	8160	21414	34026	38434	55764	89490	97456	93218	175146
29	26	560	6328	6286	44512	88280	1154	24840	25994	26078	5758	19380	12090	31636	50892	57068	83146	133356	145690	139342	260828
30	34	876	9368	9486	65776	131020	1642	36824	38480	38368	8642	28580	18008	47172	75158	84776	123176	197596	215318	205970	386724
31	58	1236	13802	13764	97060	192538	2500	54178	56660	56800	12582	42218	26384	69082	110920	124506	181274	290780	317502	303700	568798
32	76	1866	20166	20356	141714	282074	3564	79320	82884	82730	18576	61574	38738	101530	161978	182554	265284	425624	463950	443760	832834
33	122	2664	29396	29374	206524	410062	5286	115334	120644	120798	26830	89868	56226	147156	236010	265136	385974	619072	675796	646432	1211106
34	166	3900	42474	42810	298508	593800	7542	166990	174510	174330	39066	129694	81546	213644	341154	384250	558530	896052	977004	934530	1753356
35	248	5536	61184	61234	430134	854284	10988	240304	251292	251544	55956	187094	117138	306736	491602	552494	804038	1289768	1407604	1346380	2523178
36	334	8058	87622	88196	615626	1224424	15560	344314	359902	359564	80470	267604	168092	440318	703542	792158	1151786	1847690	2014952	1927370	3615350

Table 4: Multiplicities $\mathrm{mult}_R(n)$ in decomposition of $A_g(n)$.

$n \backslash g$	1A	2A	3A	4A	5A	6A	7A	8A	11A	14A	15A	23A	2B	4A	4C	3B	6B	12B	10A	12A	21A
−1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
0	20	4	2	0	0	−2	−1	−2	−2	−3	−3	−3	−4	−4	−4	−4	−4	−4	−4	−4	−4
3	−128	0	−2	0	2	6	5	8	4	7	8	10	0	16	8	4	12	8	10	10	11
4	216	−8	0	0	−4	−8	−8	−12	−4	−8	−10	−14	8	−24	−8	0	−16	−8	−12	−12	−14
7	−1026	−2	0	−2	4	16	17	30	8	19	25	32	−2	62	14	0	40	20	28	32	35
8	1616	16	−4	0	−4	−20	−22	−40	−12	−26	−34	−40	−16	−80	−16	−4	−52	−28	−36	−44	−46
11	−5504	0	4	0	6	36	47	88	18	49	69	85	0	176	24	4	108	48	70	92	95
12	8032	−32	4	0	−8	−44	−60	−112	−20	−60	−86	−110	32	−224	−32	−8	−136	−56	−88	−116	−120
15	−23550	2	−6	2	10	74	110	226	34	114	169	209	2	450	50	12	260	104	162	234	236
16	33048	56	0	0	−12	−88	−132	−284	−40	−140	−210	−256	−56	−568	−56	0	−320	−128	−196	−292	−294
19	−86400	0	0	0	20	144	239	536	60	245	380	471	0	1072	88	0	576	208	340	544	539
20	117280	−96	−8	0	−20	−168	−292	−656	−68	−292	−458	−572	96	−1312	−96	−8	−696	−240	−404	−664	−652
23	−283652	−4	10	−4	28	266	492	1180	104	500	805	996	−4	2364	140	10	1202	398	676	1194	1158
24	376608	160	12	0	−32	−308	−580	−1424	−120	−596	−968	−1190	160	−2848	−160	−24	−1432	−472	−800	−1444	−1396
27	−854528	0	−14	0	42	474	963	2464	172	973	1626	2015	0	4928	224	28	2388	728	1290	2486	2373
28	1112832	−256	0	0	−48	−544	−1134	−2944	−192	−1138	−1920	−2392	256	−5888	−256	0	−2816	−832	−1504	−2960	−2814
31	−2402298	6	0	6	62	816	1810	4934	278	1826	3155	3916	6	9862	358	0	4560	1288	2366	4960	4669
32	3082192	400	−20	0	−68	−932	−2106	−5832	−316	−2134	−3710	−4592	−400	−11664	−400	−20	−5332	−1492	−2740	−5868	−5508
35	−6378240	0	24	0	90	1368	1810	9520	440	3325	5919	7365	0	19040	560	24	8424	2216	4210	9560	8886
36	8077560	−616	24	0	−100	−1552	−2106	−11164	−488	−3836	−6886	−8596	616	−22328	−616	−48	−9776	−2512	−4844	−11204	−10380
39	−16135176	−8	−30	−8	134	2242	3303	17816	680	5886	10790	13469	−8	35640	840	60	15124	3720	7302	17874	16433
40	20213280	928	0	0	−140	−2528	−3828	−20752	−760	−6772	−12500	−15588	−928	−41504	−928	0	−17440	−4240	−8348	−20816	−19096
43	−39153024	0	0	0	186	3600	5858	32472	1036	10178	19185	24026	0	64944	1240	0	26496	6112	12370	32544	29624
44	48579872	−1376	−40	0	−208	−4040	−6724	−37584	−1144	−11624	−22060	−27684	1376	−75168	−1376	−40	−30376	−6872	−14064	−37656	−34186
47	−91617270	10	48	10	260	5680	10142	57802	1554	17244	33368	41939	10	115594	1834	48	45424	9856	20540	57904	52233
48	112732000	2016	52	0	−280	−6348	−11608	−66544	−1720	−19628	−38218	−48040	−2016	−133088	−2016	−104	−51816	−11112	−23256	−66668	−60024

Table 5: The Fourier coefficients $c_g(n)$ of the twisted elliptic genus.

g	2B	4A	4C	3B	6B	12B	10A	12A	21A
$v_g(\gamma)$	$\mathrm{e}^{-\frac{cd}{2}\pi\mathrm{i}}$	$\mathrm{e}^{-\frac{cd}{4}\pi\mathrm{i}}$	$\mathrm{e}^{-\frac{cd}{8}\pi\mathrm{i}}$	$\mathrm{e}^{-\frac{2cd}{9}\pi\mathrm{i}}$	$\mathrm{e}^{-\frac{cd}{18}\pi\mathrm{i}}$	$\mathrm{e}^{-\frac{cd}{72}\pi\mathrm{i}}$	$\mathrm{e}^{-\frac{cd}{10}\pi\mathrm{i}}$	$\mathrm{e}^{-\frac{cd}{12}\pi\mathrm{i}}$	$\mathrm{e}^{-\frac{2cd}{63}\pi\mathrm{i}}$

Table 6: Multiplier system $v_g(\gamma)$.

3 Rademacher Expansion

We set the q-series

$$-q^{\frac{1}{8}}\,\Sigma_g(\tau) = -2 + \sum_{n=1}^{\infty} A_g(n)\,q^n,$$

which can be given from

$$Z_g(z;\tau) = \chi_g\ \mathrm{ch}^{\widetilde{R}}_{\frac{1}{4},0}(z;\tau) - \frac{[\theta_{11}(z;\tau)]^2}{[\eta(\tau)]^3}\,\Sigma_g(\tau). \tag{4}$$

3.1 $\chi_g \neq 0$

We know from (1) that the BPS character $\mathrm{ch}^{\widetilde{R}}_{\frac{1}{4},0}(z;\tau)$ is a mock modular form. Then the decomposition (4) indicates that $\Sigma_g(\tau)$ is a mock theta function when $\chi_g \neq 0$. In fact we can identify that $A_g(n)$ is the Fourier coefficients of the mock theta functions on $\Gamma_0(\mathrm{ord}(g))$.

Following a method developed by Bringmann–Ono [3], we can compute the Poincaré–Maass series of the Fourier coefficients $A_g(n)$ in (2). We have

$$A_g(n) = \frac{-2\,\pi\,\mathrm{i}}{(8\,n-1)^{\frac{1}{4}}} \sum_{\substack{c=1\\ \mathrm{ord}(g)|c}}^{\infty} \frac{1}{\sqrt{c}}\, I_{\frac{1}{2}}\left(\frac{\pi\,\sqrt{8\,n-1}}{2\,c}\right) \sum_{\substack{k \bmod 4c\\ k^2=-8n+1 \bmod 8c}} \left(\frac{-4}{k}\right) \mathrm{e}^{\frac{k}{2c}\pi\mathrm{i}}, \tag{5}$$

where $\left(\frac{\bullet}{k}\right)$ is the Jacobi symbol, and $I_k(x)$ denotes the Bessel function

$$I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi\,x}}\,\sinh(x).$$

3.2 $\chi_g = 0$

In the case $\chi_g = 0$, there exists no contribution from the mock theta function in (4). Thus the q-series $\Sigma_g(\tau)$ is modular, and it can be written as as the η-product. We apply the circle method [26] to compute the Rademacher expansion of the Fourier coefficients, and we have

$$A_g(n) = \frac{4\,\pi}{(8\,n-1)^{\frac{1}{4}}} \sum_{\substack{c=1\\ \mathrm{ord}(g)|c}}^{\infty} \frac{1}{c}\, I_{\frac{1}{2}}\left(\frac{\pi\sqrt{8\,n-1}}{2\,c}\right) \sum_{\substack{d \bmod c\\ (c,d)=1}} v_g(\gamma)\,\mathrm{e}^{-3s(d,c)\pi\mathrm{i}+2\pi\mathrm{i}\frac{d}{c}n}. \tag{6}$$

Here $v_g(\gamma)$ follows from the multiplier system of the twisted elliptic genus $Z_g(z;\tau)$. See Table 6.

3.3 Asymptotics of the Number of the non-BPS Characters

Combining (5) and (6), we get

$$\log|A_g(n)| \sim \frac{2\,\pi}{\mathrm{ord}(g)}\sqrt{\frac{1}{2}\left(n-\frac{1}{8}\right)}.$$

See also [5]. This result generalizes that of [28] (see also [21]) where the entropy of the $\mathbb{Z}_N$ twisted CHL model is $1/N$ times the entropy of the untwisted model.

4 Concluding Remarks

The character decompositions of the elliptic genus for higher dimensional Calabi–Yau manifolds in terms of the superconformal characters are studied in [12, 11]. Further studies on the Mathieu moonshine may help us to study monstrous moonshine and its generalizations.

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A Notations

Throughout this article, we set $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ with τ in the upper half plane, $\tau \in \mathbb{H}$.

The Jacobi theta functions are defined by

$$\begin{aligned}
\theta_{11}(z;\tau) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \mathrm{e}^{2\pi\mathrm{i}\left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)}, \\
\theta_{10}(z;\tau) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \mathrm{e}^{2\pi\mathrm{i}\left(n+\frac{1}{2}\right)z}, \\
\theta_{00}(z;\tau) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} \mathrm{e}^{2\pi\mathrm{i}nz}, \\
\theta_{01}(z;\tau) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} \mathrm{e}^{2\pi\mathrm{i}n\left(z+\frac{1}{2}\right)}.
\end{aligned}$$

We use the Eisenstein series

$$\phi_2^{(N)}(z;\tau) = \frac{24}{N-1}\, q\frac{\partial}{\partial q} \log\left(\frac{\eta(N\tau)}{\eta(\tau)}\right).$$

We also use the Jacobi forms $\phi_{k,m}(z;\tau)$ with weight-k and index-m [18]. Amongst others, we have

$$\begin{aligned}
\phi_{0,1}(z;\tau) &= 4\left[\left(\frac{\theta_{10}(z;\tau)}{\theta_{10}(0;\tau)}\right)^2 + \left(\frac{\theta_{00}(z;\tau)}{\theta_{00}(0;\tau)}\right)^2 + \left(\frac{\theta_{01}(z;\tau)}{\theta_{01}(0;\tau)}\right)^2\right], \\
\phi_{-2,1}(z;\tau) &= -\frac{[\theta_{11}(z;\tau)]^2}{[\eta(\tau)]^6}.
\end{aligned}$$

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