Modular groups and motives

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1 Introduction

Let Γ be a *modular group* which is defined as the fundamental group of a moduli space. Then its nilpotent completion $\widehat{\Gamma}$ gives rise to extensions of certain quotients of $H_1(\Gamma)^{\otimes i}$ by themselves. When the moduli space has a natural model defined over a number field, by results of Deligne [D] and Hain [H1, 2], $\widehat{\Gamma}$ becomes a *mixed motive* which has especially Galois action and mixed Hodge structure. Furthermore, periods of this Hodge structure become iterated integrals which are represented as multiple L-values. Deligne and others (cf. [DG]) showed that for the moduli of curves of genus 0, $\widehat{\Gamma}$ has rich structure of mixed Tate motives. In this note, we consider the moduli of curves (with additional structure) of positive genus, and review a result of [I] on the motivic theory of $\widehat{\Gamma}$.

2 Teichmüller modular case

2.1

Let g and n be integers such that $3g-3+n \geq 0$, and $M_{g,n}$ be the moduli space of Riemann surfaces of genus g with n boundary components. Then $M_{g,n}$ has a natural model over $\mathbf{Q}$. Let $\Gamma_{g,n}$ be the fundamental group of $M_{g,n}$ whose base point is a point at infinity corresponding to a maximally degenerate algebraic curve, and $\Gamma_{g,n} \to Sp_{2g}(\mathbf{Z})$ be the natural homomorphism whose kernel is the Torelli group $T_{g,n}$. Note that $\Gamma_{g,n}$ has the trivial nilpotent completion for $g \geq 3$. Then Hain [H3] introduced the *relative completion* $R_{g,n}$ of $\Gamma_{g,n}$ for $\Gamma_{g,n} \to Sp_{2g}(\mathbf{Z})$ which is defined as the universal pro-algebraic group over $\mathbf{Q}$ with parallel exact sequences:

$$\begin{array}{ccccccccc} 1 & \to & T_{g,n} & \to & \Gamma_{g,n} & \to & Sp_{2g}(\mathbf{Z}) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & U_{g,n} & \to & R_{g,n} & \to & Sp_{2g} & \to & 1, \end{array}$$

where $U_{g,n}$ is pro-unipotent, and the middle downarrow is a homomorphism with Zariski dense image. Further, he showed jointly with M. Matsumoto (cf. [H4, 5, HM]) that the Lie algebra $\mathrm{Lie}\,(R_{g,n})$ of $R_{g,n}$ has motivic structure.

2.2

Theorem (cf. [I]). *Assume that $g \geq 3$, and let l be a prime. Then the $G_{\mathbf{Q}}$-module* $\mathrm{Lie}\,(R_{g,n}) \otimes \mathbf{Q}_l$ *is generated by the l-adic realizations of mixed Tate motives,* $\mathrm{Lie}\,(R_{1,1}) \otimes \mathbf{Q}_l$ *and* $\mathrm{Lie}\,(R_{1,2}) \otimes \mathbf{Q}_l$.

2.3

We give a sketch of this proof. First, using a result of Lochak [L] we show Teichmüller's Lego game proposed by Grothendieck [G] which claims that $\Gamma_{g,n}$ is an amalgamated product of $\Gamma_{0,4}$, $\Gamma_{0,5}$, $\Gamma_{1,1}$ and $\Gamma_{1,2}$, and that this product representation is $G_{\mathbf{Q}}$-equivariant under the profinite completed version. Hence by a result of Hain on the relation between $U_{g,n}$ and the unipotent completion of $T_{g,n}$, under the assumption that $g \geq 3$, we can show that the $G_{\mathbf{Q}}$-module $\mathrm{Lie}(R_{g,n}) \otimes \mathbf{Q}_l$ is generated by

$$\mathrm{Lie}(R_{0,4}) \otimes \mathbf{Q}_l,\ \mathrm{Lie}(R_{0,5}) \otimes \mathbf{Q}_l,\ \mathrm{Lie}(R_{1,1}) \otimes \mathbf{Q}_l,\ \mathrm{Lie}(R_{1,2}) \otimes \mathbf{Q}_l.$$

Furthermore, the first two $G_{\mathbf{Q}}$-modules are generated by the l-adic realizations of mixed Tate motives.

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