On Gottschling's description on Siegel's fundamental domain of degree 2 (a joint work with T. Oda)

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Introduction

Let $\{f_i \mid f_i \in \mathbb{Q}[X],\, i \in \Lambda\}$ be a finite set of ring of multi-variate polynomials of rational coefficients. For a subset L of the index set Λ, we set

$$I = I_L = \langle f_i \mid i \in L \rangle \,.$$

The ideal I is called zero-dimensional if the corresponding zero set $V(I)$ is zero-dimensional. There are several methods known to find $V(I)$ for a zero-dimensional ideal. One of the method is to determine the minimal polynomial of I. One way to obtain this is by way of Gröbner basis.

We apply this to get 0-cells (§2) of the Siegel fundamental domain $\mathcal{F}_2 \sim \Gamma\backslash\mathbb{H}_2$ of degree 2. By Siegel [10], it is known that $\mathcal{F}_2$ is described by the simultaneous solutions of finitely many inequalities of the form $|\det(CZ+D)| \geq 1$ and with the Minkowski condition. Gottschling [1] determined the minimal set of inequalities when the degree n is 2. By this fact, the boundary $\partial\mathcal{F}_2$ is described by the simultaneous zeros of polynomials $\{f_i \mid i \in L\}$ for $L \subset \Lambda$ for Λ is the index set determined by Gottschling (§2). Our result is to show all points which expected to be the 0-cells of the non-Euclidean locally polygon determined by Gottschling's data using the method described above.

Theorem. *There are* 180 *points* $p \in V(I_L) \cap \partial\mathcal{F}_2$ *for some* $L \subset \Lambda$ *such that* I_L *is zero dimensional.*

For example, the point $e_1 = \left(\begin{smallmatrix} \eta & (\eta-1)/2 \\ (\eta-1)/2 & \eta \end{smallmatrix}\right)$ with $\eta = (1+2\sqrt{2}i)/3$, which appeared already Gottschling's paper, is on the intersection of 3 "rank 1" equalities, 4 "rank 2" equalities and 2 Minkowski conditions and is strictly positive in the other 19 inequalities.

This result seems to non-Euclidean version of computation of "extreme lattices" in the theory of Voronoi [7]. The Hermite constant is obtained by one of them in Voronoi theory. There also is a notion of the Hermite constant in the case of the linear algebraic group [11] and we are just in the situation of the group being the symplectic group of degree 2. So we also expect the Hermite constant is obtained by the data obtained above. By the theorem we can show that e_1 attain the minimum $\det(Y)$ among 180 points. We remark that in 2009, Kawamura announced the determination of the Hermite constant of $\mathcal{F}_2$ [5].

Another application of the method we expect is to apply the method in general degree. In the case degree $n = 3$, there is a partial result of Kamino [4] analogous to Gottschling's index result. Optimally speaking, the 0-cells will predict the minimal set of inequalities which could possibly refine his result.

1 Siegel's fundamental domain of degree 2

Let $\mathbb{H}_2$ be the Siegel upper half space of degree 2, namely,

$$\mathbb{H}_2 = \{Z = X + \sqrt{-1}Y \mid {}^tZ = Z,\, Y\colon \text{ positive definite}\}.$$

The discrete group $\Gamma = Sp(2,\mathbb{Z})$ is a set of symplectic matrices whose entries are integers. The matrix $\gamma \in \Gamma$ acts on $\mathbb{H}_2$ discontinuously by linear fractional transform

$$\gamma \cdot Z = (AZ+B)(CZ+D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Siegel's fundamental domain $\mathcal{F}_2$ of $\mathbb{H}_2$ with respect to the action of Γ is given as follows ([10, 6]).

By definition $\mathcal{F}_2$ consists of the points determined by the following three types of inequalities.

(i) $|X_{ij}| \le 1/2$.

(ii) Y is Minkowski reduced. Specifically in degree 2 case, it is equivalent to $0 \le 2Y_{12} \le Y_{11} \le Y_{22}$.

(iii) $|\det(CZ+D)| \ge 1$ for all $\gamma = \begin{pmatrix} * & * \\ C & D \end{pmatrix} \in \Gamma$.

Siegel pointed out the finiteness result of the condition (iii). In degree 2 case, Gottschling ([1]) determined the set of 15 inequalities out of (iii) and showed it is a minimal set. In general degrees, however, it is still open that the determination of minimal set of inequalities in (iii).

To describe Gottschling's description, we prepare matrices. Let E_{ij} be a matrix unit and O the zero matrix. For simplicity, we set

$$E_1 = E_{11}, E_2 = E_{22}, I_\pm = E_1 \pm E_2, J = E_{12} + E_{21}, J_i = J + E_i, \quad (i = 1, 2).$$

Theorem 1 ([1, Satz 1,2]). *In degree* 2 *case, Condition* (iii) *given by the following matrix pairs is sufficient and no proper subset of them cannot attain the fundamental domain* $\mathcal{F}_2$.
rank$(C) = 1$:

$$(C,D) \in \{(E_1,E_1),(E_2,E_2),(E_1+E_{12}, I_+ + E_{21}),(E_1+E_{12}, -I_+ - E_{21})\},$$

rank$(C) = 2$:

$$C = I_+, \quad \pm D = O, E_1, E_2, I_\pm, J, J_1, J_2.$$

2 Index oriented model of $\mathfrak{F}_2$

Let $\partial\mathcal{F}_2$ be the boundary of $\mathcal{F}_2$. From the definition of $\mathcal{F}_2$, we are motivated to introduce the defining polynomials f_λ to describe $\partial\mathcal{F}_2$.

From Condition (i), we set

$$f_{X_1}(Z) = 1/2 - X_{11}, \quad f_{X_2}(Z) = 1/2 - X_{22}, \quad f_{X_3}(Z) = 1/2 - X_{12}.$$

From Condition (ii), we set

$$f_{Y_1}(Z) = Y_{22} - Y_{11}, \quad f_{Y_2}(Z) = Y_{11} - 2Y_{12}, \quad f_{Y_3}(Z) = Y_{12}.$$

Next consider Condition (iii) with help of Theorem 1. For simplicity we denote $\mathbf{1} = (E_1, E_1)$, $\mathbf{2} = (E_2, E_2)$ and $R = (E_1 + E_{12}, I_+ + E_{21})$ in rank 1 case of Theorem 1. In rank 2 case C being always an identity, we represent (C, D) by D. Then we put

$$f_\lambda(Z) = |\det(CZ + D)|^2 - 1, \quad \lambda \text{ stands for } (C, D).$$

We define $\theta(z) = -\overline{z}$ for $z \in \mathbb{C}$ and extend $\theta_0(Z) = (\theta(Z_{ij}))$. We define $f_{\theta_0\lambda}(Z) = f(\theta_0 Z)$.

Summarizing, let Λ be the collection of the indices

$$\Lambda = \{Y_1, Y_2, Y_3, O\} \cup \{D, \theta_0 D \mid D = R, E_i, I_\pm, J, J_i, X_k, \quad i = 1, 2, k = 1, 2, 3\}$$

so the cardinality of Λ is 28. Then $\mathcal{F}_2 = \bigcap_{\lambda\in\Lambda}\{f_\lambda(Z) \geq 0\}$. Put

$$W_\lambda = \{Z \in \mathcal{F}_2 \mid f_\lambda(Z) = 0\}.$$

We also use the extended notation $W_L = \bigcap_{\lambda \in L} W_\lambda$. Then we have

$$\partial\mathcal{F}_2 = \bigcup_{\emptyset \neq L \subset \Lambda} W_L.$$

We name "label" as a subset L of Λ. Note that the labels are inclusion-reversing, i.e., $L \subset L' \Rightarrow W_L \supset W_{L'}$.

Let us introduce a notion of 0-cells, a candidate of "vertices" in this model. Let V be the affine space which the coordinate $(X_{11}, X_{12}, X_{22}, Y_{11}, Y_{12}, Y_{22})$ lives. Put $I_L = \langle f_\lambda \mid \lambda \in L\rangle$ and consider the zero set $V(I)$. Then one has $W_L = V(I_L) \cap \mathcal{F}_2$. A label L or W_L is called 0-cell if the dimension of $V(I_L)$ is zero.

3 0-cells

In this section we define 180 points and its associated labels a priori.

First of all, we extend θ_0 on the matrix as

$$\theta_{ij}(Z) = (\ldots, -\overline{Z_{ij}}, \ldots).$$

Then $\theta_0(Z) = \theta_{11}\theta_{12}\theta_{22}(Z)$. For $\gamma = \left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$, we also define $\sigma_0(Z) = \gamma \cdot Z$, which flips Z_{11} and Z_{22}. Let Δ be the finite group generated by θ_{ij} and σ_0. We say the similarity class by the equivalence class $\Delta\backslash\mathbb{H}_2$ and write $Z \overset{\Delta}{\sim} Z'$ if $\delta(Z) = Z'$ for a $\delta \in \Delta$. We may choose representative points whose X_{ij}-coordinates are all non-negative and $X_{11} \geq X_{22}$.

Using this similarity we define 40 points $e_0, \ldots, e_{39}$ as representatives of similarity class as in Table 1. In the table, we put $\tau(x) = \sqrt{1 - x^2}$ and put ω_j an algebraic number given by a real root of Ω_j specified uniquely by indicated additional conditions in Tables 2 and 3. Their floating point expressions are given in Table 4.

Finally we define actual points using Δ. It is given in Tables 5 and 6. For given point p, we associate a label L_p. We explain this in next Section.

4 0 cells and maximal labels

We state the main result of this paper.

Theorem 2. *Let p be a point and $L = L_p$ be the label associated to p in Tables* 5 *and* 6*. Then we have*
(1) I_L *is zero dimensional.*
(2) $p \in W(L)$.
(3) L *is maximal i.e.,* $p \notin W(L')$ *for* $L' \supsetneq L$.

Proof. Once the point is given by an exact way, it is easy to check that $f_\lambda(p)$ being strictly positive for $\lambda \in \Lambda \setminus L$ by evaluation. This positivity shows the maximality of L, provided that $p \in W(L)$. It is also easy to check that I_L is zero-dimensional.

To show that p is exactly in $V(I_L)$ requires an algebraic tool to confirm it is a simultaneous roots of polynomials. We postpone this part to Section 5. □

Computation of $\det(Y)$ gives the Corollary.

Corollary 3. *The point e_1 attains* $\det(Y)$ *minimum among* 180 *points in Tables* 5, 6.

5 minimal polynomial of ideals

To show that the points indicated are zeros of given polynomials, we use algebraic methods. The propositions below are cited from [8, 9] without proofs. Let $\mathbb{Q}[X]$ be a polynomial ring for multivariate $X = (X_1, \dots, X_n)$. Let I be an ideal of $\mathbb{Q}[X]$. I is called zero dimensional if $0 < \dim \mathbb{Q}[X]/I < \infty$.

Let $f \in \mathbb{Q}[X]$. We say $m(I, f; t) \in \mathbb{Q}[t]$ is a minimal polynomial with respect to f of I when

$$\{g(t) \in \mathbb{Q}[X] \mid g(f) \in I\} = \langle m(I, f; t) \rangle.$$

Proposition 4. *If I is zero dimensional, there is a unique $m(I, f; t)$ with respect to any $f \in \mathbb{Q}[X]$.*

Proposition 5. *If we take $f(X) = X_i$ a monomial of degree* 1*, then*

$$\{p \in V(\mathbb{C}) \mid m(I, X_i; p_i) = 0 \textit{ for all } i\} \supset V(I).$$

By these propositions we have candidates of points of $V(I)$. More about minimal polynomials of I, we have

Proposition 6. *Let $G \subset I$ be a Gröbner basis of I with respect to lexicographic order $X_1 \succ X_2 \succ \cdots \succ X_n$. Then*

$$I \cap \mathbb{Q}[X_n] = \langle m(I, X_n; t) \rangle.$$

If G is reduced, then $G \cap \mathbb{Q}[X_n] = m(I, X_n; t)$.

So one of the way to obtain the minimal polynomials is to compute Gröbner basis. Finally we refer to

Proposition 7 (shape basis). *Suppose that I be zero-dimensional and that* $\dim \mathbb{Q}[X]/I = \deg(m(I, X_n; t))$. *Then there is a set of polynomials $\{g_i(t)\}$ such that*

$$G = \{X_1 - g_1(X_n), X_2 - g_2(X_n), \dots, X_{n-1} - g_{n-1}(X_n), m(I, X_n; t)\}.$$

So the problem is reduced to obtain the zeros of uni-variate polynomial $m(I, X_n; t)$. The real roots of a one variable polynomial is fairly easily obtained by Newton method. So, by Proposition above, we obtain all the real zero points of $V(I)$ when I zero-dimensional.

Back to our case, apart from obtaining shape basis we do a mixed strategy utilizing other kinds of term order such as `grlex`, `grrevlex`. They sometimes make dependency a simpler form, since there is a defect using Gröbner basis with lexicographic order (`lex`) such as its coefficient explosion phenomena or inefficiency of obtaining the basis.

6 Proof of the part $p \in V(I)$

Taking the result in Section 5 in mind, we follow the following procedure. We consider the case $p \overset{\Delta}{\sim} e_{12}$. It also clarify the situation of maximality of labels. Take $L = [\mathbf{1}, \mathbf{2}, \theta R, E_1, Y_1, Y_2]$. Then I_L can be checked zero-dimensional. The minimal polynomial $m(t) = m(I, X_{22}; t)$ of X_{22} is of degree 8 but by factorization, X_{22} should be $1/3$, $\pm 1/5$ or $\Omega_6(-X_{22}) = 0$, so $X_{11} = 1/3, \pm 1/5$ or $-\omega_6$.

Consider the case $X_{22} = -\omega_6$. Since variables other than X_{12} depends on X_{22} very simply, so the essential is the dependency to X_{12}. The shape basis says

$$\begin{aligned} &- 40411800X_{22}^7 + 2571225X_{22}^6 + 91990172X_{22}^5 + 52140601X_{22}^4 - 53576848X_{22}^3 \\ &\qquad - 50659293X_{22}^2 + 16986764X_{22} - 14988288X_{12} - 4052533 = 0. \end{aligned}$$

Taking modulo $\Omega_6(-X_{22})$, we obtain the value of X_{12}. Actually, taking another term order like `grrevlex`, we may see the basis would tell a simpler dependency. In any event we can obtain $\theta_2\theta_3 e_{12}$ and is theoretically on $V(I)$. After checking positivity, we conclude whether on $W(L)$ or not. If $X_{22} = \pm 1/5$, it also give arise to the point $Z \in V(I)$, but it should make $f_O(Z)$ negative so is not the case. On the other hand, when $X_{22} = 1/3$, it also gives a valid point $\theta_2 e_1 \in W(L)$, but we readily see that happens also in $L_2 = [\mathbf{1}, \mathbf{2}, \theta R, E_1, E_2, I_+, \theta J, Y_1, Y_2]$. and is turned out to be the maximal label of $\theta_2 e_1$.

Points by points, we check the validity of points and the label. We then conclude the Theorem.

7 Computational results

We collect other computational results. By the exhaustive search in 2^Λ, we obtain

Search Result 8. *There are* 752370 *labels out of* 2^Λ *so that* I_L $(L \subset \Lambda)$ *is zero-dimensional.*

By this fact, we check all zero dimensional I_L to obtain all $p \in W(L)$.

Procedure 9.

step 1: *Collect L's where L is zero-dimensional.*

step 2: *For each L such that I_L is zero-dimensional, do the following:*

step 2-1: *Compute the minimal polynomial $m(I, X_{ij}; t)$ $m(I, Y_{ij}; t)$. If one of them have no real roots, or if no root $|X_{ij}| \leq 1/2$, or against Minkowski condition, skip to next L.*

step 2-2: *Compute all real zeros $p_k \in V(I)$.*

step 2-3: *For all i, if $p_i \in W(L)$ then register (p_i, L).*

step 3: *For points p registered, take maximal $L_{\max}$ and output $(p, L_{\max})$, finish.*

Practically speaking, **step 2-2** by shape basis is so costly. Instead we used

step 2-2': Check $p \in V(I)$ using floating point computation by certain accuracy for all combination of real roots of the minimal polynomials in **step 2-1**.

By this way, we obtained the following result:

Search Result 10. (1) *Among zero-dimensional ideals I_L there are 2146 labels so that $W(L)$ is non-empty, actually $|V(I_L) \cap \partial\mathcal{F}_2| \leq 2$. We remark that not all of them are distinct.*
(2) *They,* 2146 *labels produce* 180 *points on the boundary of the fundamental domain $V(I_L) \cap \partial\mathcal{F}$ indicated in Table* 5.
(3) *When such points Z are not similar to $Z \overset{\Delta}{\sim} e_0, e_1$ nor e_3, then there is a unique label L so that $Z \in V(I_L)$, I_L zero-dimensional.*

Though it is not theoretical proof, our search insist that Theorem 2 is best possible, i.e, there are no 0-cells in our sense.

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Table 1: Points $p = X + \sqrt{-1}Y \pmod{\Delta}$ on $W(I)$ for zero-dimensional ideals I

p	X_{11}	X_{12}	X_{22}	Y_{11}	Y_{12}	Y_{22}
e_0	$1/2$	0	$1/2$	ω_0	0	ω_0
e_1	$1/3$	$1/3$	$1/3$	ω_1	$\omega_1/2$	ω_1
e_2	$1/2$	$1/2$	$1/2$	ω_0	$\sqrt{\omega_2+1}/2$	ω_0
e_3	$1/2$	$1/2$	$1/2$	ω_0	0	ω_0
e_4	$1/2$	$1/2$	0	1	$1/2$	1
e_5	$1/2$	0	$1/2$	1	$1/2$	1
e_6	$1/2$	$\omega_0 - 3/4$	$1/2$	ω_0	$\omega_0/2$	$(3-\omega_0)/2$
e_7	0	$1/2$	0	1	$1/2$	1
e_8	$1/2$	$1/2$	$1/2$	ω_3	$\omega_3/2$	ω_3
e_9	$1/2$	$1/2$	$1/2$	ω_0	$\omega_0/2$	ω_4
e_{10}	$1/2$	$1/2$	0	ω_0	$\omega_0/2$	$\omega_5/16$
e_{11}	$1/2$	0	$1/2$	ω_0	$\omega_0/2$	$\omega_5/16$
e_{12}	ω_6	$(1-\omega_6)/2$	ω_6	$\tau(\omega_6)$	$\tau(\omega_6)/2$	$\tau(\omega_6)$
e_{13}	$(1-3\omega_6)/2$	$(1-\omega_6)/2$	ω_6	$\tau(\omega_6)$	$\tau(\omega_6)/2$	$\tau(\omega_6)$
e_{14}	$(1-3\omega_6)/2$	ω_6	$(1-3\omega_6)/2$	$\tau(\omega_6)$	$\tau(\omega_6)/2$	$\tau(\omega_6)$
e_{15}	ω_6	$(1-3\omega_6)/2$	ω_6	$\tau(\omega_6)$	$\tau(\omega_6)/2$	$\tau(\omega_6)$
e_{16}	$1/2$	$\omega_7/4$	$\frac{\omega_8}{8} + (\omega_7+1)/2$	ω_0	$\omega_0/2$	$\omega_0\sqrt{\omega_9}/4$
e_{17}	$1/2$	$(2-\omega_7)/4$	$\omega_8/8$	ω_0	$\omega_0/2$	$\omega_0\sqrt{\omega_9}/4$
e_{18}	ω_{10}	$1/2$	ω_{10}	1	$1/2$	1
e_{19}	$1/2$	$\omega_{11}/2$	$(1+\omega_{12})/2$	ω_0	$\sqrt{1+\omega_{12}}/2$	$\tau((1+\omega_{12})/2)$
e_{20}	$(1+\omega_{12})/2$	$(1+\omega_{12})/2$	$(1+\omega_{12})/2$	$\tau((1+\omega_{12})/2)$	$\sqrt{\omega_{13}+5}/2$	$\tau((1+\omega_{12})/2)$
e_{21}	$2\omega_{14}$	$1/2$	ω_{14}	$\tau(\omega_{14})$	$\tau(\omega_{14})/2$	$\tau(\omega_{14})$
e_{22}	ω_{15}	$1/2$	ω_{16}	$\tau(\omega_{15})$	$\tau(\omega_{15})/2$	$\tau(\omega_{15}-\omega_{16})$
e_{23}	ω_{17}	$1/2$	ω_{17}	$\tau(\omega_{17})$	$\tau(\omega_{17})/2$	$\tau(\omega_{17})$
e_{24}	ω_{17}	$1/2-\omega_{17}$	$1/2$	$\tau(\omega_{17})$	$\tau(\omega_{17})/2$	$\tau(\omega_{17}/2)$
e_{25}	$1/2$	$(1-\omega_{18})/4$	$1/2$	ω_{19}	$\omega_{19}/2$	ω_{19}
e_{26}	$1/2$	$\omega_{20}/2$	$1-\omega_{20}$	$\sqrt{\omega_{21}/2}$	$\sqrt{\omega_{21}/8}$	$\sqrt{\omega_{21}/2}$
e_{27}	ω_{22}	$\omega_{23}/2$	$1/2$	$\tau(\omega_{22})$	$\tau(\omega_{22})/2$	$\tau(\omega_{22})\omega_{24}/2$
e_{28}	ω_{25}	$1/2$	ω_{26}	$\tau(\omega_{25})$	$\tau(\omega_{25})/2$	$\tau(\omega_{26})$
e_{29}	$1/2$	$1/2$	0	ω_0	$\omega_{27}/2$	1
e_{30}	$(1-\omega_{27})/2$	$1/2$	$(1-\omega_{27})/2$	$\tau((1-\omega_{27})/2)$	$\tau((1-\omega_{27})/2) - \frac{1}{2}$	$\tau((1-\omega_{27})/2)$
e_{31}	ω_{28}	$1/2$	$\omega_{28}/2$	$\tau(\omega_{28})$	$\tau(\omega_{28})/2$	$\tau(\omega_{28}/2)$
e_{32}	ω_{29}	ω_{30}	ω_{29}	$\tau(\omega_{29})$	$\tau(\omega_{29})/2$	$\tau(\omega_{29})$
e_{33}	$1-2\omega_{30}$	$1-\omega_{29}-\omega_{30}$	ω_{29}	$\tau(\omega_{29})$	$\tau(\omega_{29})/2$	$\tau(\omega_{29})$
e_{34}	$1-2\omega_{30}$	$\omega_{30}-\omega_{29}$	ω_{29}	$\tau(\omega_{29})$	$\tau(\omega_{29})/2$	$\tau(\omega_{29})$
e_{35}	ω_{31}	$(1+\omega_{31}-\omega_{32})/2$	$\omega_{32}/2$	$\tau(\omega_{31})$	$\tau(\omega_{31})/2$	$\tau(\omega_{32}/2)$
e_{36}	ω_{31}	$(1-\omega_{31}-\omega_{32})/2$	$\omega_{32}/2$	$\tau(\omega_{31})$	$\tau(\omega_{31})/2$	$\tau(\omega_{32}/2)$
e_{37}	ω_{33}	$\omega_{34}/2$	$\omega_{35}/4$	$\tau(\omega_{33})$	$\tau(\omega_{33})/2$	$\tau(\omega_{35}/4)$
e_{38}	ω_{33}	$\omega_{36}/2$	$\omega_{37}/4$	$\tau(\omega_{33})$	$\tau(\omega_{33})/2$	$\tau(1-\frac{\omega_{37}}{4}-\omega_{36}-\omega_{33})$
e_{39}	$1/2$	$\omega_{38}/2$	$\omega_{39}/2$	$\tau(\omega_{39}/2)$	$\sqrt{\omega_{40}}/2$	$\tau(\omega_{39}/2)$

Table 2: Minimal polynomials of algebraic numbers appearing in the coordinates

$\Omega_0(t) = 4t^2 - 3, \qquad \omega_0 = \sqrt{3}/2 = 0.866...$

$\Omega_1(t) = 9t^2 - 8, \qquad \omega_1 = 2\sqrt{2}/3 = 0.9428...$

$\Omega_2(t) = t^4 + 8t^2 - 192t - 176, \qquad \omega_2 = -0.8811...$

$\Omega_3(t) = 9t^4 + 4t^2 - 16, \qquad \omega_3 = \frac{1}{3}\sqrt{2\sqrt{37} - 2} = 1.0627...$

$\Omega_4(t) = 65536t^4 - 133376t^2 + 61009, \qquad \omega_4 = \sqrt{(3\sqrt{3045} + 521)/2^9} = 1.1579...$

$\Omega_5(t) = t^4 - 441t^2 + 42849, \qquad \omega_5 = 3\sqrt{(\sqrt{285} + 49)/2} = 17.2...$

$\Omega_6(t) = 8t^3 - 13t^2 + 10t - 1, \omega_6 = 0.1163...$

$\Omega_7(t) = t^4 + 22t^2 + 96t - 39, \qquad \omega_7 = 0.3739...$

$\Omega_8(t) = t^4 - 14t^3 + 11t^2 + 2250t - 3687, \qquad \omega_8 = (3 + 10\omega_7 - \omega_7^2)/4 = 1.6500...$

$\Omega_9(t) = t^4 - 326t^3 + 31019t^2 - 574886t + 93025, \qquad \omega_9 = -(\omega_7^3 + 14\omega_7^2 + 5\omega_7 - 100)/4 = 24.02...$

$\Omega_{10}(t) = 4t^4 - 8t^3 + 12t^2 - 8t + 1, \qquad \omega_{10} = 0.1593...$

$\Omega_{11}(t) = t^4 - 2t^3 - 8t + 7, \qquad \omega_{11} = -(\omega_{12}^3 - \omega_{12}^2 + 10\omega_{12} + 4)/2 = 0.7985...$

$\Omega_{12}(t) = t^4 + 8t^2 + 12t + 4, \qquad \omega_{12} = -0.5188...$

$\Omega_{13}(t) = t^4 + 8t^2 + 48t - 224, \qquad \omega_{13} = -2\omega_{12}^3 + \omega_{12}^2 - 18\omega_{12} - 14 = -4.112...$

$\Omega_{14}(t) = 9t^4 - 24t^3 + 68t^2 - 48t + 4, \qquad \omega_{14} = 0.0959...$

$\Omega_{15}(t) = 9t^4 - 24t^3 + 52t^2 - 32t + 4, \qquad \omega_{15} = 0.1670...$

$\Omega_{16}(t) = 1296t^4 - 3744t^3 + 5224t^2 - 4504t + 505, \qquad \omega_{16} = (3\omega_{15}^3 - 6\omega_{15}^2 + 16\omega_{15} - 2)/4 = 0.1299...$

$\Omega_{17}(t) = 15t^4 - 32t^3 - 4t^2 + 32t - 4, \qquad \omega_{17} = 0.1291...$

$\Omega_{18}(t) = t^6 - 2t^4 + 8t^3 - 259t^2 + 256t - 64, \qquad \omega_{18} = 0.5285...$

$\Omega_{19}(t) = 2304t^{12} + 3712t^{10} - 1776t^8 - 2848t^6 - 1016t^4 - 240t^2 + 225,$

$\omega_{19} = (1 - \omega_{18})/(2\sqrt{2\omega_{18} - 1}) = 0.9861...$

$\Omega_{20}(t) = 16t^6 - 48t^5 + 52t^4 - 8t^3 - 280t^2 + 400t - 147, \qquad \omega_{20} = 0.7642...$

$\Omega_{21}(t) = 36t^6 + 116t^5 - 111t^4 - 356t^3 - 254t^2 - 120t + 225, \qquad \omega_{21} = 1.9448...$

$\Omega_{22}(t) = 9t^6 - 2t^5 - 57t^4 + 107t^2 + 2t - 23, \qquad \omega_{22} = 0.4843...$

$\Omega_{23}(t) = 9t^6 + 44t^5 - 36t^4 - 52t^3 + 124t^2 + 24t - 12, \qquad \omega_{23} = \sqrt{2 + 2\omega_{22}} - 1 - \omega_{22} = 0.2386...$

$\Omega_{24}(t) = 12t^6 - 88t^5 + 148t^4 + 28t^3 - 120t^2 + 36t + 23,$

$\omega_{24} = \omega_{22}^5 + (37\omega_{22}^4 - 148\omega_{22}^3 - 152\omega_{22}^2 + 130\omega_{22} + 73)/36 = 2.402...$

$\Omega_{25}(t) = 7t^{12} - 16t^{11} - 44t^{10} + 368t^9 - 1804t^8 + 3840t^7 - 3744t^6 - 3200t^5$
$+ 12560t^4 - 6912t^3 - 6848t^2 + 6912t - 1344, \qquad \omega_{25} = 0.2914...$

$\Omega_{26}(t) = 7168t^{12} - 34816t^{11} + 20224t^{10} + 231936t^9 - 599760t^8 + 222816t^7 + 713128t^6 + 31320t^5$
$- 1793601t^4 + 1157252t^3 + 463714t^2 - 470548t + 4511,$

$\omega_{26} = (3997\omega_{25}^{11} - 4999\omega_{25}^{10} - 31374\omega_{25}^9 + 180128\omega_{25}^8 - 836696\omega_{25}^7 + 1270256\omega_{25}^6 - 551200\omega_{25}^5$
$- 2995952\omega_{25}^4 + 4650320\omega_{25}^3 + 1295920\omega_{25}^2 - 4073760\omega_{25} + 989888)/572416 = 0.0096...$

$\Omega_{27}(t) = t^8 + 4t^6 - 16t^5 - 42t^4 - 32t^3 + 164t^2 - 16t - 47, \qquad \omega_{27} = 0.7113...$

$\Omega_{28}(t) = 9t^8 - 24t^7 + 56t^6 + 16t^5 - 232t^4 - 416t^3 + 1760t^2 - 1088t + 144, \qquad \omega_{28} = 0.1851...$

$\Omega_{29}(t) = t^8 - 4t^7 + 16t^6 - 20t^5 - 40t^4 + 64t^3 + 28t^2 - 40t + 4, \qquad \omega_{29} = 0.1105...$

Table 3: Minimal polynomials of algebraic numbers (2)

$$\Omega_{30}(t) = 256t^8 - 512t^7 + 1280t^6 - 1536t^5 + 1888t^4 - 1120t^3 + 512t^2 + 96t - 87$$
$$\omega_{30} = (3\omega_{29}^7 - 11\omega_{29}^6 + 59\omega_{29}^5 - 88\omega_{29}^4 + 78\omega_{29}^3 - 2\omega_{29}^2 - 188\omega_{29} + 74)/132 = 0.4036...$$
$$\Omega_{31}(t) = t^{16} - 12t^{15} + 88t^{14} - 448t^{13} + 1716t^{12} - 5104t^{11} + 11896t^{10} - 21956t^9 + 32228t^8 - 37776t^7 + 35464t^6 - 25968t^5 + 14736t^4 - 5744t^3 + 1220t^2 - 120t + 4, \qquad \omega_{31} = 0.18587...$$
$$\Omega_{32}(t) = t^{16} - 8t^{15} + 64t^{14} - 328t^{13} + 1248t^{12} - 3584t^{11} + 6512t^{10} - 2048t^9 - 21472t^8 + 52224t^7 - 40448t^6 - 30848t^5 + 75008t^4 - 15360t^3 - 36352t^2 + 6144t + 256, \qquad \omega_{32} = 0.2019...$$
$$\Omega_{33}(t) = 2704t^{18} + 6656t^{17} + 14600t^{16} + 40200t^{15} - 89391t^{14} - 422918t^{13} + 347143t^{12} + 483432t^{11} - 462293t^{10} + 218930t^9 + 255381t^8 - 256272t^7 + 3376t^6 + 5844t^5 + 5256t^4 - 384t^3 - 152t^2 - 16t + 4, \qquad \omega_{33} = 0.2726...$$
$$\Omega_{34}(t) = 2704t^{18} - 14976t^{17} + 50584t^{16} - 181536t^{15} + 325297t^{14} - 275424t^{13} + 906700t^{12} - 3114868t^{11} + 12229602t^{10} - 26726196t^9 + 22096964t^8 + 20670736t^7 - 134656607t^6 - 103677140t^5 + 998831544t^4 - 1897054492t^3 + 2036832876t^2 - 398345048t - 481484588, \qquad \omega_{34} = 0.9447...$$
$$\Omega_{35}(t) = 2704t^{18} - 58032t^{17} + 283180t^{16} + 4323156t^{15} - 80836455t^{14} + 586982356t^{13} - 517856644t^{12} - 33377993888t^{11} + 333714279920t^{10} - 1388330956224t^9 + 939768309312t^8 + 16067763419136t^7 - 73817655947264t^6 + 140376670666752t^5 - 73288713666560t^4 - 240280447483904t^3 + 655739410448384t^2 - 673143031070720t + 135070115430400,$$
$$\omega_{35} = -\omega_{33}^3 + 2\omega_{34}\omega_{33}^2 + (\omega_{34}^2 + 1)\omega_{33} - 2\omega_{34} + 2 = 0.2599...$$
$$\Omega_{36}(t) = 2704t^{18} - 28288t^{17} + 184120t^{16} - 978704t^{15} + 4101273t^{14} - 13366596t^{13} + 40474872t^{12} - 121621580t^{11} + 344598850t^{10} - 956962404t^9 + 2551715452t^8 - 4971996544t^7 + 7375023873t^6 - 18590375280t^5 + 50607626740t^4 - 49880139188t^3 - 26495324860t^2 + 37239706120t - 8591618060,$$
$$\omega_{36} = 0.3995...$$
$$\Omega_{37}(t) = 2704t^{18} - 50128t^{17} + 735788t^{16} - 7886228t^{15} + 75520425t^{14} - 593880004t^{13} + 4041196380t^{12} - 22976868576t^{11} + 111905562864t^{10} - 452307143488t^9 + 1581040083008t^8 - 5022901526528t^7 + 15336331767808t^6 - 36181982117888t^5 + 38017569095680t^4 + 69513413394432t^3 - 391234354937856t^2 + 576804498702336t - 270115421552640,$$
$$\omega_{37} = \omega_{33}^3 + 2\omega_{36}\omega_{33}^2 + (\omega_{36}^2 - 1)\omega_{33} - 2\omega_{36} + 2 = 1.0514...$$
$$\Omega_{38}(t) = 36t^{18} - 480t^{17} + 4136t^{16} - 22220t^{15} + 67533t^{14} - 35042t^{13} - 766479t^{12} + 4336224t^{11} - 13308798t^{10} + 27123976t^9 - 38935546t^8 + 40272664t^7 - 29588523t^6 + 13701578t^5 - 1642203t^4 - 2665532t^3 + 2118624t^2 - 805056t + 147044, \qquad \omega_{38} = 0.76594...$$
$$\Omega_{39}(t) = 9t^{18} - 30t^{17} + 347t^{16} - 2316t^{15} + 1553t^{14} + 15838t^{13} - 69221t^{12} + 153688t^{11} + 299059t^{10} - 950530t^9 - 1504919t^8 - 3251596t^7 + 21274899t^6 + 19229826t^5 - 104885343t^4 + 22091520t^3 + 149954464t^2 - 126939648t + 28776704, \qquad \omega_{39} = 0.4794...$$
$$\Omega_{40}(t) = 1296t^{18} - 29376t^{17} - 41624t^{16} + 14552056t^{15} - 62498599t^{14} - 2452996286t^{13} + 11707386901t^{12} + 194047036776t^{11} - 879334215598t^{10} - 6258212822748t^9 + 37344996202338t^8 + 25623399000072t^7 - 715703278407651t^6 + 1044863824619722t^5 + 3139512217604881t^4 - 11114753673284056t^3 + 13509026011131304t^2 - 7566283923018880t + 1647662728153744$$
$$\omega_{40} = 0.8570...$$

Table 4: Numerical expression of Points on $(V(I) \cap \partial\mathcal{F})/\sim$

pts	coordinate expression	pts	coordinate expression
e_0	$[0.5, 0, 0.5, 0.866, 0, 0.866]$	e_{20}	$[0.240, 0.240, 0.240, 0.9706, 0.4709, 0.9706]$
e_1	$[0.333, 0.333, 0.333, 0.942, 0.471, 0.942]$	e_{21}	$[0.191, 0.5, 0.095, 0.9953, 0.4976, 0.9953]$
e_2	$[0.5, 0.5, 0.5, 0.866, 0.172, 0.866]$	e_{22}	$[0.167, 0.5, 0.1299, 0.985, 0.4929, 0.9993]$
e_3	$[0.5, 0.5, 0.5, 0.866, 0, 0.866]$	e_{23}	$[0.1291, 0.5, 0.1291, 0.9916, 0.495, 0.9916]$
e_4	$[0.5, 0.5, 0, 1, 0.5, 1]$	e_{24}	$[0.2411, 0.3794, 0.5, 0.9704, 0.4852, 0.992]$
e_5	$[0.5, 0, 0.5, 1, 0.5, 1]$	e_{25}	$[0.5, 0.117, 0.5, 0.9861, 0.4930, 0.9861]$
e_6	$[0.5, 0.1160, 0.5, 0.866, 0.433, 1.066]$	e_{26}	$[0.5, 0.3821, 0.2357, 0.9861, 0.4930, 0.9861]$
e_7	$[0, 0.5, 0, 1, 0.5, 1]$	e_{27}	$[0.4843, 0.1193, 0.5, 0.8748, 0.4374, 1.0508]$
e_8	$[0.5, 0.5, 0.5, 1.062, 0.531, 1.062]$	e_{28}	$[0.2914, 0.5, 0.0096, 0.9565, 0.4782, 0.9999]$
e_9	$[0.5, 0.5, 0.5, 0.866, 0.433, 1.157]$	e_{29}	$[0.5, 0.5, 0, 0.866, 0.3556, 1]$
e_{10}	$[0.5, 0.5, 0, 0.866, 0.433, 1.076]$	e_{30}	$[0.1443, 0.5, 0.1443, 0.9895, 0.4895, 0.9895]$
e_{11}	$[0.5, 0, 0.5, 0.866, 0.433, 1.076]$	e_{31}	$[0.1851, 0.5, 0.092, 0.9827, 0.4913, 0.9957]$
e_{12}	$[0.1163, 0.441, 0.1163, 0.9932, 0.4966, 0.9932]$	e_{32}	$[0.110, 0.4036, 0.110, 0.9938, 0.4969, 0.9938]$
e_{13}	$[0.325, 0.441, 0.1163, 0.9932, 0.4966, 0.9932]$	e_{33}	$[0.192, 0.4857, 0.110, 0.9938, 0.4969, 0.9938]$
e_{14}	$[0.325, 0.1163, 0.325, 0.9932, 0.4966, 0.9932]$	e_{34}	$[0.192, 0.293, 0.110, 0.9938, 0.4969, 0.9938]$
e_{15}	$[0.1163, 0.325, 0.1163, 0.9932, 0.4966, 0.9932]$	e_{35}	$[0.1858, 0.4919, 0.100, 0.9825, 0.4912, 0.9948]$
e_{16}	$[0.5, 0.093, 0.480, 0.866, 0.433, 1.061]$	e_{36}	$[0.1858, 0.3060, 0.100, 0.9825, 0.4912, 0.9948]$
e_{17}	$[0.5, 0.406, 0.206, 0.866, 0.433, 1.061]$	e_{37}	$[0.272, 0.472, 0.064, 0.962, 0.481, 0.997]$
e_{18}	$[0.159, 0.5, 0.159, 1, 0.5, 1]$	e_{38}	$[0.272, 0.199, 0.262, 0.962, 0.481, 0.997]$
e_{19}	$[0.5, 0.399, 0.240, 0.866, 0.346, 0.9706]$	e_{39}	$[0.5, 0.3829, 0.2397, 0.9708, 0.4628, 0.9708]$

Table 5: points p and its maximal label $L_p \subset \Lambda$

pt: p	maximal label: L_p
e_0	$[\mathbf{1}, \mathbf{2}, O, E_1, E_2, I_+, Y_1, Y_3, X_1, X_2]$
$\theta_3 e_0$	$[\mathbf{1}, \mathbf{2}, O, E_1, \theta E_2, I_-, Y_1, Y_3, X_1, \theta X_2]$
$\theta_1 e_0$	$[\mathbf{1}, \mathbf{2}, O, \theta E_1, E_2, \theta I_-, Y_1, Y_3, X_2, \theta X_1]$
$\theta_1\theta_3 e_0$	$[\mathbf{1}, \mathbf{2}, O, \theta E_1, \theta E_2, \theta I_+, Y_1, Y_3, \theta X_1, \theta X_2]$
$\theta_2 e_1$	$[\mathbf{1}, \mathbf{2}, \theta R, E_1, E_2, I_+, \theta J, Y_1, Y_2]$
$\theta_1\theta_3 e_1$	$[\mathbf{1}, \mathbf{2}, R, \theta E_1, \theta E_2, \theta I_+, J, Y_1, Y_2]$
e_2	$[\mathbf{1}, \mathbf{2}, O, Y_1, X_1, X_3, X_2]$
$\theta_3 e_2$	$[\mathbf{1}, \mathbf{2}, \theta E_2, J_1, Y_1, X_1, X_3, \theta X_2]$
$\theta_2 e_2$	$[\mathbf{1}, \mathbf{2}, I_+, \theta J, Y_1, X_1, X_2, \theta X_3]$
$\theta_2\theta_3 e_2$	$[\mathbf{1}, \mathbf{2}, E_1, \theta J_2, Y_1, X_1, \theta X_3, \theta X_2]$
$\theta_1 e_2$	$[\mathbf{1}, \mathbf{2}, \theta E_1, J_2, Y_1, X_3, X_2, \theta X_1]$
$\theta_1\theta_3 e_2$	$[\mathbf{1}, \mathbf{2}, \theta I_+, J, Y_1, X_3, \theta X_1, \theta X_2]$
$\theta_1\theta_2 e_2$	$[\mathbf{1}, \mathbf{2}, E_2, \theta J_1, Y_1, X_2, \theta X_1, \theta X_3]$
$\theta_0 e_2$	$[\mathbf{1}, \mathbf{2}, O, Y_1, \theta X_1, \theta X_3, \theta X_2]$
e_3	$[\mathbf{1}, \mathbf{2}, Y_1, Y_3, X_1, X_3, X_2]$
$\theta_3 e_3$	$[\mathbf{1}, \mathbf{2}, Y_1, Y_3, X_1, X_3, \theta X_2]$
$\theta_2 e_3$	$[\mathbf{1}, \mathbf{2}, Y_1, Y_3, X_1, X_2, \theta X_3]$
$\theta_2\theta_3 e_3$	$[\mathbf{1}, \mathbf{2}, Y_1, Y_3, X_1, \theta X_3, \theta X_2]$
$\theta_1 e_3$	$[\mathbf{1}, \mathbf{2}, Y_1, Y_3, X_3, X_2, \theta X_1]$
$\theta_1\theta_3 e_3$	$[\mathbf{1}, \mathbf{2}, Y_1, Y_3, X_3, \theta X_1, \theta X_2]$
$\theta_1\theta_2 e_3$	$[\mathbf{1}, \mathbf{2}, Y_1, Y_3, X_2, \theta X_1, \theta X_3]$
$\theta_0 e_3$	$[\mathbf{1}, \mathbf{2}, Y_1, Y_3, \theta X_1, \theta X_3, \theta X_2]$
e_4	$[\mathbf{2}, O, J_1, Y_1, Y_2, X_1, X_3]$
$\theta_3\sigma_0 e_4$	$[\mathbf{1}, \theta E_2, J, Y_1, Y_2, X_3, \theta X_2]$
$\theta_2 e_4$	$[\mathbf{2}, E_1, \theta J, Y_1, Y_2, X_1, \theta X_3]$
$\theta_2\theta_3\sigma_0 e_4$	$[\mathbf{1}, O, \theta J_2, Y_1, Y_2, \theta X_3, \theta X_2]$
$\theta_2\sigma_0 e_4$	$[\mathbf{1}, E_2, \theta J, Y_1, Y_2, X_2, \theta X_3]$
$\theta_1 e_4$	$[\mathbf{2}, \theta E_1, J, Y_1, Y_2, X_3, \theta X_1]$
$\theta_1\theta_2 e_4$	$[\mathbf{2}, O, \theta J_1, Y_1, Y_2, \theta X_1, \theta X_3]$
$\sigma_0 e_4$	$[\mathbf{1}, O, J_2, Y_1, Y_2, X_3, X_2]$
e_5	$[\theta R, E_1, E_2, Y_1, Y_2, X_1, X_2]$
$\theta_1\theta_3 e_5$	$[R, \theta E_1, \theta E_2, Y_1, Y_2, \theta X_1, \theta X_2]$
e_6	$[\mathbf{1}, O, E_2, Y_2, X_1, X_2]$
$\theta_3 e_6$	$[\mathbf{1}, O, \theta E_2, Y_2, X_1, \theta X_2]$
$\theta_2 e_6$	$[\mathbf{1}, E_1, I_+, Y_2, X_1, X_2]$
$\theta_2\theta_3 e_6$	$[\mathbf{1}, E_1, I_-, Y_2, X_1, \theta X_2]$
$\theta_1 e_6$	$[\mathbf{1}, \theta E_1, \theta I_-, Y_2, X_2, \theta X_1]$
$\theta_1\theta_3 e_6$	$[\mathbf{1}, \theta E_1, \theta I_+, Y_2, \theta X_1, \theta X_2]$
$\theta_1\theta_2 e_6$	$[\mathbf{1}, O, E_2, Y_2, X_2, \theta X_1]$
$\theta_0 e_6$	$[\mathbf{1}, O, \theta E_2, Y_2, \theta X_1, \theta X_2]$
e_7	$[\mathbf{1}, \mathbf{2}, R, Y_1, Y_2, X_3]$
$\theta_2 e_7$	$[\mathbf{1}, \mathbf{2}, \theta R, Y_1, Y_2, \theta X_3]$
e_8	$[O, Y_1, Y_2, X_1, X_3, X_2]$
$\theta_3 e_8$	$[\theta E_2, J_1, Y_1, Y_2, X_1, X_3, \theta X_2]$
$\theta_2 e_8$	$[I_+, \theta J, Y_1, Y_2, X_1, X_2, \theta X_3]$

pt: p	maximal label: L_p
$\theta_2\theta_3 e_8$	$[E_1, \theta J_2, Y_1, Y_2, X_1, \theta X_3, \theta X_2]$
$\theta_1 e_8$	$[\theta E_1, J_2, Y_1, Y_2, X_3, X_2, \theta X_1]$
$\theta_1\theta_3 e_8$	$[\theta I_+, J, Y_1, Y_2, X_3, \theta X_1, \theta X_2]$
$\theta_1\theta_2 e_8$	$[E_2, \theta J_1, Y_1, Y_2, X_2, \theta X_1, \theta X_3]$
$\theta_0 e_8$	$[O, Y_1, Y_2, \theta X_1, \theta X_3, \theta X_2]$
e_9	$[\mathbf{1}, O, Y_2, X_1, X_3, X_2]$
$\theta_3 e_9$	$[\mathbf{1}, \theta E_2, J_1, Y_2, X_1, X_3, \theta X_2]$
$\theta_2 e_9$	$[\mathbf{1}, I_+, \theta J, Y_2, X_1, X_2, \theta X_3]$
$\theta_2\theta_3 e_9$	$[\mathbf{1}, E_1, \theta J_2, Y_2, X_1, \theta X_3, \theta X_2]$
$\theta_1 e_9$	$[\mathbf{1}, \theta E_1, J_2, Y_2, X_3, X_2, \theta X_1]$
$\theta_1\theta_3 e_9$	$[\mathbf{1}, \theta I_+, J, Y_2, X_3, \theta X_1, \theta X_2]$
$\theta_1\theta_2 e_9$	$[\mathbf{1}, E_2, \theta J_1, Y_2, X_2, \theta X_1, \theta X_3]$
$\theta_0 e_9$	$[\mathbf{1}, O, Y_2, \theta X_1, \theta X_3, \theta X_2]$
e_{10}	$[\mathbf{1}, O, J_1, Y_2, X_1, X_3]$
$\theta_2 e_{10}$	$[\mathbf{1}, E_1, \theta J, Y_2, X_1, \theta X_3]$
$\theta_1 e_{10}$	$[\mathbf{1}, \theta E_1, J, Y_2, X_3, \theta X_1]$
$\theta_1\theta_2 e_{10}$	$[\mathbf{1}, O, \theta J_1, Y_2, \theta X_1, \theta X_3]$
e_{11}	$[\mathbf{1}, E_1, E_2, Y_2, X_1, X_2]$
$\theta_3 e_{11}$	$[\mathbf{1}, O, I_-, Y_2, X_1, \theta X_2]$
$\theta_1 e_{11}$	$[\mathbf{1}, O, \theta I_-, Y_2, X_2, \theta X_1]$
$\theta_1\theta_3 e_{11}$	$[\mathbf{1}, \theta E_1, \theta E_2, Y_2, \theta X_1, \theta X_2]$
$\theta_3 e_{12}$	$[\mathbf{1}, \mathbf{2}, R, \theta E_2, Y_1, Y_2]$
$\theta_2\theta_3 e_{12}$	$[\mathbf{1}, \mathbf{2}, \theta R, E_1, Y_1, Y_2]$
$\theta_1 e_{12}$	$[\mathbf{1}, \mathbf{2}, R, \theta E_1, Y_1, Y_2]$
$\theta_1\theta_2 e_{12}$	$[\mathbf{1}, \mathbf{2}, \theta R, E_2, Y_1, Y_2]$
$\theta_3 e_{13}$	$[\mathbf{2}, O, \theta E_2, J_1, Y_1, Y_2]$
$\theta_2\theta_3\sigma_0 e_{13}$	$[\mathbf{1}, O, E_1, \theta J_2, Y_1, Y_2]$
$\theta_1\theta_2 e_{13}$	$[\mathbf{2}, O, E_2, \theta J_1, Y_1, Y_2]$
$\theta_1\sigma_0 e_{13}$	$[\mathbf{1}, O, \theta E_1, J_2, Y_1, Y_2]$
$\theta_2 e_{14}$	$[\theta R, O, E_1, E_2, Y_1, Y_2]$
$\theta_1\theta_3 e_{14}$	$[R, O, \theta E_1, \theta E_2, Y_1, Y_2]$
$\theta_2 e_{15}$	$[\mathbf{1}, \mathbf{2}, \theta R, O, Y_1, Y_2]$
$\theta_1\theta_3 e_{15}$	$[\mathbf{1}, \mathbf{2}, R, O, Y_1, Y_2]$
e_{16}	$[\mathbf{1}, O, E_1, E_2, Y_2, X_1]$
$\theta_2\theta_3 e_{16}$	$[\mathbf{1}, O, E_1, I_-, Y_2, X_1]$
$\theta_1 e_{16}$	$[\mathbf{1}, O, \theta E_1, \theta I_-, Y_2, \theta X_1]$
$\theta_0 e_{16}$	$[\mathbf{1}, O, \theta E_1, \theta E_2, Y_2, \theta X_1]$
$\theta_3 e_{17}$	$[\mathbf{1}, O, \theta E_2, J_1, Y_2, X_1]$
$\theta_2 e_{17}$	$[\mathbf{1}, E_1, I_+, \theta J, Y_2, X_1]$
$\theta_1\theta_3 e_{17}$	$[\mathbf{1}, \theta E_1, \theta I_+, J, Y_2, \theta X_1]$
$\theta_1\theta_2 e_{17}$	$[\mathbf{1}, O, E_2, \theta J_1, Y_2, \theta X_1]$
$\theta_3 e_{18}$	$[R, \theta E_2, J_1, Y_1, Y_2, X_3]$
$\theta_2\theta_3 e_{18}$	$[\theta R, E_1, \theta J_2, Y_1, Y_2, \theta X_3]$
$\theta_1 e_{18}$	$[R, \theta E_1, J_2, Y_1, Y_2, X_3]$
$\theta_1\theta_2 e_{18}$	$[\theta R, E_2, \theta J_1, Y_1, Y_2, \theta X_3]$

Table 6: points and its maximal label (2)

point	maximal label	point	maximal label
$\theta_3 e_{19}$	$[\mathbf{1}, \mathbf{2}, O, \theta E_2, J_1, X_1]$	$\theta_1\theta_3 e_{27}$	$[\mathbf{1}, \theta E_1, \theta E_2, \theta I_+, Y_2, \theta X_2]$
$\theta_2 e_{19}$	$[\mathbf{1}, \mathbf{2}, E_1, I_+, \theta J, X_1]$	$\theta_3 e_{28}$	$[\mathbf{1}, \mathbf{2}, O, J_1, Y_2, X_3]$
$\theta_1\theta_3 e_{19}$	$[\mathbf{1}, \mathbf{2}, \theta E_1, \theta I_+, J, \theta X_1]$	$\theta_2\theta_3 e_{28}$	$[\mathbf{1}, \mathbf{2}, E_1, \theta J, Y_2, \theta X_3]$
$\theta_1\theta_2 e_{19}$	$[\mathbf{1}, \mathbf{2}, O, E_2, \theta J_1, \theta X_1]$	$\theta_1 e_{28}$	$[\mathbf{1}, \mathbf{2}, \theta E_1, J, Y_2, X_3]$
$\theta_2 e_{20}$	$[\mathbf{1}, \mathbf{2}, \theta R, O, E_1, E_2, Y_1]$	$\theta_1\theta_2 e_{28}$	$[\mathbf{1}, \mathbf{2}, O, \theta J_1, Y_2, \theta X_3]$
$\theta_1\theta_3 e_{20}$	$[\mathbf{1}, \mathbf{2}, R, O, \theta E_1, \theta E_2, Y_1]$	e_{29}	$[\mathbf{1}, \mathbf{2}, O, J_1, X_1, X_3]$
$\theta_3 e_{21}$	$[\mathbf{2}, R, J_1, Y_1, Y_2, X_3]$	$\theta_2 e_{29}$	$[\mathbf{1}, \mathbf{2}, E_1, \theta J, X_1, \theta X_3]$
$\theta_3\sigma_0 e_{21}$	$[\mathbf{1}, R, \theta E_2, Y_1, Y_2, X_3]$	$\theta_1 e_{29}$	$[\mathbf{1}, \mathbf{2}, \theta E_1, J, X_3, \theta X_1]$
$\theta_2\theta_3 e_{21}$	$[\mathbf{2}, \theta R, E_1, Y_1, Y_2, \theta X_3]$	$\theta_1\theta_2 e_{29}$	$[\mathbf{1}, \mathbf{2}, O, \theta J_1, \theta X_1, \theta X_3]$
$\theta_2\theta_3\sigma_0 e_{21}$	$[\mathbf{1}, \theta R, \theta J_2, Y_1, Y_2, \theta X_3]$	$\theta_3 e_{30}$	$[\mathbf{1}, \mathbf{2}, R, \theta E_2, J_1, Y_1, X_3]$
$\theta_1 e_{21}$	$[\mathbf{2}, R, \theta E_1, Y_1, Y_2, X_3]$	$\theta_2\theta_3 e_{30}$	$[\mathbf{1}, \mathbf{2}, \theta R, E_1, \theta J_2, Y_1, \theta X_3]$
$\theta_1\theta_2 e_{21}$	$[\mathbf{2}, \theta R, \theta J_1, Y_1, Y_2, \theta X_3]$	$\theta_1 e_{30}$	$[\mathbf{1}, \mathbf{2}, R, \theta E_1, J_2, Y_1, X_3]$
$\theta_1\theta_2\sigma_0 e_{21}$	$[\mathbf{1}, \theta R, E_2, Y_1, Y_2, \theta X_3]$	$\theta_1\theta_2 e_{30}$	$[\mathbf{1}, \mathbf{2}, \theta R, E_2, \theta J_1, Y_1, \theta X_3]$
$\theta_1\sigma_0 e_{21}$	$[\mathbf{1}, R, J_2, Y_1, Y_2, X_3]$	$\theta_3 e_{31}$	$[\mathbf{1}, \mathbf{2}, R, J_1, Y_2, X_3]$
$\theta_3 e_{22}$	$[\mathbf{1}, R, \theta E_2, J_1, Y_2, X_3]$	$\theta_2\theta_3 e_{31}$	$[\mathbf{1}, \mathbf{2}, \theta R, E_1, Y_2, \theta X_3]$
$\theta_2\theta_3 e_{22}$	$[\mathbf{1}, \theta R, E_1, \theta J_2, Y_2, \theta X_3]$	$\theta_1 e_{31}$	$[\mathbf{1}, \mathbf{2}, R, \theta E_1, Y_2, X_3]$
$\theta_1 e_{22}$	$[\mathbf{1}, R, \theta E_1, J_2, Y_2, X_3]$	$\theta_1\theta_2 e_{31}$	$[\mathbf{1}, \mathbf{2}, \theta R, \theta J_1, Y_2, \theta X_3]$
$\theta_1\theta_2 e_{22}$	$[\mathbf{1}, \theta R, E_2, \theta J_1, Y_2, \theta X_3]$	$\theta_3 e_{32}$	$[\mathbf{1}, \mathbf{2}, O, \theta E_2, Y_1, Y_2]$
e_{23}	$[\mathbf{1}, \mathbf{2}, O, Y_1, Y_2, X_3]$	$\theta_2\theta_3 e_{32}$	$[\mathbf{1}, \mathbf{2}, O, E_1, Y_1, Y_2]$
$\theta_2 e_{23}$	$[\mathbf{1}, \mathbf{2}, \theta J, Y_1, Y_2, \theta X_3]$	$\theta_1 e_{32}$	$[\mathbf{1}, \mathbf{2}, O, \theta E_1, Y_1, Y_2]$
$\theta_1\theta_3 e_{23}$	$[\mathbf{1}, \mathbf{2}, J, Y_1, Y_2, X_3]$	$\theta_1\theta_2 e_{32}$	$[\mathbf{1}, \mathbf{2}, O, E_2, Y_1, Y_2]$
$\theta_0 e_{23}$	$[\mathbf{1}, \mathbf{2}, O, Y_1, Y_2, \theta X_3]$	$\theta_3 e_{33}$	$[\mathbf{2}, R, \theta E_2, J_1, Y_1, Y_2]$
$\theta_2 e_{24}$	$[\mathbf{1}, E_2, I_+, \theta J, Y_2, X_2]$	$\theta_2\theta_3\sigma_0 e_{33}$	$[\mathbf{1}, \theta R, E_1, \theta J_2, Y_1, Y_2]$
$\theta_2\theta_3 e_{24}$	$[\mathbf{1}, O, E_1, \theta J_2, Y_2, \theta X_2]$	$\theta_1\theta_2 e_{33}$	$[\mathbf{2}, \theta R, E_2, \theta J_1, Y_1, Y_2]$
$\theta_1 e_{24}$	$[\mathbf{1}, O, \theta E_1, J_2, Y_2, X_2]$	$\theta_1\sigma_0 e_{33}$	$[\mathbf{1}, R, \theta E_1, J_2, Y_1, Y_2]$
$\theta_1\theta_3 e_{24}$	$[\mathbf{1}, \theta E_2, \theta I_+, J, Y_2, \theta X_2]$	$\theta_2 e_{34}$	$[\mathbf{2}, \theta R, O, E_1, Y_1, Y_2]$
e_{25}	$[O, E_1, E_2, Y_1, Y_2, X_1, X_2]$	$\theta_2\sigma_0 e_{34}$	$[\mathbf{1}, \theta R, O, E_2, Y_1, Y_2]$
$\theta_3 e_{25}$	$[O, \theta E_2, I_-, Y_1, Y_2, X_1, \theta X_2]$	$\theta_1\theta_3 e_{34}$	$[\mathbf{2}, R, O, \theta E_1, Y_1, Y_2]$
$\theta_2 e_{25}$	$[E_1, E_2, I_+, Y_1, Y_2, X_1, X_2]$	$\theta_1\theta_3\sigma_0 e_{34}$	$[\mathbf{1}, R, O, \theta E_2, Y_1, Y_2]$
$\theta_2\theta_3 e_{25}$	$[O, E_1, I_-, Y_1, Y_2, X_1, \theta X_2]$	$\theta_3 e_{35}$	$[\mathbf{1}, \mathbf{2}, R, \theta E_2, J_1, Y_2]$
$\theta_1 e_{25}$	$[O, \theta E_1, \theta I_-, Y_1, Y_2, X_2, \theta X_1]$	$\theta_1\theta_2 e_{35}$	$[\mathbf{1}, \mathbf{2}, \theta R, E_2, \theta J_1, Y_2]$
$\theta_1\theta_3 e_{25}$	$[\theta E_1, \theta E_2, \theta I_+, Y_1, Y_2, \theta X_1, \theta X_2]$	$\theta_2 e_{36}$	$[\mathbf{1}, \mathbf{2}, \theta R, O, E_1, Y_2]$
$\theta_1\theta_2 e_{25}$	$[O, E_2, \theta I_-, Y_1, Y_2, X_2, \theta X_1]$	$\theta_1\theta_3 e_{36}$	$[\mathbf{1}, \mathbf{2}, R, O, \theta E_1, Y_2]$
$\theta_0 e_{25}$	$[O, \theta E_1, \theta E_2, Y_1, Y_2, \theta X_1, \theta X_2]$	$\theta_3 e_{37}$	$[\mathbf{1}, \mathbf{2}, O, \theta E_2, J_1, Y_2]$
$\theta_3 e_{26}$	$[O, \theta E_2, J_1, Y_1, Y_2, X_1]$	$\theta_1\theta_2 e_{37}$	$[\mathbf{1}, \mathbf{2}, O, E_2, \theta J_1, Y_2]$
$\theta_2 e_{26}$	$[E_1, I_+, \theta J, Y_1, Y_2, X_1]$	$\theta_2 e_{38}$	$[\mathbf{1}, \theta R, O, E_1, E_2, Y_2]$
$\theta_2\theta_3\sigma_0 e_{26}$	$[O, E_1, \theta J_2, Y_1, Y_2, \theta X_2]$	$\theta_1\theta_3 e_{38}$	$[\mathbf{1}, R, O, \theta E_1, \theta E_2, Y_2]$
$\theta_2\sigma_0 e_{26}$	$[E_2, I_+, \theta J, Y_1, Y_2, X_2]$	$\theta_3 e_{39}$	$[\mathbf{2}, O, \theta E_2, J_1, Y_1, X_1]$
$\theta_1\theta_3 e_{26}$	$[\theta E_1, \theta I_+, J, Y_1, Y_2, \theta X_1]$	$\theta_2 e_{39}$	$[\mathbf{2}, E_1, I_+, \theta J, Y_1, X_1]$
$\theta_1\theta_3\sigma_0 e_{26}$	$[\theta E_2, \theta I_+, J, Y_1, Y_2, \theta X_2]$	$\theta_2\theta_3\sigma_0 e_{39}$	$[\mathbf{1}, O, E_1, \theta J_2, Y_1, \theta X_2]$
$\theta_1\theta_2 e_{26}$	$[O, E_2, \theta J_1, Y_1, Y_2, \theta X_1]$	$\theta_2\sigma_0 e_{39}$	$[\mathbf{1}, E_2, I_+, \theta J, Y_1, X_2]$
$\theta_1\sigma_0 e_{26}$	$[O, \theta E_1, J_2, Y_1, Y_2, X_2]$	$\theta_1\theta_3 e_{39}$	$[\mathbf{2}, \theta E_1, \theta I_+, J, Y_1, \theta X_1]$
$\theta_2 e_{27}$	$[\mathbf{1}, E_1, E_2, I_+, Y_2, X_2]$	$\theta_1\theta_3\sigma_0 e_{39}$	$[\mathbf{1}, \theta E_2, \theta I_+, J, Y_1, \theta X_2]$
$\theta_2\theta_3 e_{27}$	$[\mathbf{1}, O, E_1, I_-, Y_2, \theta X_2]$	$\theta_1\theta_2 e_{39}$	$[\mathbf{2}, O, E_2, \theta J_1, Y_1, \theta X_1]$
$\theta_1 e_{27}$	$[\mathbf{1}, O, \theta E_1, \theta I_-, Y_2, X_2]$	$\theta_1\sigma_0 e_{39}$	$[\mathbf{1}, O, \theta E_1, J_2, Y_1, X_2]$