Derivations and automorphisms on the non-commutative algebra of power series*

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1 Introduction

In this short report, we will highlight the bijective correspondence via the exponential/logarithm maps between the Lie algebra of derivations and the group of algebra automorphisms on the non-commutative algebra of formal power series in two variables. A purpose of this notes is to describe the corresponding automorphism explicitly for given any derivation and to investigate the structure in the behind.

This kind of study is started in [7] motivated by the study of multiple zeta values and developed in [5] and [6] as a purely algebraic problem. Actually, in [7], various operations on the space of multiple zeta values are translated to the language of derivations on the algebra of power series. Relations among operations were clarified by looking at the corresponding automorphisms rather than derivations themselves via the exponential map. In the story of multiple zeta values or in related topics, the non-commutative algebra of formal power series in two variables appears naturally as the completion of the tensor algebra of the space of holomorphic 1-forms on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, or as a completion of the group ring of the fundamental group of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$, both of which are isomorphic as complete augmented algebras discussed in the appendix of [10].

It is fundamental and must be useful to clarify the structure of the derivation Lie algebra, that of the automorphism group, and their relations.

In this notes we introduce the results in the references [5, 6, 7] and will explain them more. First we define a specific class of derivations on the algebra of power series and discuss the structure of the Lie algebra generated by the derivations, and then determine the corresponding class of automorphisms via the exponential map.

2 Notation

Let k be a field of characteristic zero. The algebra $R = k\langle a, b\rangle$ of non-commutative polynomials in two variables over the field k has a natural grading by defining the degree of the generators a and b are one. The ideal $\mathfrak{m}$ consisting the polynomials which does not have constant term is the unique maximal ideal of R. The completion $R^\wedge$ of R with respect to the topology induced from the filtration defined by the sequence of powers of ideal: $\mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots$ can be identified with the algebra $k\langle\langle a, b\rangle\rangle$ of the non-commutative formal power series in two variables. For any ideal $\mathfrak{i}$ of R, $\mathfrak{i}^\wedge$ denotes the closure of $\mathfrak{i}$ in $R^\wedge$. For instance $(\mathfrak{m}^n)^\wedge$ is the ideal of $R^\wedge$ consists of the power series whose smallest degree of terms is larger than or equal to n. The $R^\wedge$ is a typical example of complete augmented algebras over k by the natural projection map $R^\wedge \longrightarrow k$.

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Let Der(R^) be the space of all derivations on $R^\wedge$ over k. By definition, a derivation D is a k-linear endomorphism on $R^\wedge$ satisfies the Leibniz rule: $D(xy) = D(x)y + xD(y)$ for any elements x, y in $R^\wedge$. The Der(R^) forms a Lie algebra over k under the usual commutator bracket operation: $[D, D'] := DD' - D'D$. On the other hand, Aut(R^) denotes the set of all algebra automorphisms on $R^\wedge$ over k. An automorphism Δ is a k-linear automorphism on $R^\wedge$ satisfying $\Delta(xy) = \Delta(x)\Delta(y)$. The Aut(R^) forms a group under the composition.

Let $\mathrm{Der}^+(\mathrm{R}^\wedge)$ be the Lie subalgebra consisting of derivations D which increase the degree: $D(\mathfrak{m}^\wedge) \subset (\mathfrak{m}^2)^\wedge$, where $(\mathfrak{m}^2)^\wedge$ is the closure of $\mathfrak{m}^2$ in $R^\wedge$. Let $\mathrm{Aut}^1(\mathrm{R}^\wedge)$ be the the subgroup consisting of automorphisms Δ such that $(\mathrm{Id} - \Delta)(\mathfrak{m}^\wedge) \subset (\mathfrak{m}^2)^\wedge$. Such derivations and automorphisms induce the trivial derivation and the identity map, respectively on the associated graded algebra $\mathrm{gr}(\mathrm{R}^\wedge) = \bigoplus (\mathfrak{m}^\mathrm{n})^\wedge/(\mathfrak{m}^{\mathrm{n}+1})^\wedge$ of $R^\wedge$.

The exponential map and the logarithm map are defined by usual way and give a one-to-one correspondence between $\mathrm{Der}^+(\mathrm{R}^\wedge)$ and $\mathrm{Aut}^1(\mathrm{R}^\wedge)$:

$$\exp(D) = e^D = \sum_{m\geq 0} \frac{1}{m!} D^m, \qquad \log \Delta = -\sum_{m\geq 1} \frac{1}{m} (\mathrm{Id} - \Delta)^m$$

for $D \in \mathrm{Der}^+(\mathrm{R}^\wedge)$ and $\Delta \in \mathrm{Aut}^1(\mathrm{R}^\wedge)$. Since D and $\mathrm{Id} - \Delta$ have the degree-increasing properties, the infinite sum of both RHSs are convergent. Also here the assumption on the characteristic of k is used to define the maps.

The problem what we like to focus here is to give the complete description of the correspondence between $\mathrm{Der}^+(\mathrm{R}^\wedge)$ and $\mathrm{Aut}^1(\mathrm{R}^\wedge)$ via the exponential/logarithm maps.

3 Examples

Before beginning the subject, let's see the typical examples. Note that a derivation (resp. an automorphism) on $R^\wedge = k\langle\langle a, b\rangle\rangle$ is uniquely determined by the values on the generators. Since $R^\wedge$ is generated (topologically) freely by a and b, one can choose any elements in $(\mathfrak{m}^2)^\wedge$ as values on generators: $\mathrm{Der}^+(\mathrm{R}^\wedge) \cong (\mathfrak{m}^2)^\wedge \times (\mathfrak{m}^2)^\wedge$. We use the notation $(1-w)^{-1} = 1 + w + w^2 + \cdots$ for $w \in \mathfrak{m}^\wedge$ throughout the notes.

Example. Define the derivation $D \in \mathrm{Der}^+(\mathrm{R}^\wedge)$ by

$$D(a) = a^2, \qquad D(b) = ab.$$

Then the corresponding automorphism is characterized by

$$\exp(D)(a) = (1-a)^{-1}a, \qquad \exp(D)(b) = (1-a)^{-1}b.$$

This example can be easily checked by seeing $D^m(a) = m!a^{m+1}$ and $D^m(b) = m!a^m b$ by using induction.

Example. Define the derivation $D \in \mathrm{Der}^+(\mathrm{R}^\wedge)$ by

$$D(a) = a^2, \qquad D(b) = ab - 3ba + b^2.$$

Then the corresponding automorphism is characterized by

$$\exp(D)(a) = (1-a)^{-1}a, \qquad \exp(D)(b) = (1-a)^{-1}b(1-a-3b)^{-1}(1-a)^2.$$

This result may show the difficulty to compute the formula for D^m in general. This example is a special case of Corollary 2 in Section 6.

4 Derivations

In this section, we define a specific class of derivations of $R^\wedge = k\langle\langle a, b\rangle\rangle$ discussed in [7, 5, 6] and study the structure of the Lie algebra generated by them.

Definition 1. For $n \geq 1$ and for elements $\alpha, \beta, \gamma, \delta$ and λ in k, define the family of derivations $D_n := D_n(\lambda; \alpha, \beta, \gamma, \delta) \in \mathrm{Der}^+(\mathrm{R}^\wedge)$ by

$$D_n(a) = \lambda a^{n+1}, \qquad D_n(b) = \alpha a^{n+1} + \beta a^n b + \gamma b a^n + \delta b a^{n-1} b.$$

The derivations in Examples in Section 3 are special cases of this class. This class of derivations are defined and discussed in [7, 5] in the case of $\lambda = 0$, and in [6] in general case. In this paper, we always fix the parameters $\alpha, \beta, \gamma, \delta$ and λ in k.

Let $\mathcal{L} = \mathcal{L}(\lambda; \alpha, \beta, \gamma, \delta)$ be the Lie subalgebra of $\mathrm{Der}^+(\mathrm{R}^\wedge)$ topologically generated by all D_n for $n \geq 1$. This means that $\mathcal{L}$ is the closure of the Lie algebra generated by all D_n in $R^\wedge$. Note that $\mathcal{L}$ is graded naturally whose all graded components are 1-dimensional. The following proposition says that the family $\{D_n\}$ satisfies nice bracket relations.

Proposition 1 ([6]). *For any $m, n \geq 1$ and for fixed $\alpha, \beta, \gamma, \delta$ and λ, we have*

(i) $[D_m, D_n] = \lambda(m - n)D_{m+n}$.

(ii) $D_{n+1} = ad^{n-1}(D_1)(D_2)/(n-1)! := \lambda^{n-1}\underbrace{[D_1, [D_1, \cdots [D_1}_{n-1}, D_2]\cdots]]/(n-1)!$.

The first relation is proved by checking values of both-sides on generators a and b. (ii) is deduced from (i).

From the proposition, we know $\mathcal{L}$ is a non-abelian Lie algebra unless $\lambda = 0$ and is generated (topologically) by D_1 and D_2. As a remark, when $\lambda = 1$, the relation (i) is known as that of a classical Witt algebras in conformal field theory (cf. [11] on the Witt algebras). It is also proved that there does not exist a derivation θ of degree 1 (namely, $\theta(a)$ and $\theta(b)$ are homogeneous of degree 2) satisfies $D_2 = [\theta, D_1]$.

Since any element in $\mathcal{L}$ is an infinite linear combination of D_n' s, we use the following notation introduced in [7] and extend it to our case.

Let $k[[X]]$ be the commutative algebra of formal power series in an indeterminate X over field k. Let $L = Xk[[X]]$ be the subspace consisting of power series of constant term zero, and $G = 1 + Xk[[X]]$ be a set of power series of constant term 1. The exponential/logarithm maps give a bijective correspondence between L and G.

Definition 2. For any $f(X) = \sum_{n\geq 1} c_n X^n \in L$, define the derivation $D_f := D_f(\lambda; \alpha, \beta, \gamma, \delta) \in \mathcal{L} \subset \mathrm{Der}^+(\mathrm{R}^\wedge)$ by $D_f(\lambda; \alpha, \beta, \gamma, \delta) = \sum_{n\geq 1} c_n D_n(\lambda; \alpha, \beta, \gamma, \delta)$. The action of D_f on generators are given by

$$D_f(a) = \lambda f(a)a, \quad D_f(b) = \alpha f(a)a + \beta f(a)b + \gamma b f(a) + \delta b \frac{f(a)}{a} b.$$

Note that the asignment $D : f \mapsto D_f$ gives an linear isomorphism from L and $\mathcal{L}$. We define a group $\mathcal{G}$ by the image of $\mathcal{L}$ under the exponential map: $\mathcal{G} := \exp(\mathcal{L})$.

Our interest is the relation between $\mathcal{L}$ and $\mathcal{G}$ under the exponential/logarithm maps. What kind of automorphisms will appear corresponding to D_f?

5 Image of a

One can regard $k[[a]]$ as a subalgebra of $R^\wedge = k\langle\langle a, b\rangle\rangle$ generated topologically by a. Note that the derivation D_f defined in Section 4 can be restricted to a derivation on $k[[a]]$. We express it by same letter D_f. In this section we introduce a formula proved by L. Comtet in 1973, which gives an explicit expression of $D_f^n(a)$. This result gives an expression of $e^{D_f}(a)$.

Theorem 1 (Comtet [3]). *Let g be an arbitrary differential function. Define the functions $\{T_{n,l}\}_{l=1}^n$ for $n \geq 1$ by $(g(a)\frac{d}{da})^n = \sum_{l=1}^n T_{n,l}(\frac{d}{da})^l$. Then we have*

$$T_{n,l} = \frac{g(a)}{l!} \sum_{\substack{k_1+\cdots+k_{n-1}=n-l,\\ k_1+\cdots+k_i\leq i,\ 0\leq k_i,\\ (1\leq\forall i\leq n-1)}} \prod_{j=1}^{n-1}(j+1-k_1-\cdots-k_j)\frac{g(a)^{(k_j)}}{k_j!} \tag{1}$$

where $g(a)^{(k_j)}$ means the k_j-th derivative function of $g(a)$ in a.

Example. For simplicity put $g = g(a)$.

$$\begin{aligned}
(g\frac{d}{da})^2 &= gg'(\frac{d}{da}) + g^2(\frac{d}{da})^2,\\
(g\frac{d}{da})^3 &= (g(g')^2 + g^2g'')(\frac{d}{da}) + 3g^2g'(\frac{d}{da})^2 + g^3(\frac{d}{da})^3,\\
(g\frac{d}{da})^4 &= (g(g')^3 + 4g^2g'g'' + g^3g''')(\frac{d}{da})\\
&\quad + (7g^2(g')^2 + 4g^3g'')(\frac{d}{da})^2 + 6g^3g'(\frac{d}{da})^3 + g^4(\frac{d}{da})^4.
\end{aligned}$$

The numbers appearing in coefficients in $\{T_{n,l}\}$ relate to the number of 'rooted tree' and the theory of 'species' in combinatorics. See [1, 4, 8] for this connection.

Note that $D_f = \lambda f(a)a\frac{d}{da}$ on $k[[a]]$. Hence we can use Theorem 1 to compute $D_f^n(a)$ by putting $g(a) = \lambda f(a)a$.

Corollary 1. *For $f \in L := Xk[[X]]$ and $n \geq 1$, we have*

$${D_f}^n(a) = T_{n,1} = \lambda^n f(a)a \sum_{\substack{k_1+\cdots+k_{n-1}=n-1,\\ k_1+\cdots+k_i\leq i, 0\leq k_i,\\ (1\leq\forall i\leq n-1)}} \prod_{j=1}^{n-1}(j+1-k_1-\cdots-k_j)\frac{(f(a)a)^{(k_j)}}{k_j!},$$

where $T_{n,1}$ is the element defined in (1) in case of $g(a) := \lambda f(a)a$.

Proposition 2 ([6]). *For $f \in L = Xk[[X]]$, put $h = e^f \in G = 1 + Xk[[X]]$. Then we have*

$$e^{D_f}(a) = u_h(a)a,$$

where $u_h(a) \in 1 + ak[[a]]$ is defined by the equation $u_h(a)a = \sum_{n\geq 0}\frac{T_{n,1}}{n!}$ with $T_{0,1} = a$.

Remark. In general, we do not have a simple expression of $u_h(a)$ by means of elementary functions. See [6] for such examples and some analytic expression of $u_h(a)$ in terms of the solution to a differential equation when $k = \mathbf{R}$. In case of $f(X) = X^m$, we can show the explicit form: $u_h(a) = (1 - \lambda m a^m)^{-\frac{1}{m}}$.

6 Image of b

In this section, the automorphism $\exp(D_f)$ is detected explicitly.

Definition 3 ([5, 6]). For $h \in G = 1 + Xk[[X]]$ and for $\alpha, \beta, \gamma, \delta$ and $\lambda \in k$, we define an automorphism $\Delta_h = \Delta_h(\lambda; \alpha, \beta, \gamma, \delta) \in \mathrm{Aut}^1(\mathrm{R}^\wedge)$ by the following action on generators:

$$\Delta_h(a) = u_h(a)a, \qquad \Delta_h(b) = A_h B_h^{-1}$$

where $u_h(a)$ is defined in Proposition 2 in Section 5 and

$$\begin{aligned} A_h :&= v_h(a)^{\beta+\varepsilon}\Big[b + \frac{v_h(a)^{-\omega} - 1}{-\omega}(\alpha a - \varepsilon b)\Big] \\ &= v_h(a)^{\beta}\Big[(v_h(a)^{\varepsilon} - v_h(a)^{\varepsilon'})\alpha a - (\varepsilon' v_h(a)^{\varepsilon} - \varepsilon v_h(a)^{\varepsilon'})b\Big]/\omega \qquad (2) \\ B_h :&= v_h(a)^{-(\gamma+\varepsilon)}\Big[1 + \frac{v_h(a)^{\omega} - 1}{\omega a}(\varepsilon a - \delta b)\Big] \\ &= v_h(a)^{\beta-\lambda}\Big[(\varepsilon v_h(a)^{\varepsilon} - \varepsilon' v_h(a)^{\varepsilon'}) - \frac{v_h(a)^{\varepsilon} - v_h(a)^{\varepsilon'}}{a}\delta b\Big]/\omega \qquad (3) \end{aligned}$$

where $v_h(a) := u_h(a)^{1/\lambda} \in 1 + ak[[a]]$ if $\lambda \neq 0$, $v_h(a) := h(a)$ if $\lambda = 0$. ε and ε' are roots of the equation $T^2 + (\beta + \gamma - \lambda)T + \alpha\delta = 0$ and $\omega = \varepsilon - \varepsilon'$. Since the each expression (2), (3) is symmetric in ε and ε', $A_h \in R^\wedge$ and $B_h \in (R^\wedge)^\times$ (the unit group of $R^\wedge$).

Theorem 2 ([5, 6]). *For any $f \in L$, we set $h = e^f \in G$. Then we have*

$$\Delta_h = \exp(D_f).$$

This theorem says the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\exp} & G \\ {\scriptstyle D}\big\downarrow & & \big\downarrow{\scriptstyle \Delta} \\ \mathcal{L} & \xrightarrow{\exp} & \mathcal{G} \end{array}$$

where $\mathcal{L}$ and $\mathcal{G}$ were defined in Section 4.

Corollary 2. *For $m \geq 1$ and any parameter $s \in k$, $\exp(sD_m) \in \mathrm{Aut}^1(\mathrm{R}^\wedge)$ is determined by*

$$\begin{aligned} e^{sD_m}(a) &= (1 - s\lambda m a^m)^{-1/m}a, \\ e^{sD_m}(b) &= (1 - s\lambda m a^m)^{-\frac{\beta+\varepsilon}{\lambda m}}\Big[b + \frac{(1 - s\lambda m a^m)^{\frac{\omega}{\lambda m}} - 1}{-\omega}(\alpha a - \varepsilon b)\Big] \\ &\quad \times \Big[1 + \frac{(1 - s\lambda m a^m)^{\frac{-\omega}{\lambda m}} - 1}{\omega a}(\varepsilon a - \delta b)\Big]^{-1}(1 - s\lambda m a^m)^{-\frac{\gamma+\varepsilon}{\lambda m}}, \end{aligned}$$

where $\varepsilon, \varepsilon'$, and ω are defined in Definition 3. When $\lambda = 0$ we regard $(1 - s\lambda m a^m)^{-1/\lambda m}$ as e^{a^m}.

This corollary is a special case of the theorem for $f(X) = sX^m$.

7 Sketch of proof

In this section we sketch a proof of Theorem 2. We can find an interesting structure in the proof! For the detail, see the reference [6].

To prove the theorem, it is enough to show that for any parameters $s, s' \in k$,

(i) $\frac{d}{ds}\Delta_{h^s}|_{s=0} = D_f \quad$ for $h = e^f$,

(ii) $\Delta_{h^{s+s'}} = \Delta_{h^s}\Delta_{h^{s'}}$

because $\exp(sD_f)$ satisfies the same conditions and these conditions uniquely determine the corresponding automorphism. (ii) says that Δ_{h^s} is a one-parameter subgroup. (i) says that the tangent vector of the subgroup at $s = 0$ is D_f.

We have $\frac{d}{ds}u_{h^s}(a)|_{s=0} = \lambda f(a)$, because $e^{sD_f}(a) = u_{h^s}(a)a$ is true by Proposition 2. For (i), it is clear that $\frac{d}{ds}\Delta_{h^s}(a)|_{s=0} = \frac{d}{ds}u_{h^s}(a)a|_{s=0} = \lambda f(a)a = D_f(a)$. For $v_h(a)$, when $\lambda \neq 0$, $\frac{d}{ds}v_{h^s}(a)|_{s=0} = \frac{d}{ds}(u_{h^s}(a))^{1/\lambda}|_{s=0} = \frac{1}{\lambda}\lambda f(a) = f(a)$. When $\lambda = 0$, $\frac{d}{ds}v_{h^s}(a)|_{s=0} = \frac{d}{ds}h^s(a)|_{s=0} = f(a)$. Using this and (2) and (3), one has

$$\frac{d}{ds}A_{h^s}|_{s=0} = (\beta+\varepsilon)f(a)b + f(a)(\alpha a - \varepsilon b) = \alpha f(a)a + \beta f(a)b,$$
$$\frac{d}{ds}B_{h^s}^{-1}|_{s=0} = -\frac{f(a)}{a}(\varepsilon a - \delta b) + (\gamma+\varepsilon)f(a) = \gamma f(a) + \delta\frac{f(a)}{a}b.$$

Using these we have

$$\frac{d}{ds}\Delta_{h^s}(b)|_{s=0} = \alpha f(a)a + \beta f(a)b + \gamma b f(a) + \delta b\frac{f(a)}{a}b = D_f(b).$$

To prove (ii), we will look at carefully a Lie algebra structure (resp. a group structure) on L (resp. G).

Recall that $L = Xk[[X]]$ and $G = 1 + Xk[[X]]$ relates bijectively via the exponential-logarithm maps. We have defined the Lie algebra $(\mathcal{L}, [\ ,\ ])$ in Section 4 generated topologically by the derivations D_f for $f \in L$. Furthermore $\mathcal{G}$ is the image of $\mathcal{L}$ via the exponential map: $\mathcal{G} := \exp\mathcal{L}$. It can be shown $(\mathcal{G}, \cdot)$ forms a group under the usual composition of maps. See below about this group operation in term of Baker-Campbell-Hausdorff series.

First we introduce a new Lie algebra structure on $L = Xk[[X]]$, and a new group structure on $G = 1 + Xk[[X]]$ which make D and Δ a Lie homomorphism and a group homomorphism, respectively.

Proposition 3 ([6]). *For $f \in L$, put $f^\bullet = X\frac{df}{dX}$ (Euler operator). Then for $f, g \in L$, the bracket operation defined by $[f,g]_\bullet := \lambda(fg^\bullet - gf^\bullet)$ makes L a Lie algebra $(L, [\ ,\ ]_\bullet)$. Then $D : L \longrightarrow \mathcal{L}$ ($f \mapsto D_f$) gives a Lie algebra isomorphism i.e., $[D_f, D_g] = D_{[f,g]_\bullet}$.*

Proposition 4 ([6]). *For $g, h \in G$, define a product $*$ in G by $g*h := \exp H_\bullet(\log(g), \log(h))$ where $H_\bullet(f,g) := f + g + \frac{1}{2}[f,g]_\bullet + \frac{1}{12}([f[f,g]_\bullet]_\bullet - [g,[f,g]_\bullet]_\bullet) + \cdots$ is defined by the Baker-Campbell-Hausdorff series. Then $(G, *)$ forms a group.*

The proofs of Proposition 3,4 are in [6]. One can show that $(L, H_\bullet)$ forms a non-abelian group unless $\lambda = 0$, which is called Hausdorff group (cf. [2]) associated to the Lie algebra $(L, [\ ,\ ]_\bullet)$.

Theorem 3 ([6]). *The map Δ satisfies the homomorphism property, namely for any $h, h' \in G$, we have*

$$\Delta_{h*h'} = \Delta_h\Delta_{h'}.$$

(ii) is proved from Theorem 3 by applying $h = h^s$ and $h' = h^{s'}$, since we have $h^s * h^{s'} = h^{s+s'}$. For the proof of Theorem 3 we will show a lemma.

Lemma. *For any $h, h' \in G$, we have*

$$\Delta_h(A_{h'})B_h = A_{h*h'}, \tag{4}$$

$$\Delta_h(B_{h'})B_h = B_{h*h'}. \tag{5}$$

Using this lemma one can show Theorem 3 by checking the values on generators of both-hand-sides. The images of a has checked in Proposition 2. For b, by using (4) and (5):

$$\begin{aligned}\Delta_h(\Delta_{h'}(b)) &= \Delta_h(A_{h'}B_{h'}^{-1}) = \Delta_h(A_{h'})\Delta_h(B_{h'})^{-1} \\ &= (\Delta_h(A_{h'})B_h)(\Delta_h(B_{h'})B_h)^{-1} = A_{h*h'}B_{h*h'}^{-1} = \Delta_{h*h'}(b).\end{aligned}$$

This completes the proof of Theorem 2.

For the proof of Lemma, first we show the following:

$$u_{h*h'}(a) = u_{h'}(u_h(a)a)u_h(a). \tag{6}$$

One has $e^{D_f}e^{D_g} = e^{D_{H_\bullet(f,g)}}$ by the definition of $H_\bullet$. Put $f = \log(h), g = \log(h') \in L$. From Proposition 2 we have

$$\begin{aligned}e^{D_f}(e^{D_g}(a)) &= e^{D_f}(u_{h'}(a)a) = u_{h'}(e^{D_f}(a))e^{D_f}(a) = u_{h'}(u_h(a)a)u_h(a)a, \\ e^{D_{H_\bullet(f,g)}}(a) &= u_{e^{H_\bullet(f,g)}}(a)a = u_{h*h'}(a)a.\end{aligned}$$

Comparing both equations, we get (6). For simplicity we write v_h (resp. u_h) instead of $v_h(a)$ (resp. $u_h(a)$). Hence we can write

$$\begin{aligned}A_h &= v_h^\beta\big[(v_h^\varepsilon - v_h^{\varepsilon'})\alpha a - (\varepsilon' v_h^\varepsilon - \varepsilon v_h^{\varepsilon'})b\big]/\omega, \\ B_h &= v_h^{\beta-\lambda}\big[(\varepsilon v_h^\varepsilon - \varepsilon' v_h^{\varepsilon'}) - \frac{v_h^\varepsilon - v_h^{\varepsilon'}}{a}\delta b\big]/\omega\end{aligned}$$

and $\Delta_h(a) = u_h a$, $\Delta_h(b) = A_h B_h^{-1}$. Next we prove that $F := \Delta_h(v_{h'}) = v_{h*h'}v_h^{-1}$. By (6), when $\lambda \neq 0$ we have

$$\begin{aligned}F = \Delta_h(v_{h'}) &= \Delta_h(u_{h'}^{1/\lambda}) = (\Delta_h(u_{h'}(a)))^{1/\lambda} = (u_{h'}(u_h(a)a))^{1/\lambda} \\ &= (u_{h*h'}(a)u_h(a)^{-1})^{1/\lambda} = (u_{h*h'}u_h^{-1})^{1/\lambda} = v_{h*h'}v_h^{-1}.\end{aligned}$$

When $\lambda = 0$, note that Δ_h is the identity map on $k[[a]]$ and $h(a) * h'(a)$ is usual product $h(a)h'(a)$. Hence we have $F = \Delta_h(v_{h'}) = \Delta_h(h'(a)) = h'(a) = h(a)h'(a)h(a)^{-1} = v_{h*h'}v_h^{-1}$. Using this we obtain

$$\begin{aligned}\Delta_h(A_{h'})B_h &= \Delta_h\left(v_{h'}^\beta\big[(v_{h'}^\varepsilon - v_{h'}^{\varepsilon'})\alpha a - (\varepsilon' v_{h'}^\varepsilon - \varepsilon v_{h'}^{\varepsilon'})b\big]\right)B_h/\omega \\ &= F^\beta\big[(F^\varepsilon - F^{\varepsilon'})\alpha\Delta_h(a)B_h - (\varepsilon' F^\varepsilon - \varepsilon F^{\varepsilon'})A_h\big]/\omega \\ &= v_{h*h'}^\beta\Big[(F^\varepsilon - F^{\varepsilon'})\alpha u_h a v_h^{-\lambda}\big\{(\varepsilon v_h^\varepsilon - \varepsilon' v_h^{\varepsilon'}) - \frac{v_h^\varepsilon - v_h^{\varepsilon'}}{a}\delta b\big\} \\ &\qquad - (\varepsilon' F^\varepsilon - \varepsilon F^{\varepsilon'})\big\{(v_h^\varepsilon - v_h^{\varepsilon'})\alpha a - (\varepsilon' v_h^\varepsilon - \varepsilon v_h^{\varepsilon'})b\big\}\Big]\Big/\omega^2 \\ &= v_{h*h'}^\beta\Big[\big\{(F^\varepsilon - F^{\varepsilon'})(\varepsilon v_h^\varepsilon - \varepsilon' v_h^{\varepsilon'}) - (\varepsilon' F^\varepsilon - \varepsilon F^{\varepsilon'})(v_h^\varepsilon - v_h^{\varepsilon'})\big\}\alpha a \\ &\qquad - \big\{(F^\varepsilon - F^{\varepsilon'})(v_h^\varepsilon - v_h^{\varepsilon'})\alpha\delta - (\varepsilon' F^\varepsilon - \varepsilon F^{\varepsilon'})(\varepsilon' v_h^\varepsilon - \varepsilon v_h^{\varepsilon'})\big\}b\Big]\Big/\omega^2 \\ &= v_{h*h'}^\beta\big[(v_{h*h'}^\varepsilon - v_{h*h'}^{\varepsilon'})\alpha a - (\varepsilon' v_{h*h'}^\varepsilon - \varepsilon v_{h*h'}^{\varepsilon'})b\big]/\omega = A_{h*h'}.\end{aligned}$$

The proof of (5) is the almost same as (4). Thus we conclude the lemma.

8 Non-abelian cohomology

In this section we show that the elements u_h and B_h, defined in Proposition 2 and Definition 3 respectively, satisfy the 1-cocycle condition for some non-abelian group cohomology.

We have proved that $G = 1 + Xk[[X]]$ forms a group under the product $*$, and acts on $R^\wedge$ by Δ_h as algebra automorphisms. In particular G acts on $(R^\wedge)^\times$ (the unit group of $R^\wedge$) from left as group automorphisms.

We recall the definition of (non-abelian) group cohomology. Let G be arbitrary group for a while, and N a group which G acts from left by group automorphisms: $G \times N \longrightarrow N$, $(h, n) \mapsto hn$. Both G and N can be non-abelian. The set of 1-cocycle is defined by

$$Z^1(G, N) := \{v : G \longrightarrow N \,|\, v(hh') = (hv(h'))v(h)\}.$$

If N is a non-abelian group, $Z^1(G, N)$ does not have natural group structure. Two cocycles v, v' are cohomologous, $v \sim v'$, if and only if there exists $n \in N$ such that $v(h) = (hn)v'(h)n^{-1}$ for all $h \in G$. The quotient set is the 1-st cohomology of G with coefficient N, is denoted by $H^1(G, N)$.

Let us return to our situation: $G = 1 + Xk[[X]]$ and $N = (R^\wedge)^\times$. By (6) and (5), we have the following:

Proposition 5 ([6]). *The $u : h \mapsto u_h(a)$ and $B : h \mapsto B_h$ satisfy the 1-cocycle conditions, that is u, $B \in Z^1((G, *), (R^\wedge)^\times)$.*

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