Half integral weight mock modular forms

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1 Introduction

1.1 Mock Theta Functions

Ramanujan's last letter to Hardy (January 12, 1920):

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$
$$\phi(q) := 1 + \sum_{n=1}^{\infty} \frac{(-q)^{n^2}}{(1+q^2)^2(1+q^4)^2\cdots(1+q^{2n})^2},$$
$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(1-q)^2(1-q^3)^2\cdots(1-q^{2n+1})^2}.$$

$$2\phi(-q) - f(q) = \theta_4(q)\prod_{n=1}^{\infty}(1+q^n).$$

Question. *How do the mock theta functions fit in the theory of modular forms?*

1.2 Rank Generating Function

Definition 1. The *rank* of a partition is its largest part minus its number of parts.

$$N(m,n) := \#\{\text{partitions of } n \text{ with rank } m\}.$$

If $R(w;q) = \sum N(m,n)w^m q^n$, then

$$R(w;q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n},$$

where

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Theorem 1 (Zwegers, 2002). $q^{-1/24}f(q) = q^{-1/24}R(-1;q)$ *is the holomorphic part of a real analytic modular form of weight* $1/2$.

Theorem 2 (Bringmann-Ono, 2006). *If* $\zeta \neq 1$ *is a root of unity of odd order, then* $q^{-1/24}R(\zeta;q)$ *is the holomorphic part of a weak Maass form of weight* $1/2$.

Theorem 3 (Zagier, 2006). *If* $\zeta = e^{2\pi i\alpha} \neq 1$ *is a root of unity, then* $q^{-1/24}R(\zeta;q)$ *is a mock modular form of weight* $1/2$ *with shadow proportional to*

$$(\zeta^{-1/2} - \zeta^{1/2})\sum_{n\in\mathbb{Z}}\left(\frac{12}{n}\right)n\sin(\pi n\alpha)q^{n^2/24}.$$

2 Definitions

2.1 Harmonic Maass Forms

A *harmonic weak Maass form of weight k on a subgroup* $\Gamma \subset SL_2(\mathbb{Z})$ is any smooth function $h : \mathbb{H} \to \mathbb{C}$ satisfying:

1. For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in \mathbb{H}$,

$$h(Az) = \begin{cases} (cz+d)^k h(z) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz+d)^k \ h(z) & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases}$$

   where $\epsilon_d = 1$ if $d \equiv 1 \pmod 4$ and i if $d \equiv 3 \pmod 4$.

2. $\Delta_k h := [-y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)]h = 0.$

3. The function $h(z)$ has at most linear exponential growth at all cusps.

If $\Gamma_0(N) \subset SL_2(\mathbb{Z})$ is a congruence subgroup, then we let

$$\begin{aligned} S_k(N) &:= \{\text{weight } k \text{ cusp forms on } \Gamma_0(N)\}, \\ M_k(N) &:= \{\text{weight } k \text{ hol. modular forms on } \Gamma_0(N)\}, \\ M_k^!(N) &:= \{\text{weight } k \text{ weakly hol. modular forms on } \Gamma_0(N)\}, \\ H_k(N) &:= \{\text{weight } k \text{ harmonic Maass forms on } \Gamma_0(N)\}. \end{aligned}$$

Suppose $k \in \frac{1}{2}\mathbb{Z}^+$.

Theorem 4 (Bruinier-Funke, 2004). *If $w \in \frac{1}{2}\mathbb{Z}$ and $\xi_w := 2iy^w \overline{\frac{\partial}{\partial \bar{z}}}$, then*

$$\xi_k : H_k(N) \longrightarrow S_{2-k}(N).$$

Moreover, this map is **surjective**.

2.2 Mock Modular Forms

Zagier: Mock Modular Forms

$$h = f + g^*,$$

$h :=$ a harmonic weak Maass form of weight k,
$f :=$ a mock modular form of weight k,
$g :=$ the shadow of f, weight $2-k$ modular form.

$$g^*(z) = (i/2)^{k-1} \int_{-\bar{z}}^{i\infty} (\tau + z)^{-k} \overline{g(-\overline{\tau})} \ d\tau = \sum_{n>0} n^{k-1} \overline{b_n} \beta_k(4ny) q^{-n},$$

where $g = \sum_{n>0} b_n q^n \in M_{2-k}$ and $\beta_k(t)$ is the incomplete gamma function:

$$\beta_k(x) = \int_x^\infty t^{-k} e^{-\pi t} dt$$

Let

$$\mathbb{M}_k(N) := \{\text{weight } k \text{ mock modular forms on } \Gamma_0(N)\}.$$

Then

$$\phi : \mathbb{M}_k(N) \cong H_k(N).$$

For $f \in \mathbb{M}_k(N)$ with shadow g, $\phi(f) = f + g^*$.

For $h \in H_k(N)$, we get

$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g^*}{\partial \bar{z}} = y^{-k}\bar{g}.$$

$$\phi^{-1}(h) = h - g^*.$$

$$\mathfrak{M}_{k,\ell} := \{F | F(\gamma z) = \rho(z\gamma)(cz+d)^k (c\bar{z}+d)^\ell F(z)\}$$

for $\gamma \in \Gamma$.

$$\mathfrak{M}_k = \mathfrak{M}_{k,0} \xrightarrow{\partial/\partial \bar{z}} \mathfrak{M}_{k,2} \xrightarrow{\cdot y^k} \mathfrak{M}_{0,2-k} \xrightarrow{\partial/\partial z} \mathfrak{M}_{2,2-k}$$

$$\cup\uparrow \qquad \nearrow 0$$

$$\overline{M_{2-k}}$$

$$H_k = \{F \in \mathfrak{M}_k | \frac{\partial}{\partial z}(y^k \frac{\partial}{\partial \bar{z}}) = 0\}.$$

The composite map

$$\mathfrak{M}_k \xrightarrow{y^k \partial/\partial \bar{z}} \mathfrak{M}_{0,2-k} \xrightarrow{y^{2-k}\partial/\partial z} \mathfrak{M}_k$$

is the Laplace operator in weight k.

2.3 Mock Jacobi Forms

Zwegers: Appell-Lerch function

$q^{-1/24} f(q)$ is a mock modular form of weight 1/2 with shadow $\sum_{n\equiv 1 \pmod 6} nq^{n^2/24}$, a unary theta series of weight 3/2.

Set

$$\vartheta(z) := \vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} (-1)^n e^{\pi i(2n+1)z} q^{\frac{n(n+1)}{2}+\frac{1}{8}}$$
$$= q^{1/8} x^{-1/2} \prod_{n=1}^{\infty} (1-q^n)(1-xq^{n-1})(1-x^{-1}q^n),$$

where $q = e^{2\pi i\tau}$ and $x = e^{2\pi iz}$.

Definition 2.

$$\mu(u,v) := \mu(u,v;\tau) := \frac{a^{1/2}}{\vartheta(v)} \sum_{n=-\infty}^{\infty} \frac{(-b)^n q^{n(n+1)/2}}{1-aq^n},$$

where $a = e^{2\pi iu}$ and $b = e^{2\pi iv}$ for complex values u and v.

Mock Jacobi form of weight 1/2

- μ with elliptic variables at torsion points multiplied by a certain power of q is a mock modular form.
- $\mu(u,v;\tau) + \frac{1}{2}C(u-v;\tau)$ is a non-holomorphic Jacobi form.
- For a, $b \in \mathbb{R}$, define a unary theta series of weight 3/2, $g_{a,b}(\tau) := \sum_{\nu\in a+\mathbb{Z}} \nu e^{\pi i\nu^2\tau + 2\pi i\nu b}$. Then for $a \in (-\frac{1}{2}, \frac{1}{2})$,

$$\int_{-\bar{\tau}}^{i\infty} \frac{g_{a+\frac{1}{2},b+\frac{1}{2}}(z)}{\sqrt{-i(z+\tau)}} dz = -e^{-\pi ia^2\tau + 2\pi ia(b+\frac{1}{2})} C(a\tau - b).$$

Theorem 5 (Zagier).

$$\frac{q^{-1/24}R(e^{2\pi i\alpha};q)}{e^{-\pi i\alpha}-e^{\pi i\alpha}} = \frac{\eta^3(3\tau)}{\eta(\tau)\vartheta(3\alpha;3\tau)} - q^{-1/6}e^{-2\pi i\alpha}\mu(3\alpha,-\tau;3\tau) + q^{-1/6}e^{2\pi i\alpha}\mu(3\alpha,\tau;3\tau).$$

3 Universal Mock Theta Functions

$$g_3(w;q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(w;q)_{n+1}(q/w;q)_{n+1}}.$$

All mock theta functions of odd order are related to $g_3(w;q)$. (Hickerson)

Example.

$$f(-q) = -4qg_3(q,q^4) + \frac{(q^2;q^2)_\infty^7}{(q)_\infty^3(q^4;q^4)_\infty^3}.$$

$$g_2(w;q) := \sum_{n=0}^{\infty} \frac{(-q)_n q^{n(n+1)/2}}{(w;q)_{n+1}(q/w;q)_{n+1}}.$$

All mock theta functions of even order are related to $g_2(w;q)$. (McIntosh)

Theorem 6 (K., 2009). *For any root of unity $\zeta \neq 1$, $\zeta g_2(\zeta;q) + \frac{1}{2}$ is a mock modular form of weight $1/2$ with shadow proportional to $\sum_{n=-\infty}^{\infty}(-1)^n n\zeta^{-2n}q^{n^2}$.*

Proof. 1. $e^{2\pi i\alpha}g_2(e^{2\pi i\alpha};q) = \dfrac{\eta^4(2\tau)}{\eta^2(\tau)\vartheta(2\alpha;2\tau)} + e^{2\pi i\alpha}q^{-1/4}\mu(2\alpha,\tau;2\tau).$

2. $e^{2\pi i\alpha}q^{-1/4}\dfrac{\mathcal{C}(2\alpha-\tau;2\tau)}{2} = \dfrac{1}{2} - \dfrac{i}{2}\displaystyle\int_{-2\bar{\tau}}^{i\infty} \frac{g_{0,-2\alpha+\frac{1}{2}}(z)}{\sqrt{-i(z+2\tau)}}\,dz.$

□

Theorem 7 (K., 2009). *For any root of unity $\zeta = e^{2\pi i\alpha} \neq 1$, $q^{-1/24}(\zeta^{1/2}+\zeta^{3/2}g_3(\zeta;q))$ is a mock modular form of weight $1/2$ with shadow proportional to $\sum_{n=-\infty}^{\infty}(\frac{12}{n})n\sin(\pi n\alpha)q^{n^2/24}$.*

Conclusion. All mock theta functions are mock modular forms of weight $1/2$ with shadows, unary theta series of weight $3/2$.

Theorem 8 (K., 2009). *If $w = e^{2\pi i\alpha}$ for any complex number $\alpha \neq n\tau + m/2$ for integers m and n, then we have a meromorphic Jacobi form*

$$w(g_2(w;q) + g_2(-w;q)) = \frac{2\eta^4(2\tau)}{\eta^2(\tau)\vartheta(2\alpha;2\tau)},$$

which is a weight $1/2$ weakly holomorphic modular form when it is multiplied by q^{-r^2}, if $\alpha = r\tau + s$ with r, $s \in \mathbb{Q}$.

This is a generalization of Bringman-Ono-Rhoades (2008) who proved for the case when $w = q^r$ for a rational number r.

4 Traces of Singular Moduli

4.1 Generating function of class numbers

$$H(N) := \text{Hurwitz-Kronecker class number}$$
$$= \begin{cases} -\frac{1}{12}, & \text{if } N = 0; \\ \sum_{f^2 \mid N} h'(-\frac{N}{f^2}), & \text{if } N > 0. \end{cases}$$

Let

$$\mathcal{H}(\tau) = \sum_{N=0}^{\infty} H(N) q^N = -\frac{1}{12} + \frac{1}{3}q^3 + \frac{1}{2}q^4 + \cdots .$$

Then $\mathcal{H}(\tau)$ is a mock modular form of weight 3/2 with shadow $\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}$. That is, $\mathcal{H}(\tau)$ is the holomorphic part of the Zagier-Eisenstein series $\mathcal{F}(\tau)$,

$$\mathcal{F}(\tau) := \sum_{N=0}^{\infty} H(N) q^N + \frac{1+i}{16\pi} \int_{-\bar{\tau}}^{i\infty} \frac{\theta(z)}{(z+\tau)^{3/2}} dz$$
$$= \sum_{N=0}^{\infty} H(N) q^N + y^{-1/2} \sum_{n \in \mathbb{Z}} \beta(4\pi n^2 y) q^{-n^2}.$$

4.2 Trace of singular moduli

Zagier's Modified trace of j

Let $\mathcal{Q}_D$ be the set of positive definite integral binary quadratic forms of discriminant $-D$.

We define a generalized Hilbert class polynomial $\mathcal{H}_D$ by

$$\mathcal{H}_D(X) = \prod_{Q \in \mathcal{Q}_D / \Gamma} (X - j(\tau_Q))^{1/w_Q},$$

where $w_Q = |\Gamma_Q|$ be the order of the isotropy subgroup of Q in Γ and τ_Q is the CM point associated to Q.

We note that

$$w_Q = \begin{cases} 2, & \text{if } Q \sim_\Gamma [a, 0, a]; \\ 3, & \text{if } Q \sim_\Gamma [a, a, a]; \\ 1, & \text{otherwise.} \end{cases}$$

Then we have the q-expansion

$$\mathcal{H}_D(j(\tau)) = q^{-H(D)}(1 - \mathbf{MT}_J(D) q + O(q^2)),$$

where $\mathbf{MT}_J(D) := \sum_{Q \in \mathcal{Q}_D / \Gamma(1)} \frac{J(\tau_Q)}{w_Q}$, $J = j - 744$.

Modularity of the trace of singular moduli

$$\frac{j(\frac{-1+\sqrt{-3}}{2}) - 744}{3} = -248, \quad \frac{j(i) - 744}{2} = 492, \quad j(\frac{1+\sqrt{-7}}{2}) - 744 = -4119.$$

$$g_1(\tau) := q^{-1} - 2 + 248q^3 - 492q^4 + 4119q^7 - 7256q^8 + \cdots$$
$$= -\frac{\eta(\tau)^2 E_4(4\tau)}{\eta(2\tau)\eta(4\tau)^6}$$

is a weakly holomorphic modular form of weight 3/2.

Question. *Is the generating function of the Galois traces of another algebraic integers a weakly holomorphic modular form? (Joint work with D. Jeon and C.H. Kim.)*

5 Class Invariants

5.1 Definition

Write

$$\theta := \begin{cases} \frac{i\sqrt{D}}{2}, & \text{if } D \equiv 0 \pmod 4; \\ \frac{-1+i\sqrt{D}}{2}, & \text{if } D \equiv 3 \pmod 4. \end{cases} \tag{1}$$

In his Lehrbuch der Algebra, H. Weber calls the value of a modular function $f : \mathbb{H} \to \mathbb{C}$ at θ a *class invariant* if we have

$$K(f(\theta)) = K(j(\theta))$$

and gives several examples such as a holomorphic cube root $\gamma_2 : \mathbb{H} \to \mathbb{C}$ of j-function and a modular function $\mathfrak{f}_2 : \mathbb{H} \to \mathbb{C}$ of level 48.

The function values $\zeta_3\gamma_2(\theta)$ and $\zeta_{48}\mathfrak{f}_2(\theta)$ at $\theta = \frac{-1+\sqrt{-23}}{2}$ are both class invariants.

These values have minimal polynomials:

$$\begin{aligned}
\mathcal{H}_{-23}^{\zeta_{48}\mathfrak{f}_2(\theta)}(X) &= X^3 - X - 1 \in \mathbb{Z}[X], \\
\mathcal{H}_{-23}(X) &= X^3 + 3491750X^2 - 5151296875X + 12771880859375 \in \mathbb{Z}[X], \\
\mathcal{H}_{-71}(X) &= X^7 + 313645809715X^6 - 3091990138604570X^5 \\
&\quad + 98394038810047812047812049302X^4 \\
&\quad - 823534263439730779968091389X^3 \\
&\quad + 5138800366453976780323726329446X^2 \\
&\quad - 425319473946139603274605151187659X \\
&\quad + 737707086760373111335771424100608l263 \in \mathbb{Z}[X], \\
\mathcal{H}_{-71}^{\zeta_3\gamma_2(\theta)}(X) &= X^7 + 6745X^6 - 327467X^5 + 51857115X^4 \\
&\quad + 2319299751X^3 + 41264582513X^2 \\
&\quad - 307873876442X + 903568991567 \in \mathbb{Z}[X], \\
\mathcal{H}_{-71}^{\zeta_{48}\mathfrak{f}_2(\theta)}(X) &= X^7 + X^6 - X^5 - X^4 - X^3 + X^2 + 2X - 1 \in \mathbb{Z}[X].
\end{aligned}$$

5.2 A cube root of j

The j-function has a holomorphic cube root $\gamma_2 : \mathbb{H} \to \mathbb{C}$ defined by

$$\gamma_2(\tau) = \frac{12g_2(\tau)}{\eta^8(\tau)}$$

with integral Fourier expansion. (g_2 here is not a universal mock theta function discussed in Section 5, but a classical invariant function.)

The function γ_2 is a modular function of level 3 (i.e., invariant on $\Gamma(3)$) and the generating matrices $S, T \in \Gamma$ given by $S(\tau) = -1/\tau$ and $T(\tau) = \tau + 1$ act via

$$\gamma_2 \circ S = \gamma_2 \quad \gamma_2 \circ T = \zeta_3^{-1}\gamma_2.$$

It is well known that if D is a discriminant not divisible by 3, then $\zeta_3^B\gamma_2(\theta)$ is a class invariant and its minimal polynomial has integer coefficients, where $B=0$ or 1 as defined in Theorem 9 below.

Galois conjugates of $\gamma_2(\theta)$

Using Stevenhagen-Gee method that applies Shimura reciprocity theorem, we obtain the Galois conjugates of $\gamma_2(\theta)$:

Theorem 9. *Suppose $-D$ is an imaginary quadratic discriminant such that $3\nmid D$. We let $\theta=\frac{-B+i\sqrt{D}}{2}$ as in (1) and let $Q=[a,b,c]$ be a primitive quadratic form of discriminant $-D$. The action of the form class group on $\zeta_3^B\gamma_2(\theta)$ is given by the formula*

$$(\zeta_3^B\gamma_2)^{[a,-b,c]}(\theta)=\begin{cases}\zeta_3^{ab}\gamma_2(\tau_Q), & \text{if } 3\nmid a;\\ \zeta_3^{-bc}\gamma_2(\tau_Q), & \text{if } 3\mid a \text{ and } 3\nmid c;\\ \gamma_2(\tau_Q), & \text{if } 3\mid a \text{ and } 3\mid c.\end{cases}$$

For $-D$ that is congruent to a square modulo $4N^2$ and for a fixed solution $\beta \pmod{2N^2}$ of $\beta^2\equiv -D \pmod{4N^2}$, we define

$$\begin{aligned}\mathcal{Q}_{D,(N)}&=\{[Na,b,Nc]\in\mathcal{Q}_D\},\\ \mathcal{Q}_{D,(N),\beta}&=\{[Na,b,Nc]\in\mathcal{Q}_{D,(N)}\mid b\equiv\beta \pmod{2N^2}\}\end{aligned}$$

on which $\Gamma_0^0(N)$ acts.

Theorem 10. *There is a canonical bijection between $\mathcal{Q}_{D,(N),\beta}/\Gamma_0^0(N)$ and $\mathcal{Q}_D/\Gamma(1)$ for D satisfying the Heegner condition (D is not divisible as a discriminant by the square of any prime dividing N.)*

Example. Let $-D=-23\equiv 7^2 \pmod{36}$ and $\beta=7$.

$$\begin{aligned}\mathcal{Q}_{23}/\Gamma(1)&=\{[1,1,6],[2,1,3],[2,-1,3]\},\\ \mathbf{GT}(j(\theta))&=j(\tau_{[1,1,6]})+j(\tau_{[2,1,3]})+j(\tau_{[2,-1,3]}),\\ \mathbf{GT}(\zeta_3\gamma_2(\theta))&=\zeta_3\gamma_2(\tau_{[1,1,6]})+\zeta_3^2\gamma_2(\tau_{[2,1,3]})+\zeta_3^{-2}\gamma_2(\tau_{[2,-1,3]}).\end{aligned}$$

Considering

$$\mathcal{Q}_{23,(3),7}/\Gamma_0^0(3)=\{[6,25,27],[9,25,18],[3,7,6]\},$$

we obtain also

$$\mathbf{GT}(\zeta_3\gamma_2(\theta))=\gamma_2(z_{[6,25,27]})+\gamma_2(z_{[9,25,18]})+\gamma_2(z_{[3,7,6]}),$$

which is the modular trace of γ_2. Thus we may write the Hilbert class polynomial of $\zeta_3\gamma_2$ as

$$\mathcal{H}_{-23}^{\zeta_3\gamma_2}(X)=\prod_{Q\in\mathcal{Q}_{23,(3),7}/\Gamma_0^0(3)}(X-\gamma_2(\tau_Q)).$$

Then if the discriminant $-D$ is congruent to a square modulo 36 but not divisible by 3, we may define the modified Galois trace of $\zeta_3^B\gamma_2(\theta)$ by

$$\mathbf{GT}_{\zeta_3^B\gamma_2}(\theta):=\sum_{Q\in\mathcal{Q}_{D,(3),\beta}/\Gamma_0^0(3)}\gamma_2(\tau_Q)$$

that becomes a real Galois trace when $-D$ is a fundamental discriminant. Therefore,

$$\mathbf{GT}_{\zeta_3^B\gamma_2}(\theta)=\mathbf{MT}_{\gamma_2}(D).$$

Theorem 11. *Assume that D is a discriminant that is divisible by 3, then* $\mathbf{MT}_{\gamma_2}(D) = 0$.

Consequently, the generating function of **GT** for $\zeta_3^B\gamma_2$ coincides with that of **MT** of γ_2.

Theorem 12 (Bruinier-Funke, 2006). *The modular traces of the values of an arbitrary modular function at Heegner points are Fourier coefficients of the holomorphic part of a harmonic weak Maass form of weight* $3/2$.

If the constant coefficients of the modular function at all cusps vanish, then the generating series of the modular traces is a weakly holomorphic modular form of weight $3/2$.

γ_2 has zero constant coefficient at all cusps.

Theorem 13. *The Galois traces of* $\zeta_3^B\gamma_2(\theta)$ *are Fourier coefficients of a weakly holomorphic modular form of weight* $3/2$ *on* $\Gamma_0(36)$.

Theorem 14. *The generating function of the Galois traces of* $\zeta_{48}\mathfrak{f}_2(\theta)$ *is a weakly holomorphic modular form of weight* $3/2$ *on* $\Gamma_0(9216)$.

6 Concluding Remarks

This article surveys the recent work of authors in [1] and [2].

References

[1] Kang, S.-Y., Mock Jacobi forms in basic hypergeometric series, *Compositio Math.* **145** (2009), 553-565.

[2] Jeon, D., Kang, S.-Y. and Kim, C. H., Traces of class invariants and Hilbert class polynomials for orders, preprint.