Spectral theory on 3-dimensional hyperbolic space and Hermitian modular forms (a joint work with R. Matthes)

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1 Introduction

In [11], Imai discovered how one can apply the spectral theory on the upper half-plane to Siegel modular forms of degree two. We generalize this method to Hermitian modular forms with actual applications. More precisely, we present three main results: (1) A 3-dimensional analogue of Katok-Sarnak's correspondence; (2) An analytic proof of a Hermitian Saito-Kurokawa lift by means of a converse theorem; (3) An explicit formula for the Fourier coefficients of a certain Hermitian-Eisenstein series.

Our main object is to study a unimodular invariant Fourier series $F(Z)$ on the Hermitian upper half-space $H_2 = \{Z \in M_2(\mathbf{C}); (Z - {}^t\overline{Z})/(2i) > O\}$ of degree 2. Recall that the so-called hermitian imaginary part $Y = (Z - {}^t\overline{Z})/(2i)$ of $Z \in H_2$ is the set $\mathcal{P}_2$ of all 2 by 2 positive definite hermitian matrices and that it can be parametrized by determinant and 3-dimensional hyperbolic space $\mathbb{H}^3$. In view of this fact combined with the unimodular invariance and the principle of analytic continuations, studying $F(iY)$ $(Y \in \mathcal{P}_2)$ as a function on $SL_2(\mathcal{O}) \setminus \mathbb{H}^3$ by the spectral decomposition turns out to be an useful approach in order to study $F(Z)$. Here $\mathcal{O}$ is the ring of integers of an imaginary quadratic field. A certain integral formula describes the spectral coefficients by the associated Koecher-Maass series. In some actual applications, an important result is Katok-Sarnak type correspondence for automorphic functions on $\mathbb{H}^3$.

2 Katok-Sarnak type correspondence

We refer to [6] as a basic reference for automorphic functions on 3-dimensional hyperbolic space. Let $K = \mathbf{Q}(i)$ be the Gaussian number field, $\mathcal{O} = \mathbf{Z}[i]$ the ring of all integers, $\mathcal{D}^{-1} = (2i)^{-1}\mathcal{O}$ the inverse different and $\chi_K = (\frac{-4}{})$ the Kronecker symbol of K. Let $\mathbb{H}^3 = \{P = z + rj; z \in \mathbf{C}, r > 0\}$ be 3-dimensional hyperbolic space. An automorphic function on $\mathbb{H}^3$ is any function $\mathcal{U}(P)$ on $\mathbb{H}^3$ satisfying the following three conditions.

(G-i) $\mathcal{U}(\gamma P) = \mathcal{U}(P)$ for all $\gamma \in SL_2(\mathcal{O})$.

(G-ii) $\mathcal{U}(P)$ is a C^2-function on $\mathbb{H}^3$ with respect to x, y, r, where $P = x + yi + rj \in \mathbb{H}^3$. It satisfies a differential equation $-\Delta\mathcal{U} = \lambda\mathcal{U}$ with some $\lambda \in \mathbf{C}$, where $\Delta = r^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}) - r\frac{\partial}{\partial r}$.

(G-iii) $\mathcal{U}(P)$ is of polynomial growth as r tends to ∞.

*The author is supported by JSPS Grant-in-Aid for Young Scientists (Start-up) 21840036.

See [6] p.3 for the action of $SL_2(\mathbf{C})$ on $\mathbb{H}^3$. As an example, there is an Eisenstein series $E(P,t)$ defined for $\Re(t) > 1$ by

$$E(P,t) = \frac{1}{4} \sum_{\substack{c,d\in\mathcal{O} \\ (c,d)=\mathcal{O}}} \left(\frac{r}{|cz+d|^2 + |c|^2 r^2} \right)^{1+t},$$

where (c,d) is a fractional ideal generated by c, d. $E(P,t)$ has a meromorphic continuation to the whole complex t-plane and it is holomorphic for $\Re(t) > 0$ except for a simple pole at $t = 1$.

Denote by $L_2 = \{T = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right); a, d \in \mathbf{Z}, b \in \mathcal{D}^{-1}\}$ the set of all half-integral hermitian matrices of size two. Put $L_2^+ = \{T \in L_2; T > O\}$. The group $SL_2(\mathcal{O})$ acts on each set by $T \to [U]T = UT\,{}^t\overline{U}$. To any positive definite $T = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) \in L_2^+$, we associate the point $P_T = b/d + (\sqrt{\det T}/d)j \in \mathbb{H}^3$. While to any indefinite $T = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) \in L_2$, we associate the geodesic hyperplane $S_T = \{P = z + rj \in \mathbb{H}^3; a + b\bar{z} + \bar{b}z + d(|z|^2 + r^2) = 0\}$. Moreover, we denote by $E(T) = \{U \in SL_2(\mathcal{O}); [U]T = T\}$ the unit group of T. Recall that $P_{[\sigma]T} = \sigma P_T$ and $S_{[\sigma]T} = \sigma S_T$ for $\sigma \in SL_2(\mathbf{C})$. The following is a 3-dimensional analogue of Katok-Sarnak [12] and Duke-Imamoḡlu [5].

Theorem 1. *Let $\mathcal{U}(P)$ be a spectral eigenfunction on $\mathbb{H}^3$ such that $-\Delta\mathcal{U} = (1-\mu^2)\mathcal{U}$ with some complex number μ. In the case of cusp eigenfunctions, there exists a real analytic cusp form $\varphi(\tau)$ on $H_1 = \{\tau = u + iv; v > 0\}$ of weight -1, character χ_K with respect to $\Gamma_0(4)$, that is*

$$\varphi(\gamma\tau) = \chi_K(d)|c\tau + d|(c\tau+d)^{-1}\varphi(\tau), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),$$

such that the Fourier expansion $\varphi(\tau) = \sum_{0\neq l\in\mathbf{Z}} b_{\mathcal{U}}(l) W_{-\mathrm{sgn}(l)/2,\ \mu/2}(4\pi|l|v)e(lu)$ satisfies

$$b_{\mathcal{U}}(l) = l^{-1} \sum_{T\in SL_2(\mathcal{O})\backslash L_2^+,\ 4\det T = l} \mathcal{U}(P_T)/\sharp E(T) \quad \text{for} \quad l > 0, \tag{1}$$

$$b_{\mathcal{U}}(l) = C_{\mathcal{U}}|l|^{-1} \sum_{T\in SL_2(\mathcal{O})\backslash L_2,\ 4\det T = l} \int_{E(T)\backslash \mathcal{S}_T} \mathcal{U}(P)d\sigma \quad \text{for} \quad l < 0$$

with a constant $C_{\mathcal{U}}$, where $d\sigma$ is hyperbolic measure on $\mathcal{S}_T$ and $W_{\alpha,\beta}(v)$ is the usual Whittaker function. In the case of non-cusp eigenfunctions, there exists a real analytic Eisenstein series $\varphi(\tau)$ of weight -1, character χ_K with respect to $\Gamma_0(4)$ whose Fourier coefficients are given by the same formulas for l such that all of $T \in L_2$ with $4\det T = l$ are not zero-forms. Moreover $\varphi(\tau)$ satisfies the plus condition, that is, if $\chi_K(l) = 1$ then $b_{\mathcal{U}}(l) = 0$ for any integer l.

The proof is based on [7], [17], [19], [20]. The measure $d\sigma$ is given explicitly in [7], [17].

3 Hermitian modular forms

Let $U(2,2) = \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_4(\mathbf{C});\ {}^t\overline{M}JM = J\}$, $J = \left(\begin{smallmatrix} O_2 & -I_2 \\ I_2 & O_2 \end{smallmatrix}\right)$. This group acts on the Hermitian upper half-space $H_2 = \{Z \in M_2(\mathbf{C}); (Z - {}^t\overline{Z})/(2i) > O\}$ by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)Z = (AZ+B)(CZ+D)^{-1}$. We denote by $\Gamma_2 = U(2,2) \cap M_4(\mathcal{O})$ the full Hermitian modular group of degree two. For any natural number N, the congruence subgroup $\Gamma_0^{(2)}(N)$ is defined to be $\Gamma_0^{(2)}(N) = \{\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2; C \equiv O_2 \pmod{N\mathcal{O}}\}$.

For $k \in \mathbf{Z}$ let ω be a character on $\mathcal{O}^\times$ such that $\omega(i) = i^{-k}$ and ψ a character on $(\mathbf{Z}/N\mathbf{Z})^\times$ such that $\psi(-1) = (-1)^k$. Then put $\rho(\epsilon d) = \omega(\epsilon)\psi(d)$ for $\epsilon \in \mathcal{O}^\times, d \in (\mathbf{Z}/N\mathbf{Z})^\times$. Using this ρ, we define a character on $\Gamma_0^{(2)}(N)$ by $\rho(\gamma) = \rho(\det D)$ for $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(2)}(N)$. Note here that $\det D \in \mathbf{Z} \cup i\mathbf{Z}$ for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2$. For an even natural number k, denote by $M_k(\Gamma_0^{(2)}(N), \rho)$ the space of all holomorphic functions $f(Z)$ on H_2 which satisfy

$$f(\gamma Z) = \rho(\gamma) \det(CZ + D)^k f(Z), \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(2)}(N).$$

Since k is even, the condition (A') in [14] p.92 holds, namely the character ρ is trivial on the principal congruence subgroup of level N. Any $f \in M_k(\Gamma_0^{(2)}(N), \rho)$ has a Fourier expansion $f(Z) = \sum_{T \in L_2 \geq 0} A(T, f) e(\mathrm{tr}(TZ))$, where the sum is extended over all half-integral semi-positive definite hermitian matrices of size two. As in Theorem 1, L_2^+ denotes the set of all positive definite $T \in L_2$. There exists a constant C such that $|A(T, f)| \leq C(\det T)^k$ for all $T \in L_2^+$.

Recall that the so-called hermitian imaginary part $Y = (Z - {}^t\overline{Z})/(2i)$ of $Z \in H_2$ is a positive definite hermitian matrix of size two. Denote by $\mathcal{P}_2$ the set of all positive definite hermitian matrices of size two and by $\mathcal{SP}_2$ the determinant one part of $\mathcal{P}_2$. We identify $\mathcal{SP}_2$ with 3-dimensional hyperbolic space $\mathbb{H}^3$ by

$$W = \begin{pmatrix} (|z|^2 + r^2)r^{-1} & zr^{-1} \\ \overline{z}r^{-1} & r^{-1} \end{pmatrix} \to P_W = z + rj.$$

Any automorphic function $\mathcal{U}(P)$ on $\mathbb{H}^3$ gives a function on $\mathcal{P}_2$ by setting $\mathcal{U}(Y) = \mathcal{U}(P_Y)$, where P_Y corresponds to $(\det Y)^{-1/2} Y$, in other words $Y \in \mathcal{P}_2$ is identified with $P_Y \in \mathbb{H}^3$ by

$$Y = \begin{pmatrix} a & b \\ \overline{b} & d \end{pmatrix} \to P_Y = b/d + (\sqrt{\det Y}/d)j.$$

Put moreover $\hat{\mathcal{U}}(Y) = \mathcal{U}(Y^{-1})$. Recall that $P_{[\sigma]Y} = \sigma P_Y$ for $Y \in \mathcal{P}_2, \sigma \in SL_2(\mathbf{C})$. Note that $\mathrm{tr}(TX), \mathrm{tr}(TY) \in \mathbf{R}$ for hermitian matrices T, X, Y.

Take a Fourier series

$$F(Z) = \sum_{T \in L_2^+} A(T, F) e(\mathrm{tr}(TZ)), \quad Z \in H_2. \tag{2}$$

Here we assume that $A(T, F) = A([U]T, F)$ for any $U \in GL_2(\mathcal{O})$ and $A(T, F) = O((\det T)^{\delta_1})$ with a positive constant δ_1. The series defining $F(Z)$ converges absolutely and uniformly in any domain $Y \geq Y_0 > O$ and bounded there. Moreover there exist positive constants C_1, C_2, δ_3, l such that

$$|F(iY)| \leq (C_1(\det Y)^{-(l+1)} + C_2(\det Y)^{-l}) e^{-\delta_3 (\det Y)^{1/2}}, \quad Y \in \mathcal{P}_2.$$

The needed reduction theory to prove this is presented in Theorem 4.12 [14] p.35.

For any $Y \in \mathcal{P}_2$ put $Y = uW$ with $u = (\det Y)^{1/2} > 0$ and $W \in \mathcal{SP}_2$. Assuming $\Re(s)$ to be sufficiently large, set

$$\tilde{F}_s(P) = \int_0^\infty F(iuW) u^{2s-1} du, \quad P = P_W.$$

By the assumptions on the Fourier coefficients, this satisfies $\tilde{F}_s(\gamma P) = \tilde{F}_s(P)$ for all $\gamma \in SL_2(\mathcal{O})$. It is easy to see that

$$\tilde{F}_s(P) = (2\pi)^{-2s} \Gamma(2s) \tilde{f}_s(P), \quad \tilde{f}_s(P) = \sum_{T \in L_2^+} A(T, F) \mathrm{tr}(TW)^{-2s} \tag{3}$$

and that the series defining $\tilde{f}_s(P)$ converges absolutely and uniformly on $\mathbb{H}^3 \times \{s = \sigma + it; \sigma_1 \leq \sigma\}$ with sufficiently large σ_1. These are bounded on $\mathbb{H}^3 \times \{s = \sigma + it; \sigma_1 \leq \sigma \leq \sigma_2\}$ with sufficiently large σ_1. Taking Stirling's formula into account, we can get $F(iuW)$ back by Mellin inversion from $\tilde{F}_s(P_W)$, that is,

$$F(iy^{1/2}W) = \frac{1}{2\pi i}\int_{\Re(s)=\sigma} 2\tilde{F}_s(P)y^{-s}ds, \quad \sigma >> 0.$$

Using $\Delta e^{-2\pi uB} = (-2\pi u)\{3B - 2\pi u(B^2 - 4\det T)\}e^{-2\pi uB}$ with $B = \mathrm{tr}(TW)$, we can deduce $\tilde{F}_s(P), \Delta\tilde{F}_s(P) \in L^2(SL_2(\mathcal{O}) \setminus \mathbb{H}^3)$ for sufficiently large $\Re(s)$. Using Theorem 3.4 (3) [6] p.267, we have the spectral decomposition of $\tilde{F}_s(P)$ ($\Re(s) >> 0$)

$$\tilde{F}_s(P) = \sum_m D^*(F, \overline{\mathcal{U}}_m, s)\mathcal{U}_m(P) + \frac{1}{2\pi}\int_{-\infty}^{\infty} D^*(F, \overline{E}(\ , it), s)E(P, it)dt, \tag{4}$$

where $\{\mathcal{U}_m\}$ consists of the constant function $\pi/\sqrt{2\zeta_K(2)}$ and an orthonormal system of cusp eigenfunctions, and the spectral coefficient $D^*(F, \overline{\mathcal{U}}, s)$ with respect to $\mathcal{U}(P)$ is given by

$$D^*(F, \overline{\mathcal{U}}, s) = \int_{\mathcal{F}} \tilde{F}_s(P)\overline{\mathcal{U}}(P)\frac{dxdydr}{r^3}, \quad P = (x + yi) + rj \in \mathbb{H}^3$$

where $\mathcal{F}$ is a fundamental domain of $SL_2(\mathcal{O}) \setminus \mathbb{H}^3$. As we will see in Proposition 1 below, this is the so-called Koecher-Maass series twisted by $\mathcal{U}(P)$ (cf. [16], [11] in the Siegel case).

For $f(Z) = \sum_{T \in L_2 \geq O} A(T, f)e(\mathrm{tr}(TZ)) \in M_k(\Gamma_0^{(2)}(N), \rho)$ and a spectral eigenfunction $\mathcal{U}$, we define $D^*(f, \mathcal{U}, s)$ by $D^*(F, \mathcal{U}, s)$, where $F(Z) = \sum_{T \in L_2^+} A(T, f)e(\mathrm{tr}(TZ))$ (the non-degenerate part of f).

Proposition 1. (1) *If $\mathcal{U}(P)$ has an eigenvalue $1 - \mu^2$ of $-\Delta$, then*

$$D^*(F, \mathcal{U}, s) = \pi(2\pi)^{-2s}\Gamma(s - 1/2 + \mu/2)\Gamma(s - 1/2 - \mu/2) \sum_{T \in SL_2(\mathcal{O}) \setminus L_2^+} \frac{A(T, F)\hat{\mathcal{U}}(T)}{\epsilon(T)(\det T)^s}$$

for sufficiently large $\Re(s)$, where the summation extends over all $T \in L_2^+$ modulo the action $T \to [U]T = UT\,{}^t\overline{U}$ of the group $SL_2(\mathcal{O})$ and $\epsilon(T) = \sharp\{U \in SL_2(\mathcal{O}); [U]T = T\}$ is the order of the unit group of T.
(2) *For $f \in M_k(\Gamma_0^{(2)}(N), \rho)$ and a spectral eigenfunction $\mathcal{U}$, $D^*(f, \mathcal{U}, s)$ has a meromorphic continuation to all s. It satisfies a functional equation*

$$N^s D^*(f, \overline{\mathcal{U}}, s) = (-1)^k N^{k-s} D^*(g, \hat{\overline{\mathcal{U}}}, k - s),$$

where $g(Z) = N^{-k}(\det Z)^{-k} f(-(NZ)^{-1})$.

We note here that Ibukiyama [10] established a general theory of Koecher-Maass series with Grössencharacter (suitable automorphic forms) associated with modular forms on tube domains including convergences of the series, determination of the gamma factor, meromorphic continuations and functional equations.

If the Fourier coefficients $A(T, F)$ satisfy a Maass type relation, then $D(F, \mathcal{U}, s)$ is a convolution product of two Dirichlet series (cf. Satz 3 [3] and Lemma 3 [5]).

Proposition 2. *Let χ be a multiplicative function on $\mathbf{N}$. Suppose that there exists a function α on the set of all natural numbers satisfying*

$$A(T, F) = \sum_{d|e(T)} \chi(d)d^{k-1}\alpha((4\det T)/d^2),$$

where $e(T) = \max\{q \in \mathbf{N}; q^{-1}T \in L_2^+\}$. *Then for any automorphic function* $\mathcal{U}$ *on* $\mathbb{H}^3$ *whose eigenvalue of* $-\Delta$ *is* $1-\mu^2$, *we have*

$$D^*(F,\overline{\mathcal{U}},s) = \pi(2\pi)^{-2s}\Gamma(s-1/2+\overline{\mu}/2)\Gamma(s-1/2-\overline{\mu}/2)4^sL(2s-k+1,\chi)\sum_{l\geq 1}\frac{\alpha(l)b_{\hat{\overline{\mathcal{U}}}}(l)l}{l^s}$$

for sufficiently large $\Re(s)$, *where* $b_{\mathcal{U}}(l)$ *is as in* (1).

4 Hermitian analogue of Saito-Kurokawa lift

As discovered by Duke-Imamoḡlu [5] in the case of the Saito-Kurokawa lift, Theorem 1 and Proposition 2 allow us to analyze each spectral coefficient of $F(iY)$ by the Rankin-Selberg method. We can reprove a Hermitian analogue of Saito-Kurokawa lift by means of a converse theorem. This lifting was discovered by Kojima [13].

Suppose that a natural number k is divisible by 4. Take a cusp form of weight $k-1$, character χ_K on $\Gamma_0(4)$ belonging to the plus space in the sense of Kojima [13]

$$g(\tau) = \sum_{l\geq 1,\, \chi_K(l)\neq 1} c(l)e(l\tau) \in S_{k-1}(\Gamma_0(4),\chi_K), \quad \tau\in H_1. \tag{5}$$

Put $\alpha^*(l) = c(l)/(\chi_K(-l)+1)$ and define a function on H_2 by

$$F(Z) = \sum_{T\in L_2^+}\left(\sum_{d|e(T)} d^{k-1}\alpha^*((4\det T)/d^2)\right)e(\mathrm{tr}(TZ)). \tag{6}$$

Theorem 2. *$F(Z)$ is a modular form of weight k on the full Hermitian modular group* $\Gamma_2 = \{\gamma\in M_4(\mathcal{O}); {}^t\overline{\gamma}J\gamma = J\}$, $J = \left(\begin{smallmatrix} O_2 & -I_2 \\ I_2 & O_2\end{smallmatrix}\right)$.

An analogous converse theorem to the Siegel case given in Theorem 2 [5] is the following.

Proposition 3. *Suppose that k is divisible by 4 and take $F(Z)$ as in* (2). *If* $D^*(F,\overline{\mathcal{U}},s)$ *and* $D^*(F,\hat{\overline{\mathcal{U}}},s)$ *are entire and bounded in every vertical strip in s and satisfy* $D^*(F,\hat{\overline{\mathcal{U}}},s) = D^*(F,\overline{\mathcal{U}},k-s)$ *for any spectral eigenfunction $\mathcal{U}$ on* $\mathbb{H}^3$, *then $F(Z)$ is a Hermitian modular form of weight k on* Γ_2.

Proof. We follow Ibukiyama's proof [9] of Theorem 2 [5]. Put $B = \mathrm{tr}(TW)$. Using the identities $\Delta B = 3B$ and $r^2((B_x)^2+(B_y)^2+(B_r)^2) = B^2-4\det T$, we have

$$\begin{aligned}\Delta^2 e^{-2\pi uB} = (-2\pi u)e^{-2\pi uB}\{&9B-2\pi u(8(B^2-4\det T)+15B^2)\\ &+(2\pi u)^2 10B(B^2-4\det T)-(2\pi u)^3(B^2-4\det T)^2\}.\end{aligned}$$

It follows that $\Delta^2\tilde{f}_s(P)$ is bounded on $\mathbb{H}^3\times\{s=\sigma+it;\sigma_1\leq\sigma\leq\sigma_2\}$ with sufficiently large σ_1 and that $\Delta^2\tilde{f}_s(P)\in L^2(SL_2(\mathcal{O})\setminus\mathbb{H}^3)$, where $\tilde{f}_s$ is as in (3) (see Proposition 2.5 [11] p.910 and we refer to Theorem 4.12 [14] p.35 for the reduction theory in the Hermitian case). Suppose that $-\Delta\mathcal{U}_m = \lambda_m\mathcal{U}_m$. Since $-\Delta$ is symmetric (Theorem 1.7 [6] p.136), we have $(\tilde{f}_s,\mathcal{U}_m) = \lambda_m^{-2}(\Delta^2\tilde{f}_s,\mathcal{U}_m)$ for cusp eigenfunctions $\mathcal{U}_m$, where (,) denotes the inner product on $SL_2(\mathcal{O})\setminus\mathbb{H}^3$, see (1.2) [6] p.133. Using an elementary inequality about geometric and arithmetic means, the above relation and Schwarz' inequality, we deduce that

$$|(\tilde{F}_s,\mathcal{U}_m)\mathcal{U}_m(P)| \leq |(2\pi)^{-2s}\Gamma(2s)|2^{-1}\{\lambda_m^{-2}(\Delta^2\tilde{f}_s,\Delta^2\tilde{f}_s)+\lambda_m^{-2}|\mathcal{U}_m(P)|^2\}.$$

Then Corollaries 5.3 and 5.5 [6] p.182 implies that the sum on the right-hand side of (4) converges absolutely and uniformly in $L \times \{s = \sigma + it; \sigma_1 \leq \sigma \leq \sigma_2\}$, where L is a compact subset of $\mathbb{H}^3$ and σ_1 is sufficiently large, and it is of rapid decay as $|t| \to \infty$ by Stirling's formula. We also note that there exists a positive constant δ_2 such that $E(P, it) = O(|t|^{\delta_2})$ as $|t| \to \infty$. Accordingly, we can apply Mellin inversion of (4) term by term.

The standard procedure (employing Stirling's formula and the Phragmen-Linderöf theorem, shifting the path of integration, using the functional equation) gives $F(iu^{-1}W^{-1}) = u^{2k}F(iuW)$ and $F(-(iY)^{-1}) = \det(iY)^k F(iY)$. In fact, $D^*(F, \hat{\overline{\mathcal{U}}}, s)$ is the spectral coefficient of $\tilde{F}_s(P_{W^{-1}})$ with respect to $\mathcal{U}(P_W)$. In view of Lemma 1.7 [14] p.79 and Lemma 1.6 [14] p.48, we complete the proof of Proposition 3. □

Proof of Theorem 2. By the Rankin-Selberg method, we get an integral representation of $D^*(F, \overline{\mathcal{U}}, s)$. This integral representation allows us to check the assumptions in Proposition 3. Note here that $b_{\hat{\mathcal{U}}}(l) = b_{\mathcal{U}}(l)$. □

5 Hermitian-Eisenstein series

5.1 A main result

Another example is an application to a Hermitian-Eisenstein series. Theorem 1 and Proposition 2 make it possible to determine every spectral coefficients of the non-degenerate part of a certain Hermitian-Eisenstein series. Using a Maass lift, we can construct a Hermitian modular form with the same spectral coefficients. Consequently, the Hermitian-Eisenstein series coincides with this image of the Maass lift. This fact yields an explicit form of the Fourier coefficients of the Hermitian-Eisenstein series.

Suppose that $k > 4$ is even and N is a natural number. Denote by Γ_2 the full Hermitian modular group and put $\Gamma_\infty^{(2)} = \{\gamma \in \Gamma_2; C = O_2\}$. A Hermitian-Eisenstein series of weight k, degree two and character ρ on $\Gamma_0^{(2)}(N)$ is defined to be

$$E_{k,\overline{\rho}}^{(2)}(Z) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty^{(2)} \backslash \Gamma_0^{(2)}(N)} \overline{\rho}(\det D) \det(CZ + D)^{-k}, \qquad Z \in H_2.$$

It has a Fourier expansion indexed by semi-positive definite $T \in L_2$. If $N = 1$, some explicit forms of the Fourier coefficients are obtained by Krieg [15] and Nagaoka [22].

Theorem 3. *Suppose that $N > 1$ is a square-free odd natural number and the above ψ is a primitive Dirichlet character mod N. The T-th Fourier coefficient of the Hermitian-Eisenstein series for any positive definite $T \in L_2$ is given by*

$$A(T, E_{k,\overline{\rho}}^{(2)}) = \frac{(-2\pi i)^k \tau_N(\overline{\psi})}{N^k \Gamma(k) L(k, \overline{\psi})} \sum_{d|e(T)} \psi(d) d^{k-1} e_{\overline{\rho}}^{\infty}((4 \det T)/d^2),$$

where $\tau_N(\psi) = \sum_{r=1}^{N} \psi(r) e^{2\pi i r/N}$ is the Gauss sum, $\Gamma(s)$ is the gamma function, $L(s, \psi)$ is the Dirichlet L-function of ψ, and $e_{\overline{\rho}}^{\infty}(t)$ has the form

$$e_{\overline{\rho}}^{\infty}(t) = \frac{2^{2-k}\pi^{k-1}}{i^k \Gamma(k-1)} t^{k-2} \frac{\gamma_{2,\overline{\psi}}(t, k-1)}{L(k-1, \chi_K \overline{\psi})} \prod_{odd\ prime\ p} \gamma_{p,\overline{\psi}}(t, k-1) \prod_{prime\ p|N} C_{\overline{\psi},p}^{\infty}(t).$$

Here for a prime q and a natural number l_q such that q^{l_q} is the exact power of q dividing t,

$$\gamma_{p,\psi}(t,k-1) = \frac{1-(\chi_K(p)\psi(p)p^{2-k})^{l_p+1}}{1-\chi_K(p)\psi(p)p^{2-k}}, \quad (p \neq 2),$$

$$\gamma_{2,\psi}(t,k-1) = \begin{cases} 1, & \text{for } l_2 = 0, \\ 1+\chi_K(-t/2^{l_2})(\psi(2)2^{2-k})^{l_2}, & \text{for } l_2 \geq 1, \end{cases}$$

$$C^{\infty}_{\psi,p}(t) = \psi_p(4)\frac{\psi_p^*(p^{l_p+1})}{p^{(k-1)(l_p+1)}}\chi_K(p)^{l_p+1}\overline{\psi_p}(t/p^{l_p})p^{l_p}\tau_p(\psi_p),$$

where $\tau_p(\psi_p) = \sum_{r=0}^{p-1}\psi_p(r)e^{2\pi i r/p}$ is the Gauss sum, ψ_p are the primitive Dirichlet characters mod p so that $\psi = \prod_{prime\ p|N}\psi_p$ and $\psi_p^* = \prod_{prime\ q|(N/p)}\psi_q$.

In the following subsections, we give a sketch of the proof.

5.2 Hermitian Jacobi forms

In this section we recall some basic facts on Hermitian Jacobi forms. We refer to [8], [23] for more details. Let $H_1 = \{\tau = u+iv; v>0\}$ be the upper half-plane. The action of $SL_2(\mathbf{R})$ on H_1 is denoted by $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\tau = \frac{a\tau+b}{c\tau+d}$. Put $U(1,1) = \{\epsilon M; \epsilon \in S^1, M \in SL_2(\mathbf{R})\}$. For integers k and m, there is an action of the Jacobi group $U(1,1) \ltimes (\mathbf{C}^2 \times S^1)$ for functions ϕ on $H_1 \times \mathbf{C}^2$ given by

$$\phi|_{k,m}\xi(\tau,z,w) = \epsilon^{-k}(c\tau+d)^{-k}e^m\left(\frac{-c(z+\lambda\tau+\mu)(w+\overline{\lambda}\tau+\overline{\mu})}{c\tau+d} + \mathcal{N}(\lambda)\tau + \overline{\lambda}z + \lambda w\right)$$
$$\times s^m\phi\left(M\tau, \frac{\epsilon(z+\lambda\tau+\mu)}{c\tau+d}, \frac{\overline{\epsilon}(w+\overline{\lambda}\tau+\overline{\mu})}{c\tau+d}\right),$$

where $\xi = \left(\epsilon\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), (\lambda,\mu), s\right) \in U(1,1) \ltimes (\mathbf{C}^2 \times S^1), (\tau,z,w) \in H_1 \times \mathbf{C}^2$ and $e^m(x) = e^{2\pi i m x}$.

For a square-free odd natural number N, denote by $\Gamma_0^{(1)}(N)$ the congruence subgroup

$$\Gamma_0^{(1)}(N) = \left\{\epsilon\begin{pmatrix} a & b \\ c & d\end{pmatrix}; \epsilon \in \mathcal{O}^\times, \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbf{Z}), c \equiv 0 \pmod{N}\right\}.$$

Let ω be a character on $\mathcal{O}^\times$ such that $\omega(i) = i^{-k}$ and ψ a primitive Dirichlet character on $(\mathbf{Z}/N\mathbf{Z})^\times$ such that $\psi(-1) = (-1)^k$. Put $\rho(\epsilon d) = \omega(\epsilon)\psi(d)$ for $\epsilon \in \mathcal{O}^\times, d \in (\mathbf{Z}/N\mathbf{Z})^\times$. Then $\rho(\epsilon\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)) = \rho(\epsilon d)$ is a character on $\Gamma_0^{(1)}(N)$. For natural numbers $k>4$ and m, we denote by $J_{k,m}(\Gamma_0^{(1)}(N),\rho)$ the space consisting of all holomorphic functions ϕ on $H_1 \times \mathbf{C}^2$ satisfying the following two conditions.

(J-i) $\phi|_{k,m}\xi = \rho(\gamma)\phi$ for all $\xi = (\gamma,(\lambda,\mu)) \in \Gamma^J = \Gamma_0^{(1)}(N) \ltimes \mathcal{O}^2$.

(J-ii) For each $M \in SL_2(\mathbf{Z})$, $\phi|_{k,m}M$ has a Fourier expansion of the form

$$\phi|_{k,m}M(\tau,z,w) = \sum_{\substack{n\in\mathbf{Z},\alpha\in\mathcal{D}^{-1} \\ nm-\nu\mathcal{N}(\alpha)\geq 0}} c_M(n,\alpha)q^{n/\nu}\zeta_1^{\alpha}\zeta_2^{\overline{\alpha}},$$

where $q^\beta = e^{2\pi i\beta\tau}$, $\zeta_1^\alpha = e^{2\pi i\alpha z}$, $\zeta_2^\alpha = e^{2\pi i\alpha w}$ and ν is a natural number depending on M.

The Maass lift $\mathcal{M}$ from the space $J_{k,1}(\Gamma_0^{(1)}(N),\rho)$ to the space $M_k(\Gamma_0^{(2)}(N),\rho)$ is defined as follows. For $\phi \in J_{k,1}(\Gamma_0^{(1)}(N),\rho)$ and any natural number m, we define the operator V_m by

$$\phi|_{k,1}V_m(\tau,z,w)$$
$$= m^{k-1} \sum_{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_0(N)\backslash M_2^*(m)} \psi(a)(c\tau+d)^{-k} e\left(-\frac{cmzw}{c\tau+d}\right)\phi\left(M\tau, \frac{mz}{c\tau+d}, \frac{mw}{c\tau+d}\right),$$

where $M_2^*(m) = \{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbf{Z}); \det M = m, c \equiv 0 \pmod N, (a,N)=1\}$. It is easy to see that $\phi|_{k,1}V_m$ transforms like an element in $J_{k,m}(\Gamma_0^{(1)}(N),\rho)$. Moreover, put

$$\phi_0(\tau) = \left\{ \frac{N^k\Gamma(k)L(k,\overline{\psi})}{(-2\pi i)^k \tau_N(\overline{\psi})} + \sum_{n\geq 1}\left(\sum_{d|n}\psi(d)d^{k-1}\right)q^n \right\} c(0,0).$$

Since ψ is primitive, this is an Eisenstein series on $\Gamma_0(N)$ for the cusp ∞.

Proposition 4.

$$\mathcal{M}\phi(\begin{pmatrix} \tau' & z \\ w & \tau \end{pmatrix}) := \phi_0(\tau) + \sum_{m\geq 1}\phi|_{k,1}V_m(\tau,z,w)e(m\tau') \in M_k(\Gamma_0^{(2)}(N),\rho).$$

5.3 Hermitian Jacobi Eisenstein series

In this section we give some Fourier developments of Hermitian Jacobi Eisenstein series on $\Gamma^J = \Gamma_0^{(1)}(N) \ltimes \mathcal{O}^2$ associated with the cusps 0 and ∞. With the previous notation, suppose moreover that $N > 1$ is square-free odd. For $G \subset \Gamma_0^{(1)}(1) \ltimes \mathcal{O}^2$, put $G_\infty = \left\{\left(\epsilon\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right), (0,\mu)\right) \in G\right\}$. For any cusp κ of $\Gamma_0(N)$, take $g \in SL_2(\mathbf{Z})$ such that $g(\infty) = \kappa$. The Hermitian Jacobi Eisenstein series of weight k and index 1 associated with κ is defined by

$$E^\kappa_{k,1,\rho}(\tau,z,w) = \sum_{\gamma\in(g\Gamma^J g^{-1})_\infty\backslash g\Gamma^J} \rho(g^{-1}\gamma)1|_{k,1}\gamma.$$

One easily has $E^\kappa_{k,1,\rho}|_{k,1}\gamma = \overline{\rho}(\gamma)E^\kappa_{k,1,\rho}$ for all $\gamma \in \Gamma^J$. We choose $g = I_2$ (resp. $g = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$) so that $g(\infty) = \infty$ (resp. $g(\infty) = 0$).

The Fourier coefficients of $E^\kappa_{k,1,\rho}$ for $\kappa \in \{\infty, 0\}$ can be computed in the same way as in Theorem 2.1 [22]. Define $\theta_\alpha(\tau,z,w)$ by

$$\theta_\alpha(\tau,z,w) = \sum_{\beta\in\alpha+\mathcal{O}} q^{\mathcal{N}(\beta)}\zeta_1^{\overline{\beta}}\zeta_2^{\beta}, \tag{7}$$

where $q^\beta = e^{2\pi i\beta\tau}$, $\zeta_1^\alpha = e^{2\pi i\alpha z}$, $\zeta_2^\alpha = e^{2\pi i\alpha w}$.

Proposition 5. *For $\kappa \in \{\infty, 0\}$ the Fourier development of $E^\kappa_{k,1,\rho}$ is given by*

$$E^\kappa_{k,1,\rho}(\tau,z,w) = \delta_{\kappa,\infty}\theta_0(\tau,z,w) + \sum_{\substack{t>0,\alpha\in\mathcal{D}^{-1} \\ t\equiv -4\mathcal{N}(\alpha) \pmod 4}} e^\kappa_\rho(t)q^{(t+4\mathcal{N}(\alpha))/4}\zeta_1^\alpha\zeta_2^{\overline{\alpha}},$$

$$e^\infty_\rho(t) = \alpha_{k,4}t^{k-2}B_\psi(t)\prod_{prime\ p|N} C^\infty_{\psi,p}(t), \quad e^0_\rho(t) = \alpha_{k,4}\psi(-1)t^{k-2}B_{\overline{\psi}}(t).$$

Here $\alpha_{k,4} = 2^{2-k}\pi^{k-1}i^{-k}\Gamma(k-1)^{-1}$ *and* $\delta_{i,j}$ *is Kronecker's delta,*

$$B_\psi(t) = \frac{\gamma_{2,\psi}(t,k-1)}{L(k-1,\chi_K\psi)} \prod_{odd\ prime\ p} \gamma_{p,\psi}(t,k-1),$$

where $\gamma_{q,\psi}(t,k-1)$ *and* $C^\infty_{\psi,p}(t)$ *are as in Theorem* 3.

Since N is square-free, $\{\infty,0\}\bigcup\{1/\mu; 1<\mu<N, \mu|N\}$ is a set of representatives of non-equivalent cusps of $\Gamma_0(N)$. As elements in $SL_2(\mathbf{Z})$ transforming ∞ to the cusps, we take

$$\sigma_\infty = I_2,\ \sigma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \sigma_\mu = \begin{pmatrix} 1 & \alpha \\ \mu & N\beta/\mu \end{pmatrix}, \tag{8}$$

where integers α and β are chosen so that $N\beta/\mu - \alpha\mu = 1$. For the cusp κ, we will also use symbols σ_κ instead of (8).

Proposition 6. (1) *The Fourier development of* $E^0_{k,1,\rho}|_{k,1}\sigma_0$ *is given by*

$$E^0_{k,1,\rho}|_{k,1}\sigma_0(\tau,z,w) = \theta_0(\tau,z,w) + \sum_{\substack{t>0,\alpha\in\mathcal{D}^{-1} \\ t\equiv -4N\mathcal{N}(\alpha) \pmod 4}} a^0_\rho(Nt)q^{(t+4N\mathcal{N}(\alpha))/(4N)}\zeta_1^\alpha\zeta_2^{\overline{\alpha}},$$

$$a^0_\rho(Nt) = \alpha_{k,4}\chi_K(N)t^{k-2}B_{\overline{\psi}}(Nt) \prod_{prime\ p|N} C^\infty_{\overline{\psi},p}(t). \tag{9}$$

Here the notation is the same as in Proposition 5.
(2) *For* $\kappa = 1/\mu$ *with* $1<\mu<N$ *dividing* N*, the coefficient functions* $H^\kappa_\alpha(\tau)$ *of the theta expansion*

$$E^0_{k,1,\rho}|_{k,1}\sigma_\kappa(\tau,z,w) = \sum_{\alpha\in\mathcal{D}^{-1}/\mathcal{O}} H^\kappa_\alpha(\tau)\theta_\alpha(\tau,z,w) \tag{10}$$

are of rapid decay as $\Im\tau$ *tends to* ∞.

5.4 Proof of Theorem 3

We can determine an explicit form of the Koecher-Maass series associated with Hermitian-Eisenstein series by a method similar to [21]. For a spectral eigenfunction $\mathcal{U}$ corresponding to the eigenvalue $-(1-\mu^2)$ of Δ, the Koecher-Maass series of $E^{(2)}_{k,\overline{\rho}}$ has the form ($\Re(s) >> 0$)

$$D^*(E^{(2)}_{k,\overline{\rho}},\overline{\mathcal{U}},s) = \frac{\tau_{k,2}4^{-k+2}N^{-k}\pi^{1-2s}\tau_N(\overline{\psi})}{\alpha_{k,4}L(k,\overline{\psi})} \times \Gamma(s-1/2+\overline{\mu}/2)\Gamma(s-1/2-\overline{\mu}/2)L(2s-k+1,\psi)\sum_{l=1}^\infty \frac{e^\infty_{\overline{\rho}}(l)b_{\hat{\overline{\mathcal{U}}}}(l)l}{l^s},$$

where $\tau_{k,2} = (-1)^k(2\pi)^{2k-1}\{2\Gamma(k)\Gamma(k-1)\}^{-1}$ and $\alpha_{k,4}$ is as in Proposition 5.

Set

$$F = E^{(2)}_{k,\overline{\rho}} - \frac{(-2\pi i)^k\tau_N(\overline{\psi})}{N^k\Gamma(k)L(k,\overline{\psi})}\mathcal{M}E^\infty_{k,1,\overline{\rho}}.$$

Applying the Siegel operator Φ defined by $\Phi F(\tau) = \lim_{\lambda\to+\infty} F\left(\begin{smallmatrix} i\lambda & 0 \\ 0 & \tau\end{smallmatrix}\right) \in M_k(\Gamma_0(N),\rho)$, we have $\Phi F = 0$ so that the Fourier expansion of $F(Z)$ has only the terms indexed by L_2^+. Moreover, $D^*(F,\overline{\mathcal{U}},s) = 0$ for any spectral eigenfunctions $\mathcal{U} = \mathcal{U}_m, E(\ ,it)$. Hence $\tilde{F}_s(P) = 0$ in (4) and so $F(Z) = 0$ by Mellin inversion and the principle of analytic continuations (Lemma 1.6 [14] p.48). This completes the proof of Theorem 3.

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