On the anabelian geometry of hyperbolic curves over finite fields

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In my talk in the "3rd Fukuoka Symposium on Number Theory" I reported on the recent progress I made in collaboration with Akio Tamagawa in the study of the anabelian geometry of hyperbolic curves over finite fields. Below are the main results I discussed in my talk.

Let X be a proper, smooth, and geometrically connected curve over a finite field $k = k_X$ of characteristic $p = p_X > 0$. Write $K = K_X$ for the function field of X. Let S be a (possibly empty) finite set of closed points of X, and set $U = U_S \overset{\text{def}}{=} X - S$. We assume that U is hyperbolic. Let η be a base point of X with value in the generic point of X. Then η determines an algebraic closure $\bar{k}$ of k, and a separable closure K^{sep} of K. Denote by $\overline{U} \overset{\text{def}}{=} U \times_k \bar{k}$ the geometric fiber of U, $G_k \overset{\text{def}}{=} \text{Gal}(\bar{k}/k)$ the absolute Galois group of k, and by $\pi_1(U)$ the étale fundamental group of U with base point η. Then $\pi_1(U)$ sits naturally in the following exact sequence:

$$1 \to \pi_1(\overline{U}) \to \pi_1(U) \to G_k \to 1,$$

where $\pi_1(\overline{U})$ is the étale fundamental group of $\overline{U}$ with base point η. Also, denote by $\overline{X} \overset{\text{def}}{=} X \times_k \bar{k}$ the geometric fiber of X, and $K_{\overline{X}}$ the function field of $\overline{X}$. Let $G_{K_X} \overset{\text{def}}{=} \text{Gal}(K^{\text{sep}}/K)$ be the absolute Galois group of K_X. Then G_{K_X} sits naturally in the following exact sequence:

$$1 \to G_{K_{\overline{X}}} \to G_{K_X} \to G_k \to 1$$

where $G_{K_{\overline{X}}} \overset{\text{def}}{=} \text{Gal}(K^{\text{sep}}/K_{\overline{X}})$ is the absolute Galois group of $K_{\overline{X}}$.

According to the anabelian (respectively, birational anabelian) philosophy of Grothendieck (cf. [Gr]) the isomorphy type of U as a scheme (respectively, K_X as a field) should be determined by the isomorphy type of $\pi_1(U)$ as a profinite group (respectively, G_{K_X} as a profinite group). The following result is fundamental in the anabelian geometry of hyperbolic curves over finite fields.

Theorem 1 (Tamagawa, Mochizuki). *Let U, V be hyperbolic curves over finite fields k_U, k_V, respectively. Let*

$$\alpha : \pi_1(U) \overset{\sim}{\to} \pi_1(V)$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\sim} & \tilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are isomorphisms, and the vertical arrows are the profinite étale universal coverings determined by the profinite groups $\pi_1(U)$, $\pi_1(V)$, respectively.

Theorem 1 implies in particular the following birational version of the Grothendieck anabelian conjecture for hyperbolic curves over finite fields, which was already proved by Uchida.

Theorem 2 (Uchida). *Let X, Y be proper, smooth, and geometrically connected curves over finite fields k_X, k_Y, respectively. Let K_X, K_Y be the function fields of X, Y, respectively. Let G_{K_X}, G_{K_Y} be the absolute Galois groups of K_X, K_Y, respectively. Let*

$$\alpha : G_{K_X} \xrightarrow{\sim} G_{K_Y}$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of field extensions:

$$\begin{array}{ccc} (K_X)^{\sim} & \xrightarrow{\sim} & (K_Y)^{\sim} \\ \uparrow & & \uparrow \\ K_X & \xrightarrow{\sim} & K_Y \end{array}$$

in which the horizontal arrows are isomorphisms, and the vertical arrows are the field extensions corresponding to the Galois groups G_{K_X}, G_{K_Y}, respectively.

Theorem 2 was first proven by Uchida (cf. [Uc]). Theorem 1 was proven by Tamagawa (cf. [Ta, Theorem (4.3)]) in the affine case (together with a certain tame version), and more recently by Mochizuki (cf. [Mo, Theorem 3.12]) in the proper case. It implies in particular that one can embed a suitable category of hyperbolic curves over finite fields into the category of profinite groups via the fundamental group functor. It is essential in the anabelian philosophy of Grothendieck, as was formulated in [Gr], to be able to determine the image of this functor. Recall that the full structure of the profinite group $\pi_1(\overline{U})$ is unknown (for any single example of U which is hyperbolic). Hence, a fortiori, the structure of $\pi_1(U)$ is unknown (the closed subgroup $\pi_1(\overline{U})$ of $\pi_1(U)$ can be reconstructed group theoretically from the isomorphy type of $\pi_1(U)$ (cf. [Ta, Proposition 3.2]). (Even if we replace the fundamental groups $\pi_1(\overline{U})$, $\pi_1(U)$ by the tame fundamental groups $\pi_1^t(\overline{U})$, $\pi_1^t(U)$, respectively, the situation is just the same.) The full structure of the absolute Galois group G_{K_X} is also unknown, though one knows the structure of the closed subgroup $G_{K_{\overline{X}}}$ of G_{K_X} by a result of Pop and Harbater. Namely $G_{K_{\overline{X}}}$ is a profinite free group on Card($\bar{k}$)-generators (cf. [Po], [Ha]). Thus, the problem of determining the image of the above functor seems to be quite difficult, at least for the moment. It is quite natural to address the following question:

Question 1. Is it possible to prove any result analogous to the above Theorems 1 and 2 where $\pi_1(U)$ (respectively, G_{K_X}) is replaced by some (continuous) quotient of $\pi_1(U)$ (respectively, G_{K_X}) whose structure is better understood?

The first quotients that come into mind are the following. Let $\mathfrak{Primes}$ be the set of all prime integers. Let $\Sigma = \Sigma_U \subset \mathfrak{Primes}$ be a set of prime integers not containing the characteristic p. Let $\mathcal{C}$ (respectively, $\mathcal{C}^l$) be the full class of finite groups whose cardinality is divisible only by primes in Σ (respectively, finite l-groups, where $l \neq p$ is a fixed prime number). Let Δ_U be the maximal pro-$\mathcal{C}$ quotient of $\pi_1(\overline{U})$. For a profinite group Γ, Γ^l stands for the maximal pro-l (i.e., pro-$\mathcal{C}^l$) quotient of Γ. Here, the structure of Δ_U is well understood: Δ_U is isomorphic to the pro-Σ completion of a certain well-known finitely generated discrete group (i.e., either a free group or a surface group). Let $\Pi_U \overset{\text{def}}{=} \pi_1(U)/\operatorname{Ker}(\pi_1(\overline{U}) \twoheadrightarrow \Delta_U)$ be the corresponding quotient of $\pi_1(U)$. We shall refer to Π_U as the geometrically pro-Σ

étale fundamental group of U. In a similar way we can define the maximal pro-Σ quotient $G^{\Sigma}_{K_{\overline{X}}}$ of $G_{K_{\overline{X}}}$ and the corresponding quotient $G^{(\Sigma)}_{K_X}$ of G_{K_X}, which we will refer to as the geometrically pro-Σ quotient of the absolute Galois group G_{K_X}.

Question 2. Is it possible to prove any result analogous to the above Theorem 1 (respectively, Theorem 2) where $\pi_1(U)$ is replaced by Π_U (respectively, G_{K_X} is replaced by $G^{(\Sigma)}_{K_X}$), for a given set of prime integers $\Sigma \subset \mathfrak{Primes}$ not containing the characteristic p?

The first set Σ to consider is the set $\Sigma \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \{\text{characteristic} = p\}$. In this case we have the following results.

Theorem 3 (A Prime-to-p Version of Grothendieck's Anabelian Conjecture for Hyperbolic Curves over Finite Fields). *Let U, V be hyperbolic curves over finite fields k_U, k_V, respectively. Let $\Sigma_U \stackrel{\text{def}}{=} \mathfrak{Primes} - \{\text{char}(k_U)\}$, $\Sigma_V \stackrel{\text{def}}{=} \mathfrak{Primes} - \{\text{char}(k_V)\}$, and write Π_U, Π_V for the geometrically pro-Σ_U étale fundamental group of U, and the geometrically pro-Σ_V étale fundamental group of V, respectively. Let*

$$\alpha : \Pi_U \xrightarrow{\sim} \Pi_V$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\sim} & \tilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\sim} & V \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups Π_U, Π_V, respectively.

Theorem 3 was proven by Saïdi and Tamagawa (cf. [Sa-Ta, Corollary 3.10]). As a consequence of Theorem 3 one can deduce the following prime-to-characteristic version of Uchida's theorem (cf. [Sa-Ta, Corollary 3.11]). In the case where $\Sigma \stackrel{\text{def}}{=} \mathfrak{Primes} \setminus \{\text{characteristic} = p\}$ we will refer to $G^{(\Sigma)}_K$ as the geometrically prime-to-characteristic quotient of G_K.

Theorem 4 (A Prime-to-p Version of Uchida's Theorem on Isomorphisms between Galois Groups of Function Fields). *Let X, Y be proper, smooth, and geometrically connected curves over finite fields k_X, k_Y, respectively. Let K_X, K_Y be the function fields of X, Y, respectively. Let G_{K_X}, G_{K_Y} be the absolute Galois groups of K_X, K_Y, respectively, and let G'_{K_X}, G'_{K_Y} be their geometrically prime-to-characteristic quotients. Let*

$$\alpha : G'_{K_X} \xrightarrow{\sim} G'_{K_Y}$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of field extensions:

$$\begin{array}{ccc} (K_X)^{\sim} & \xrightarrow{\sim} & (K_Y)^{\sim} \\ \uparrow & & \uparrow \\ K_X & \xrightarrow{\sim} & K_Y \end{array}$$

in which the horizontal arrows are isomorphisms, and the vertical arrows are the extensions corresponding to the groups G'_{K_X}, G'_{K_Y}, respectively.

In a recent work together with Akio Tamagawa we proved the following refined version of Uchida's theorem which is stronger than Theorem 4.

Theorem 5 (A Refined Version of Uchida's Theorem on Isomorphisms between Galois Groups of Function Fields). *Let X, Y be proper, smooth, and geometrically connected curves over finite fields k_X, k_Y, respectively. Let K_X, K_Y be the function fields of X, Y, respectively. Let G_{K_X}, G_{K_Y} be the absolute Galois groups of K_X, K_Y, respectively. Let $\Sigma = \Sigma_X \subset \mathfrak{Primes}$ be a set of primes which is k_X-large, meaning that the Σ-cyclotomic character $\chi_\Sigma : G_{k_X} \to \prod_{l \in \Sigma} \mathbb{Z}_l^\times$ is injective. Let $G_{K_X}^{(\Sigma)}$ (respectively. $G_{K_Y}^{(\Sigma)}$) be the geometrically pro-Σ quotient of G_{K_X} (respectively. G_{K_Y}). Let*

$$\alpha : G_{K_X}^{(\Sigma)} \xrightarrow{\sim} G_{K_Y}^{(\Sigma)}$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of field extensions:

$$\begin{array}{ccc} (K_X)^{\sim} & \xrightarrow{\sim} & (K_Y)^{\sim} \\ \uparrow & & \uparrow \\ K_X & \xrightarrow{\sim} & K_Y \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the field extensions corresponding to the Galois groups $G_{K_X}^{\Sigma}$, $G_{K_Y}^{\Sigma}$, respectively.

We can also prove the following refined version of Theorem 3.

Theorem 6 (A Refined Version of Grothendieck's Anabelian Conjecture for Hyperbolic Curves over Finite Fields). *Let X, Y be proper hyperbolic curves over finite fields k_X, k_Y, respectively. Let $\Sigma = \Sigma_X \subset \mathfrak{Primes}$ be a set of primes which is X-large, meaning that the pro-Σ representation $\rho_\Sigma : G_{k_X} \to \prod_{l \in \Sigma} GL_{2g}(\mathbb{Z}_l)$ arising from the action on the Tate module of the Jacobian of $\overline{X}$ is injective (here g denotes the genus of X). Write Π_X, Π_Y for the geometrically pro-Σ étale fundamental group of X, and the geometrically pro-Σ étale fundamental group of Y, respectively. Let*

$$\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y$$

be an isomorphism of profinite groups. Then α arises from a uniquely determined commutative diagram of schemes:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\sim} & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & Y \end{array}$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups Π_X, Π_Y, respectively.

At the moment of writing this report we do not know if a pro-l version of the above theorems hold, namely if the above Theorems 5 and 6 hold in the case where $\Sigma = \{l\}$ consists of a single prime integer l which is different from the characteristic p. It is very important for the anabelian geometry of hyperbolic curves over finite fields to know whether such a version holds or not.

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