On Siegel Eisenstein series of degree 2 for low weights

Keiichi Gunji (POSTECH)

Introduction

Let Γ be a congruence subgroup of $SL(2,\mathbb{R})$, and $M_k(\Gamma)$ the space of elliptic modular forms of weight k with respect to Γ. For the space $S_k(\Gamma)$ of cusp forms, the Riemann-Roch theorem tells us its dimension if $k \geq 3$. One can also compute $\dim S_2(\Gamma)$, because it equals to the genus of the modular curve $\Gamma\backslash\mathbb{H}_1^*$. However no general methods to compute $\dim S_1(\Gamma)$ are known, even in the case of $\Gamma = \Gamma_0(N)$.

On the other hand for a complement space $E_k(\Gamma)$ of $S_k(\Gamma)$ in $M_k(\Gamma)$, we call it the *space of Eisenstein series*, the dimension formula is known even in the low weight case. We shall explain that. Let $\Gamma = \Gamma(N)$ be the principal congruence subgroup of level N, P the set of upper triangular elements in $SL(2,\mathbb{Z})$. First assume $k \geq 3$. Then the infinite series

$$E^k_{\Gamma(N)}(z) = \sum_{P\cap\Gamma(N)\backslash\Gamma(N)} (cz+d)^{-k}$$

converges absolutely and uniformly on $\mathbb{H}_1$, and $E^k_{\Gamma(N)} \in M_k(\Gamma(N))$. Moreover we can show

$$M_k(\Gamma(N)) = S_k(\Gamma(N)) \oplus \left\langle E^k_{\Gamma(N)}|_k\gamma \;\middle|\; \gamma \in SL(2,\mathbb{Z}) \right\rangle_{\mathbb{C}}. \tag{0.1}$$

As a consequence

$$\dim E_k(\Gamma(N)) = \{\text{number of the cusps}\} = \frac{1}{2}N^2\prod_{p|N}(1-p^{-2}).$$

For the low weight case, E. Hecke considered in [He] the following series:

$$E^k_{\Gamma(N)}(z,s) = \sum_{P\cap\Gamma(N)\backslash\Gamma(N)} (cz+d)^{-k}|cz+d|^{-2s},$$

with $s \in \mathbb{C}$. The right hand side converges when $2\,\mathrm{Re}(s) + k > 2$, and it has a meromorphic continuation for whole s-plane. If $k = 2$, then $E^2_{\Gamma(N)}(z,0)$ is *not* holomorphic in z, however $E^2_{\Gamma(N)}|_2\gamma(z,0) - E^2_{\Gamma(N)}(z,0) \in M_k(\Gamma(N))$ for any $\gamma \in SL(2,\mathbb{Z})$. If $k = 1$ then $E^1_{\Gamma(N)}|_1\gamma(z,0) \in M_1(\Gamma(N))$ for any $\gamma \in SL(2,\mathbb{Z})$, but the functions $\{E^1_{\Gamma(N)}|_1\gamma \mid \gamma \in \Gamma(N)\backslash SL(2,\mathbb{Z})/P\}$ are not linear independent. In both cases we have the same decomposition formula (0.1), and

$$\dim E_k(\Gamma(N)) = \begin{cases} \{\text{number of the cusps}\} - 1 & k = 2; \\ \frac{1}{2}\{\text{number of the cusps}\} & k = 1. \end{cases}$$

In this report, we consider Siegel modular forms of degree 2, and give a dimension formula of the space of Siegel Eisenstein series.

1 Notation and setting

We use the following notation.

- $\mathbb{H}_g = \{Z \in M_g(\mathbb{C}) \mid {}^tZ = Z,\ \mathrm{Im}(Z) > 0\}$.
- $\Gamma^g = Sp(g,\mathbb{Z}) = \{\gamma \in GL(2g,\mathbb{Z}) \mid {}^t\gamma J_g \gamma = J_g\}$, with $J_g = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$.
- For $\gamma \in \Gamma^g$, the (g,g)-matrices $A_\gamma, \dots, D_\gamma$ are defined by $\gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix}$.
- $\Gamma_0^g(N) = \{\gamma \in \Gamma^g \mid C_\gamma \equiv 0 \bmod N\}$, $\Gamma^g(N) = \{\gamma \in \Gamma^g \mid \gamma \equiv 1_{2g} \bmod N\}$.
- The space of Siegel modular forms is defined by

  $$M_k(\Gamma^g(N)) = \{f\colon \mathbb{H}_g \xrightarrow{\text{hol}} \mathbb{C} \mid f|_k\gamma = f,\ \forall\gamma \in \Gamma^g(N)\}$$

  with $f|_k\gamma(Z) = \det(C_\gamma Z + D_\gamma)^{-k} f(\gamma\langle Z\rangle)$, $\gamma\langle Z\rangle = (A_\gamma Z + B_\gamma)(C_\gamma Z + D_\gamma)^{-1}$.
  If $g = 1$ we also requires the holomorphic condition at each cusp.
- Let ψ be a Dirichlet character modulo N.

  $$M_k(\Gamma_0^g(N), \psi) = \{f \in M_k(\Gamma^g(N)) \mid f|_k\gamma = \psi(\det D_\gamma) f,\ \forall\gamma \in \Gamma_0^g(N)\}.$$

We consider the following decomposition of the space of Siegel modular forms:

$$M_k(\Gamma^g(N)) = L_k(\Gamma^g(N)) \oplus E_k(\Gamma^g(N)),$$

with

$$L_k(\Gamma^g) = \{f \in M_k(\Gamma^g(N)) \mid \text{Fourier constant term of } f|_k\gamma \text{ vanishes for all } \gamma \in \Gamma^g\ \},$$

and assume that $E_k(\Gamma^g(N))$ is closed under the action of Γ^g. Such a decomposition is not unique but exists, since the action of Γ^g factors through the finite group $G = Sp(g,\mathbb{Z}/N) \simeq \Gamma^g/\Gamma^g(N)$. We put $E_k(\Gamma_0^g(N),\psi) = M_k(\Gamma_0^g(N),\psi) \cap E_k(\Gamma^g(N))$. The aim of this report is to give the dimensions and generators of the space $E_k(\Gamma^g(N))$.

2 Siegel Eisenstein series

From now on we assume $N \geq 3$. Let $P_0 = \{\gamma \in \Gamma^g \mid C_\gamma = 0\}$, ψ be a Dirichlet character modulo N such that $\psi(-1) = (-1)^k$. The Siegel Eisenstein series are defined by

$$E_{N,\psi}^k(Z,s) = \sum_{\gamma \in P_0\backslash\Gamma_0^g(N)} \psi(\det D_\gamma) \det(C_\gamma Z + D_\gamma)^{-k} |\det(C_\gamma Z + D_\gamma)|^{-2s}.$$

The right hand side converges absolutely and uniformly on $\mathbb{H}_g$ for $2\,\mathrm{Re}(s) + k > g+1$. If $k \geq g+2$, then $E_{N,\psi}^k(Z) := E_{N,\psi}^k(Z,0) \in M_k(\Gamma_0^2(N), \bar{\psi})$. Moreover

$$\begin{aligned} E_{\Gamma^g(N)}^k(Z,s) &= \sum_{\gamma \in P_0 \cap \Gamma_0^g(N)\backslash\Gamma_0^g(N)} \det(C_\gamma Z + D_\gamma)^{-k} |\det(C_\gamma Z + D_\gamma)|^{-2s} \\ &= \frac{2}{\phi(N)} \sum_{\psi(-1)=(-1)^k} E_{N,\psi}^k(Z,s), \end{aligned}$$

here ϕ is Euler's function.

When $k \geq g+2$, it is easy to show that we have a decomposition

$$M_k(\Gamma^g(N)) = L_k(\Gamma^g(N)) \oplus \left\langle E^k_{\Gamma^g(N)}|_k\gamma \;\middle|\; \gamma \in \Gamma^g \right\rangle_{\mathbb{C}},$$

and $\dim E_k(\Gamma^g(N)) = \{$number of the 0-dimensional cusps of $\Gamma^g(N)\backslash\mathbb{H}_g\}$. In particular for an odd prime number p,

$$\dim E_k(\Gamma^2(p)) = \frac{1}{2}(p^4-1), \quad k \geq 4.$$

Now we consider the low weight case. First we know the following fact

Theorem 2.1. *The Siegel Eisenstein series $E^k_{N,\psi}(Z,s)$ has a meromorphic continuation to whole s-plane.*

This theorem is originated by Langlands [La]. There are following questions.

(1) For each $Z \in \mathbb{H}_g$, $E^k_{N,\psi}(Z,s)$ is regular at $s=0$?

(2) $E^k_{N,\psi}(Z,0)$ is holomorphic in Z?

(3) Calculate the dimension of $E_k(\Gamma^g(N))$ (or $E_k(\Gamma^g_0(N),\psi)$).

These questions and answers are given by G. Shimura [Sh2], except for (3). Instead of (3), he considered the algebraicity of the Fourier coefficients. To solve the questions, Shimura considered the Fourier expansion of $E^k_{N,\psi}|_kJ_g(Z,s)$. However to solve the question (3), we have to know the value of Siegel Eisenstein series at each 0-dimensional cusp, especially the Fourier expansion of $E^k_{N,\psi}(Z,s)$.

3 Fourier expansions of Eisenstein series

From now on, let $g=2$ and $N=p$ be an odd prime number. Put $\mathbf{e}(X) = e^{2\pi i\,\mathrm{Tr}(X)}$ for a square matrix X, $A[B] := {}^tBAB$, $\Lambda_{1,2} = \{(q_1,q_2) \in \mathbb{Z}^2/\{\pm 1\} \mid (q_1,q_2)=1\}$, and

$$\mathrm{Sym}^g(\mathbb{Z})^* = \{h \in \mathrm{Sym}^g(\mathbb{Q}) \mid \mathrm{Tr}(hA) \in \mathbb{Z},\ \forall A \in \mathrm{Sym}^g(\mathbb{Z})\}.$$

Then the (1st-version of) Fourier expansion of $E^k_{p,\psi}$ is given by

$$E^k_{p,\psi}(Z,s) = 1 + \sum_{m\in\mathbb{Z}}\sum_{(q_1,q_2)\in\Lambda_{1,2}} S_1(\psi,m,k+2s)\xi_1(Y[\left(\begin{smallmatrix} q_1 \\ q_2 \end{smallmatrix}\right)],m,k+s,s)\,\mathbf{e}(m\begin{pmatrix} q_1^2 & q_1q_2 \\ q_1q_2 & q_2^2 \end{pmatrix}X)$$
$$+ \sum_{h\in\mathrm{Sym}^2(\mathbb{Z})^*} S_2(\psi,h,k+2s)\xi_2(Y,h,k+s,s)\,\mathbf{e}(hX).$$

We explain the notation. First

$$\xi_g(Y,h,\alpha,\beta) = \int_{\mathrm{Sym}^g(\mathbb{R})} \det(X+iY)^{-\alpha}\det(X-iY)^{-\beta}\mathbf{e}(-hX)\,dX,$$

with $\mathrm{Sym}^g(\mathbb{R}) \ni Y > 0$, $\alpha,\beta \in \mathbb{C}$. Here the branches of the complex power $\det(X+iY)^{-\alpha}$, $\det(X-iY)^{-\beta}$ are defined suitably. This function is called *a confluent hypergeometric function* and studied deeply in [Sh1]. We have

Theorem 3.1 (Shimura, [Sh1, (4.34.K), Theorem 4.2]). *For $h \in \mathrm{Sym}^g(Q)^*$ with $\operatorname{sgn} h = (p,q,r)$,*

$$\begin{aligned}\xi_g(Y,h;\alpha,\beta) &= i^{g(\beta-\alpha)} 2^* \pi^* \Gamma_r(\alpha+\beta-\frac{g+1}{2})\Gamma_{g-q}(\alpha)^{-1}\Gamma_{g-p}(\beta)^{-1} \\ &\times \det(Y)^{\frac{g+1}{2}-\alpha-\beta} d_+(hY)^{\alpha-\frac{g+1}{2}+\frac{q}{4}} d_-(hY)^{\beta-\frac{g+1}{2}+\frac{q}{4}}\omega(2\pi Y,h,\alpha,\beta).\end{aligned} \tag{3.1}$$

Here $d_+(x)$ (resp. $d_-(x)$) denotes the products of positive (resp. negative) eigenvalues of x and $\Gamma_m(s) = \pi^{m(m-1)/4}\prod_{k=0}^{m-1}\Gamma(s-k/2)$. Moreover $\omega(2\pi Y,h,\alpha,\beta)$ is an entire function in α and β.

Next we explain $S_g(\psi,h,s)$, which is called the *Siegel series.* For $T \in \mathrm{Sym}^g(\mathbb{Q})$, choose $U, V \in SL(g,\mathbb{Z})$ such that

$$T = U \begin{pmatrix} \nu_1/\delta_1 & & \\ & \ddots & \\ & & \nu_g/\delta_g \end{pmatrix} V, \quad (\nu_i,\delta_i)=1,\ \delta_i>0,$$

and put $\delta(T) = \prod \delta_i,\ \nu(T) = \prod \nu_i$. Then the (generalized) Siegel series are defined by

$$\begin{aligned} S_g(\psi,h,s) &= \sum_{\substack{T\in \mathrm{Sym}^g(\mathbb{Q}) \bmod 1 \\ p\mid\delta_i}} \psi(\nu(T))\delta(T)^{-s}\mathbf{e}(hT) \\ &= \prod_{q:\ \text{primes}} S_g^q(\psi,h,s), \end{aligned}$$

where

$$S_g^q(\psi,h,s) = \begin{cases} \displaystyle\sum_{T\in \mathrm{Sym}^g(\mathbb{Q})_q \bmod 1} \psi(\delta(T))\delta(T)^{-s}\mathbf{e}(hT) & q \neq p; \\ \displaystyle\sum_{\substack{T\in \mathrm{Sym}^g(\mathbb{Q})_p \bmod 1 \\ p\mid\delta_i(T),\forall i}} \psi(\nu(T))\delta(T)^{-s}\mathbf{e}(hT) & q = p, \end{cases}$$

$$\mathrm{Sym}^g(\mathbb{Q})_q = \bigcup_n \frac{1}{q^n}\mathrm{Sym}^g(\mathbb{Z}).$$

If $q \neq p$, $S_g^q(\psi,h,s)$ are already studied by many mathematicians e.g. Kaufhold ($g=2$), Siegel, Kitaoka and finally H. Katsurada ([Kat]) gave an explicit formula for any g.

Theorem 3.2 ([Kau, (2.10), Hirfssatz 10]).

$$\prod_{q\neq p} S_2^q(\psi,h,s) = \begin{cases} \dfrac{L(s-2,\psi)L(2s-3,\psi^2)}{L(s,\psi)L(2s-2,\psi^2)} & h=0; \\[2ex] \dfrac{L(2s-3,\psi^2)}{L(s,\psi)L(2s-2,\psi^2)}\displaystyle\prod_{q\neq p} F_q & \operatorname{rank} h = 1; \\[2ex] \dfrac{L(s-1,\psi\chi_h)}{L(s,\psi)L(2s-2,\psi^2)}\displaystyle\prod_{q\neq p} G_q & \operatorname{rank} h = 2. \end{cases}$$

Here $L(\psi,s)$ is the Dirichlet L-function, χ_h is the quadratic character associated with $\mathbb{Q}(\sqrt{-\det 2h})/\mathbb{Q}$, F_q and G_q are polynomials in q^{-s} such that $F_q = G_q = 1$ for all but finite q.

Remark 3.1. As is already mentioned, in [Sh2] Shimura considered the Fourier expansion of $E^k_{p,\psi}|_k J_2(Z,s)$. It is essentially given by

$$\sum_{h\in \mathrm{Sym}^2(\mathbb{Z})^*} \left(\prod_{q\neq p} S_2^q(\psi,h,k+2s)\right) \xi_2(Y,h,k+s,s)\,\mathbf{e}(hX),$$

which does not contain the term $S_g^p(\psi,h,s)$. This is the main reason why he considered the twisted function.

To consider the Fourier expansion of $E^k_{p,\psi}(Z,s)$, we have to calculate $S_g^p(\psi,h,s)$. Since it is easy if $g=1$, we consider the case $g=2$ in the next section.

4 Euler p-factor for Siegel series

The following lemmas are crucial for our main result.

Lemma 4.1.

$$S_2^p(\psi,0,s) = \begin{cases} 0 & \psi^2 \not\equiv 1; \\ \psi(-1)\,\dfrac{(p-1)p^{1-2s}}{1-p^{3-2s}} & \psi^2\equiv 1, \psi\not\equiv 1; \\ \dfrac{p^{3-2s}(1+p^{1-s})}{(1-p^{2-s})(1-p^{3-2s})} & \psi\equiv 1. \end{cases}$$

Lemma 4.2. *Assume that ψ is a non-trivial character. Then for $h=\begin{pmatrix} t & 0\\ 0 & 0\end{pmatrix}$ with* $\mathrm{ord}_p\, t = m$*,*

$$S_2(\psi,h,s) = \begin{cases} 0 & \psi^2\not\equiv 1; \\ a(p^{-s}) + \dfrac{b(p^{-s})}{1-p^{3-2s}} & \psi^2\equiv 1, \end{cases}$$

with

$$a(p^{-s}) = \psi(-1)\left(\frac{p-1}{p^2} + \sum_{k=1}^{m+1} p^{(3-2s)k}\right),$$
$$b(p^{-s}) = \psi(-1)(p-1)p^{(3-2s)m+4-4s}.$$

Lemma 4.3. *If* $\mathrm{rank}\, h = 2$*, then $S_2^p(\psi,h,s)$ is a polynomial in p^{-s}.*

The polynomial in Lemma 4.3 can be given explicitly, however we omit it. We shall give only the proof of Lemma 4.1, others are proved in a similar way.

Proof of Lemma 4.1. We use the following facts, whose proofs can be found in [Ma, §11, 12].

(i) Let $\mathfrak{M}_g = \{(C,D)\in M_{g,2g}(\mathbb{Z}) \mid C\,{}^tD = D\,{}^tC,\ CX+DY = 1_g,\ \exists X,Y\in M_g(\mathbb{Z})\}$ (the set of *symmetric co-prime pair*) and $\mathfrak{M}_g^r = \{(C,D)\in\mathfrak{M}_g \mid \mathrm{rank}\, C = r\}$. Then the map

$$GL(g,\mathbb{Z})\backslash\mathfrak{M}_g^g \longrightarrow \mathrm{Sym}^g(\mathbb{Q}),\quad (C,D)\longmapsto C^{-1}D$$

is bijective. We have $\delta(C^{-1}D) = |\det C|$ and $\nu(C^{-1}D) = \pm\det D$.

(ii) If (C, D) is symmetric i.e. $C\,{}^tD = D\,{}^tC$, then there exists $(C', D') \in \mathfrak{M}_g$ and $M \in M_g(\mathbb{Z})$ such that $C = MC'$ and $D = MD'$.

Let $\widetilde{\mathfrak{M}}_g(p) = \{(C, D) \in M_{g,2g}(\mathbb{Z}) \mid \det C = p^a,\ C \equiv 0 \bmod p,\ C\,{}^tD = D\,{}^tC\}$. The above facts show that

$$S_2^p(\psi, h, s) = \sum_{\substack{C \\ (C,D)\in SL(2,\mathbb{Z})\backslash\widetilde{\mathfrak{M}}_2(p)}} \sum_{D \bmod C} \psi(\det D)(\det C)^{-s}\mathbf{e}(hC^{-1}D).$$

Indeed if (C, D) is *not* co-prime then $\psi(\det D) = 0$ by (ii). Put $T(k, l) = \begin{pmatrix} p^k & 0 \\ 0 & p^{k+l} \end{pmatrix}$. A representative set of $SL(2, \mathbb{Z})\backslash SL(2, \mathbb{Z})T(k, l)SL(2, \mathbb{Z})$ is given by $T(k, 0)$ if $l = 0$, while it is given by

$$\left\{T(k,l)V \;\middle|\; V = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},\ u \in \mathbb{Z}/p^l\mathbb{Z}\right\} \cup \left\{T(k,l)V \;\middle|\; V = \begin{pmatrix} pu & 1 \\ -1 & 0 \end{pmatrix},\ u \in \mathbb{Z}/p^{l-1}\mathbb{Z}\right\},$$

if $l \geq 1$. For such $C = T(k, l)V$, D runs through the set

$$\left\{\begin{pmatrix} a & b \\ p^l b & d \end{pmatrix} {}^tV^{-1} \;\middle|\; a, b \in \mathbb{Z}/p^k\mathbb{Z},\ d \in \mathbb{Z}/p^{k+l}\mathbb{Z}\right\}.$$

Thus we have

$$S_2^p(\psi, 0, s) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\sum_{a,b,d}\sum_{u} p^{-(2k+l)s}\psi(ad - p^l b^2).$$

We calculate the case $\psi \not\equiv 1$. Then it is easy to see that the summation for $l \geq 1$ vanishes because of the term $\sum_{a,d}\psi(ad)$. Hence we have

$$\begin{aligned} S_2^p(\psi, 0, s) &= \sum_{k=1}^{\infty} p^{-2ks} \sum_{a,b,d\in\mathbb{Z}/p^k} \psi(ad - b^2) \\ &= \sum_{k=1}^{\infty} p^{(3-2s)k-3} \sum_{a,b,d\in\mathbb{Z}/p} \psi(ad - b^2). \end{aligned}$$

For the term $a = 0$ is given by

$$\sum_{k=1}^{\infty} p^{(3-2s)k-3} \sum_{b,d\in\mathbb{Z}/p} \psi(-b^2) = \begin{cases} \psi(-1)\sum_{k=1}^{\infty} p^{(3-2s)k-2}(p-1) & \psi^2 \equiv 1; \\ 0 & \psi^2 \not\equiv 1. \end{cases}$$

On the other hand if $a \neq 0$ then we change the variable $d \mapsto d + a^{-1}b^2$, which becomes

$$\sum_{k=1}^{\infty} p^{(3-2s)k-3} \sum_{a,b,d} \psi(ad) = 0.$$

This proves our lemma. □

Remark 4.1. Summering the above lemmas, Theorem 3.1 and Theorem 3.2 the explicit formula of the Fourier expansion of $E_{p,\psi}^k(Z, s)$ are given in the case $g = 2$. Such a formula for $k \geq 4$ are already given in [Miz] using another method.

As a consequence we can give an another proof of the following theorem.

Theorem 4.4 ([Sh2, Theorem 10.4]). *If* $\psi^2 \not\equiv 1$ *then* $E^2_{p,\psi}(Z,0) \in M_2(\Gamma^2_0(p), \overline{\psi})$*. Moreover the Fourier constant term of* $E^2_{p,\psi}(Z,0)$ *is* 1.

The word "another" means that we can show the above theorem by considering the Fourier expansion of $E^2_{p,\psi}(Z,s)$. We remark that the information of the Fourier constant term of $E^2_{p,\psi}(Z,0)$ is not given in [Sh2].

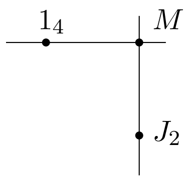
5 The dimension of the space of Eisenstein series of weight 2

An important application of the above is to calculate the dimension of the space of Siegel Eisenstein series for low weights, i.e. to answer the question (3) raised in Section 2. In the case of $k = 1$, the answer is already given in [Gu], thus we consider the case $k = 2$.

Theorem 5.1. *For* $\psi(-1) = 1$,

$$\dim E_2(\Gamma^2_0(p), \psi) = \begin{cases} 1 & \psi \equiv 1; \\ 3 & \psi = (\frac{\cdot}{p}); \\ 2 & \psi^2 \not\equiv 1. \end{cases}$$

Proof. For $f \in M_k(\Gamma^2(p))$, let $C_0(f)$ be the Fourier constant term of f. The structure of the boundary of the Satake compactification of $\Gamma^2_0(p)\backslash\mathbb{H}_2$ is given by the following figure,



where $M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $J_2 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$.

Assume that $\psi^2 \not\equiv 1$. We use the following lemma.

Lemma 5.2 ([Gu, Lemma 3.7]). *Assume that* $\psi^2 \not\equiv 1$*. Then for any* $f \in M_k(\Gamma^2_0(p), \psi)$*,* $C_0(f|_k M) = 0$.

The above figure and the lemma show that $\dim E_2(\Gamma^2_0(p)) \leq 2$. We already know $E^2_{p,\overline{\psi}}(Z) = E^2_{p,\overline{\psi}}(Z,0) \in M_2(\Gamma^2_0(p), \psi)$ by Theorem 4.4. Put

$$F(Z) = \sum_{T \in \mathrm{Sym}^2(\mathbb{F}_p)} E^2_{p,\psi} \bigg|_2 \begin{pmatrix} 0 & 1_2 \\ -1_2 & T \end{pmatrix} (Z),$$

then $F(Z) \in M_2(\Gamma^2_0(p), \psi)$. Since $C_0(E^2_{p,\psi}|_2 J_2)$ is already calculated in [Sh2], we have

$$C_0(E^2_{p,\overline{\psi}}|_2\gamma) = \begin{cases} 1 & \gamma = 1_4; \\ 0 & \gamma = M \text{ or } J_2, \end{cases} \qquad C_0(F|_2\gamma) = \begin{cases} 1 & \gamma = J_2; \\ 0 & \gamma = 1_4 \text{ or } M, \end{cases}$$

which shows that $\dim E_2(\Gamma_0^2(p),\psi)=2$.

Next we consider the case $\psi^2\equiv 1$. In this case $E^2_{p,\psi}(Z,s)$ is *not* regular at $s=0$. However as is shown in [BS, Proposition 5.2 b)], the function

$$\widetilde{E}^2_{p,\psi}(Z,s):=L(2+2s,\psi)L(2+4s,\psi^2)\det(Y)^s E^2_{p,\psi}(Z,s)$$

is regular at $s=-1/2$ and $\widetilde{E}^2_{p,\psi}(Z):=\widetilde{E}^2_{p,\psi}(Z,-1/2)\in M_2(\Gamma_0^2(p),\psi)$. For the trivial ψ, the above figure shows that $\dim E_2(\Gamma_0^2(p))\le 1$ because of the fact $\dim E_2(\Gamma_0^1(p))=1$. Thus we have $\dim E_2(\Gamma_0^2(p))=1$.

Finally we consider the case $\psi=(\frac{\cdot}{p})$. We need to know the value $C_0(\widetilde{E}^2_{p,\psi}|_2\gamma)$ for $\gamma=1_4$, J_2 and M, but it is difficult to calculate the Fourier expansion of $\widetilde{E}^2_{p,\psi}|_2M$, since we cannot show that the "Siegel series" have an Euler product expression there. However the image of Siegel operator $\Phi(\widetilde{E}^2_{p,\psi})$ can be written by elliptic Eisenstein series, thus we can compute the value $C_0(\widetilde{E}^2_{p,\psi}|_2M)=C_0(\Phi(\widetilde{E}^2_{p,\psi})|_2J_1)$. Now the following three functions in $M_2(\Gamma_0^2(p),\psi)$

$$\widetilde{E}^2_{p,\psi},\ F:=\sum_{T\in\mathrm{Sym}^2(\mathbb{F}_p)}\widetilde{E}^2_{p,\psi}|_2\alpha(T),\ G:=\sum_{c_1,d_2\in\mathbb{Z}/p}\widetilde{E}^2_{p,\psi}|_2\gamma(c_1,d_2)+\sum_{d_1\in\mathbb{Z}/p}\widetilde{E}^2_{p,\psi}|_2\delta(d_1),$$

with

$$\alpha(T)=\begin{pmatrix}0&1_2\\-1_2&T\end{pmatrix},\ \gamma(c_1,d_2)=\begin{pmatrix}0&0&0&-1\\-1&0&0&0\\c_1&1&0&d_2\\0&0&-1&c_1\end{pmatrix},\ \delta(d_1)=\begin{pmatrix}0&0&-1&0\\0&1&0&0\\1&0&d_1&0\\0&0&0&1\end{pmatrix},$$

are linearly independent and $\langle\widetilde{E}^2_{p,\psi},F,G\rangle_{\mathbb{C}}\cap L_2(\Gamma_0^2(p),\psi)=\{0\}$. This concludes our theorem. □

Finally we state our main result.

Theorem 5.3. *We have*

$$\dim E_2(\Gamma(p))=\frac{1}{2}(p-3)(p+1)(p^2+1)+\frac{1}{2}p(p^2+1).$$

Outline of the proof of Theorem 5.3. We use the theory of representations of finite groups. Let $G=Sp(2,\mathbb{F}_p)$. Put

$$\overline{P}_0=\left\{\begin{pmatrix}A&B\\0&D\end{pmatrix}\in G\ \middle|\ \det D\in\{\pm1\}\right\},\quad H=\left\{\begin{pmatrix}A&B\\0&D\end{pmatrix}\in G\right\},$$

which are the images of P_0 and $\Gamma_0^2(p)$ respectively under the canonical map $\Gamma^2\to G$. We define the characters u_0 of $\overline{P}_0$ and $\widetilde{\psi}$ of H by

$$u_0(\gamma)=\det D_\gamma\in\{\pm1\},\quad \widetilde{\psi}(\gamma)=\psi(\det D_\gamma)$$

for a Dirichlet character ψ modulo p. By [Gu, Lemma 3.3, 3.4], we can show that the representation of G on $E_k(\Gamma^2(p))$ is isomorphic to a sub-representation of

$$\mathrm{Ind}^G_{\overline{P}_0}(u_0^k)=\bigoplus_{\psi(-1)=(-1)^k}\mathrm{Ind}^G_H(\widetilde{\psi}).$$

The Frobenius reciprocity law says

$$\mathrm{Hom}_G\big(\mathrm{Ind}^G_H(\widetilde{\psi}),E_k(\Gamma^2(p))\big)\simeq\mathrm{Hom}_H\big(\widetilde{\psi},E_k(\Gamma^2(p))\big)\simeq E_k(\Gamma_0^2(p),\overline{\psi}),\tag{5.1}$$

thus $\dim E_2(\Gamma_0^2(p),\psi)$, which is given in Theorem 5.1, equals to the number of irreducible representations of G in $E_2(\Gamma^2(p))$. All the irreducible characters of G are given in [Sr]. Actually we have the following decomposition. Fix a generator ξ of $\mathbb{F}_p^\times$ and define the Dirichlet character ψ_l modulo p by $\psi_l(\xi^x) = \mathbf{e}(xl/(p-1))$. Note that $\psi_l(-1) = 1$ if and only if l is even. Then

$$\mathrm{Ind}_H^G(\widetilde{\psi_l}) = \begin{cases} 1_G \oplus \underbrace{\theta_9}_{\frac{1}{2}p(p+1)^2} \oplus \underbrace{\theta_{11}}_{\frac{1}{2}p(p^2+1)} & l = 0; \\ \underbrace{\theta_3}_{\frac{1}{2}(p^2+1)} \oplus \underbrace{\theta_4}_{\frac{1}{2}(p^2+1)} \oplus \underbrace{\Phi_9}_{p(p^2+1)} & l = (p-1)/2; \\ \underbrace{\chi_8(|l|)}_{(p+1)(p^2+1)} & -(p-3)/2 \le l \le (p-3)/2,\ l \neq 0. \end{cases}$$

By Theorem 5.1 and (5.1), we have $\mathrm{Ind}_H^G(\psi) \subset E_2(\Gamma^2(p))$ for $\psi \not\equiv 1$. If $\psi \equiv 1$, we can show that $\{\widetilde{E}^2_{p,1}|_2\gamma(Z) \mid \gamma \in \Gamma^2\}$ generates a $\frac{1}{2}p(p^2+1)$-dimensional space in $M_2(\Gamma^2(p))/L_2(\Gamma^2(p))$. This complete the theorem. □

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