Siegel modular forms modulo p

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Abstract

Using moduli theory of abelian varieties and a recent result of Böcherer-Nagaoka on liftings of the generalized Hasse invariant, we study Siegel modular forms over a field of characteristic $p > 0$, and over a ring in which p is nilpotent more generally. We describe the ring structure of Siegel full modular forms of degree 2, and show the congruence property on Siegel modular forms of any level.

1 Introduction

Let k be a perfect field of characteristic $p > 0$, denote by $W(k)$ the ring of Witt vectors over k, and put $W_m(k) = W(k)/(p^m)$. The aim of this paper is to study the ring structures, relations via the reduction map, and the congruence property of Siegel modular forms (denoted by SMFs for short) over these rings. More precisely, we consider the commutative diagram:

$$\begin{array}{ccc} \{\text{SMFs over } W(k)\} & \longrightarrow & \{\text{power series over } W(k)\} \\ \downarrow & & \downarrow \\ \{\text{SMFs over } W_m(k)\} & \longrightarrow & \{\text{power series over } W_m(k)\} \\ \downarrow & & \downarrow \\ \{\text{SMFs over } k\} & \longrightarrow & \{\text{power series over } k\}, \end{array}$$

where the rightarrows denote the Fourier expansion maps, and the downarrows denote the reduction maps, and we study

Problem

- What is the ring structure of Siegel modular forms over $W_m(k)$?
- When is the reduction map $\{\text{SMFs over } W(k)\} \to \{\text{SMFs over } W_m(k)\}$ surjective ?
- What is the structure of the ideal given by the kernel of the Fourier expansion map over $W_m(k)$? (Note that the map is injective over $W(k)$.)

In this paper, using moduli theory of abelian varieties mainly constructed by Mumford [FKM], Faltings-Chai [FC], and a result of Böcherer-Nagaoka [BN] on liftings of the generalized Hasse invariant h_{p-1}, we prove the following results:

1. If $p > 3$, then the ring of Siegel full modular forms over $W_m(k)$ (more generally, over any $\mathbb{Z}[1/6]$-algebra) of degree 2 is described, and the reduction map is surjective (cf. Theorem 1).
2. Assume that $p > g + 2$. Then for Siegel modular forms (not congruent to 0 modulo p) over $W_m(k)$ of degree g having the same Fourier expansion, their weights are congruent modulo $(p-1)p^{m-1}$, and their differences belong to the ideal generated by

$$1 - (\text{a power of } \theta_{p-1} \bmod (p^m)),$$

where θ_{p-1} denotes Böcherer-Nagaoka's lift of h_{p-1} (cf. Theorem 2).

We will briefly mention the proof of these results. The first result gives a generalization of Igusa's result [Ig1,2] in the complex coefficients case, and it follows from algebraic and moduli theoretic interpretations of results of Freitag [F], Hammond [H], Nagaoka [N] and Ibukiyama [I]. The second result is a generalization of the congruence property on elliptic modular forms shown by Swinnerton-Dyer [Sw] and Serre [S], and it follows from Katz' argument [K] by using the irreducibility of the Igusa tower proved by Faltings-Chai [FC].

2 Moduli and modular forms

We review results of Mumford [FKM] and of Faltings-Chai [FC] on moduli and modular forms. For positive integers g, n, let ζ_n be a primitive n-th root of 1, and let $M_{g,n}$ be the moduli stack (which becomes the fine moduli scheme when $n \geq 3$) over $\mathbb{Z}[1/n, \zeta_n]$ of principally polarized abelian schemes of relative dimension g with symplectic level n structure. Then $M_{g,n}(\mathbb{C})$ is a complex orbifold of dimension $g(g+1)/2$, and it is represented as the quotient space $\mathcal{H}_g/\Gamma_{g,n}$ of the Siegel upper half space $\mathcal{H}_g$ by the integral symplectic group $\Gamma_{g,n} = \mathrm{Ker}\left(Sp_g(\mathbb{Z}) \to Sp_g(\mathbb{Z}/n\mathbb{Z})\right)$ of degree g and level n. Then there exists a universal abelian scheme $\mathcal{A}$ with 0-section s over $M_{g,n}$, and the *Hodge line bundle* λ is defined by $\det\left(s^*\left(\Omega_{\mathcal{A}/M_g}\right)\right)$ which corresponds to the automorphic factor over $M_{g,n}(\mathbb{C})$. Faltings-Chai [FC] constructed a smooth compactification $\overline{M}_{g,n}$ of $M_{g,n}$ associated with a good cone decomposition of positive semi-definite symmetric bilinear forms on $\mathbb{R}^g$, and a semi-abelian scheme $\mathcal{G}$ with 0-section s over $\overline{M}_{g,n}$ extending $\mathcal{A} \to M_g$. Then $\overline{\lambda} \overset{\mathrm{def}}{=} \det\left(s^*\left(\Omega_{\mathcal{G}/\overline{M}_g}\right)\right)$ gives an extension of λ to $\overline{M}_{g,n}$, and

$$M_{g,n}^* \overset{\mathrm{def}}{=} \mathrm{Proj}\left(\bigoplus_{h \geq 0} H^0\left(\overline{M}_{g,n}, \overline{\lambda}^{\otimes h}\right)\right)$$

is a projective scheme over $\mathbb{Z}[\zeta_n, 1/n]$ called *Satake's minimal compactification.*

For any $\mathbb{Z}[1/n, \zeta_n]$-algebra R, we define the R-module $S_{g,h,n}(R)$ of *Siegel modular forms over R of degree g, weight h and level n* by

$$S_{g,h,n}(R) = H^0\left(\overline{M}_{g,n}, \overline{\lambda}^{\otimes h} \otimes_{\mathbb{Z}[1/n,\zeta_n]} R\right),$$

and the (graded) *ring $S_{g,n}^*(R)$ of Siegel modular forms of degree g and level n* by

$$S_{g,n}^*(R) = \bigoplus_{h \geq 0} S_{g,h,n}(R).$$

Then by Koecher's principle, $S_{g,h,n}(R) = H^0\left(M_{g,n}, \lambda^{\otimes h} \otimes R\right)$ if $g > 1$, and by Serre's GAGA, $S_{g,h,n}(\mathbb{C})$ becomes the space of holomorphic functions on $\mathcal{H}_g$ with the automorphy condition of weight h for $\Gamma_{g,n}$ (and the cusp condition if $g = 1$). When $n = 1$, deleting n in the above notation we put

$$\begin{cases} M_g = M_{g,1}, \ \overline{M}_g = \overline{M}_{g,1}, \ M_g^* = M_{g,1}^*, \\ S_{g,h}(R) = S_{g,h,1}(R), \ S_g^*(R) = S_{g,1}^*(R), \end{cases}$$

and call elements of $S_{g,h}(R)$ Siegel *full* modular forms.

By evaluating Siegel modular forms on Mumford's semi-abelian scheme:

$$\mathbb{G}_m^g / \langle (q_{ij} = q_{ji})_{1 \leq i \leq g} \mid 1 \leq j \leq g \rangle$$

with a polarization and a symplectic level n structure over

$$A_{g,n} = \mathbb{Z}\left[1/n, \zeta_n, q_{ij}^{\pm 1/n}(i \neq j)\right]\left[\left[q_{11}^{1/n}, ..., q_{gg}^{1/n}\right]\right]$$

given in [Mu1], for each 0-dimensional cusp, we can associate an R-linear ring homomorphism

$$F_R : S_{g,n}^*(R) \longrightarrow A_{g,n} \otimes_{\mathbb{Z}[1/n,\zeta_n]} R$$

which is called the *Fourier (q-)expansion map* and satisfies the following (cf. [FC, Chapter V]):

- F_R is functorial for R,
- $F_{\mathbb{C}}$ becomes the classical Fourier expansion,
- F_R is injective on each $S_{g,h,n}(R)$, and further for $f \in S_{g,h,n}(R)$ and a sub $\mathbb{Z}[1/n, \zeta_n]$-algebra R' of R,

$$F_R(f) \in A_{g,n} \otimes_{\mathbb{Z}[1/n,\zeta_n]} R' \iff f \in S_{g,h,n}(R')$$

  which is called the *q-expansion principle.*

Remark 1. Nagaoka [N] called the elements of

$$F_{\mathbb{C}}\left(S_g^*(\mathbb{Z}_p \cap \mathbb{Q})\right) \bmod(p) \ \subset\ F_{\mathbb{F}_p}\left(S_g^*(\mathbb{F}_p)\right) \ \subset\ A_g \otimes \mathbb{F}_p$$

mod p Siegel modular forms of degree g which are power series of q_{ij} over $\mathbb{F}_p$, and make a subring of the image by $F_{\mathbb{F}_p}$ of the ring of our Siegel full modular forms over $\mathbb{F}_p$.

Remark 2. The tautological bundle λ^* is ample on $M_{g,n}^*$, and hence sufficiently large integers h satisfy the vanishing condition:

$$H^1\left(M_{g,n}^*, (\lambda^*)^{\otimes h} \otimes_{\mathbb{Z}[1/n,\zeta_n]} W(k)\right) = \{0\}.$$

Then by the exact sequence

$$H^0(M_{g,n}^*, (\lambda^*)^{\otimes h} \otimes W(k))/p \ \rightarrow\ H^0(M_{g,n}^*, (\lambda^*)^{\otimes h} \otimes k) \ \rightarrow\ H^1(M_{g,n}^*, (\lambda^*)^{\otimes h} \otimes W(k)),$$

one can see that the reduction map $S_{g,h,n}(W(k)) \rightarrow S_{g,h,n}(k)$ is surjective if $h \gg 0$ and p is prime to n. However, the explicit lower bound of h satisfying the vanishing condition seems not known.

3 Modular forms of degree 2

In this section, we assume that $g = 2$ and consider Siegel full modular forms over a $\mathbb{Z}[1/6]$-algebra R of degree 2. By Igusa's results [Ig1,2], there are Eisenstein series $E_4 \in S_{2,4}(\mathbb{Z})$, $E_6 \in S_{2,6}(\mathbb{Z})$, and cusp forms $\chi_{10} \in S_{2,10}(\mathbb{Z})$, $\chi_{12} \in S_{2,12}(\mathbb{Z})$, $\chi_{35} \in S_{2,35}(\mathbb{Z})$ which are all normalized and hence *primitive*, i.e., not congruent to 0 modulo any prime. Denote by the same symbol the elements of $S_2^*(R)$ obtained from the above modular forms naturally, i.e., given by the image of these forms tensored with the unit of R by the natural R-linear homomorphism $S_2^*(\mathbb{Z}) \otimes R \rightarrow S_2^*(R)$.

Theorem 1. *Let R be as above. Then the even part*

$$S_2^{\mathrm{even}}(R) \stackrel{\mathrm{def}}{=} \bigoplus_{h\geq 0} S_{2,2h}(R)$$

of $S_2^*(R)$ is generated by the four modular forms E_4, E_6, χ_{10}, χ_{12}, and

$$S_2^*(R) = S_2^{\mathrm{even}}(R) \oplus \chi_{35} \cdot S_2^{\mathrm{even}}(R).$$

In particular, if $p > 3$, then the reduction map $S_2^*(\mathbb{Z}_p) \to S_2^*(\mathbb{F}_p)$ modulo p is surjective.

Proof. First, using a moduli theoretic interpretation of results of Igusa [Ig1], Freitag [F], Hammond [H] and Nagaoka [N], we will prove that

$$S_2^{\mathrm{even}}(R) = R\left[E_4, E_6, \chi_{10}, \chi_{12}\right].$$

Let $\pi : \mathcal{C} \to X_2$ be the universal curve over the moduli stack of stable curves of genus 2, and let $\omega_{\mathcal{C}/X_2}$ be its relative dualizing sheaf (cf. [DM]). Then the map

$$S^2\left(\pi_*\left(\omega_{\mathcal{C}/X_2}\right)\right) \ni s \otimes s' \mapsto s \cdot s' \in \pi_*\left(\omega_{\mathcal{C}/X_2}^{\otimes 2}\right)$$

($S^2(*)$ denotes the symmetric tensor product of $*$ with itself) obtained by taking products of local sections, and Mumford's isomorphism (cf. [Mu2], Theorem 5.10) give

$$\begin{aligned} \det\left(\pi_*\left(\omega_{\mathcal{C}/X_2}\right)\right)^{\otimes 3} &\cong \det\left(\pi_*\left(\omega_{\mathcal{C}/X_2}^{\otimes 2}\right)\right) \otimes \mathcal{O}_{X_2}\left(\Delta_1\right)^{\otimes(-1)} \\ &\cong \det\left(\pi_*\left(\omega_{\mathcal{C}/X_2}\right)\right)^{\otimes 13} \otimes \mathcal{O}_{X_2}\left(\Delta_0 + 2\Delta_1\right)^{\otimes(-1)}, \end{aligned}$$

where Δ_0 denotes the locus in X_2 consisting of self-intersecting curves of genus 2, and Δ_1 denotes that of unions of two curves of genus 1. Then the isomorphism gives

$$\mathcal{O}_{X_2} \stackrel{\sim}{\to} \det\left(\pi_*\left(\omega_{\mathcal{C}/X_2}\right)\right)^{\otimes 10} \otimes \mathcal{O}_{X_2}\left(\Delta_0 + 2\Delta_1\right)^{\otimes(-1)},$$

and the image of $1 \in \mathcal{O}_{X_2}$ becomes $\chi_{10} \in S_{2,10}(\mathbb{Z})$ up to sign by the uniqueness of the Siegel cusp form of degree 2 and weight 10. Therefore, the divisor $\mathrm{div}(\chi_{10})$ of χ_{10} is $\Delta_0 + 2\Delta_1$. Since $\overline{M}_2$ parametrizes principally polarized semi-abelian schemes with proper general fiber of relative dimension 2, there is a morphism $X_2 \to \overline{M}_2$ sending each stable curve of genus 2 to its generalized Jacobian variety with canonical polarization. Then combining the pullback by this morphism and the restriction to Δ_1, we have an R-linear homomorphism W_R : $S_2^*(R) \to S^2\left(S_1^*(R)\right)$ called the *Witt operator* which satisfies that

- if $f \in S_{2,h}(R)$, then $W(f)$ is a sum of symmetric products of $g_i, h_i \in S_{1,h}(R)$, and its Fourier expansion $F_R\left(W(f)\right)$ is $F_R(f)|_{q_{12}=q_{21}=1}$.
- $W(E_i) = E_i^{(1)}(q_{11}) \cdot E_i^{(1)}(q_{22})$ $(i = 4, 6)$, and $W(\chi_{12}) = 12\Delta(q_{11}) \cdot \Delta(q_{22})$, where $E_i^{(1)}$ and Δ are the normalized elliptic Eisenstein series of weight i and cusp form of weight 12 respectively (cf. [N, p.414]).

By the definition of W, for any $f \in \mathrm{Ker}\left(W_R\right)$ of even weight, its Fourier expansion $F_R(f)$ is divided by $q_{11} \cdot q_{22}$ and by $(q_{12} - 1)^2 = (q_{21} - 1)^2$ because it is invariant under the transform $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \mapsto \begin{pmatrix} q_{22} & q_{21}^{-1} \\ q_{12}^{-1} & q_{11} \end{pmatrix}$ which is induced from $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in Sp_2(\mathbb{Z})$.

Therefore, $\mathrm{div}(f) \geq \Delta_0 + 2\Delta_1$, and hence f is divided by $\chi_{10} \in S_{2,10}(R)$. On the other hand, the moduli stack $M_1 \otimes \mathbb{Z}[1/6]$ of elliptic curves over $\mathbb{Z}[1/6]$-schemes is the quotient of the affine line $\mathbb{A}^1_{\mathbb{Z}[1/6]}$ such that $\mathbb{A}^1_{\mathbb{Z}[1/6]} \to M_1 \otimes \mathbb{Z}[1/6]$ is a double cover and cyclic of order $4, 6$ at the unique zeros of $E_6^{(1)}, E_4^{(1)}$ respectively. Hence the ring $S_1^*(R)$ of elliptic modular forms over R is generated by $E_4^{(1)}$ and $E_6^{(1)}$. Therefore, by Nagaoka's argument in the proof of [N, Theorem 4.3],

$$S_2^{\mathrm{even}}(R) = R\left[E_4, E_6, \chi_{10}, \chi_{12}\right].$$

Second, following Ibukiyama's proof [I] of Igusa's result [Ig1], we will prove that any Siegel full modular form over R of degree 2 and odd weight is divided by χ_{35}, and hence the quotient belongs to $S_2^{\mathrm{even}}(R)$. Following [Ig1, II, p.398], we identify $Sp_2(\mathbb{Z}/2\mathbb{Z})$ with the symmetric group S_6 of degree 6 via the action on odd characteristics, and denote by $\Gamma_{2,e}$ the subgroup of $\Gamma_2 = Sp_2(\mathbb{Z})$ of index 2 which is defined as the inverse image of the alternating subgroup A_6 of S_6 by the surjective homomorphism $\Gamma_2 \to Sp_2(\mathbb{Z}/2\mathbb{Z}) \cong S_6$. Let $\Gamma_{1,e}$ be the subgroup of $\Gamma_1 = SL_2(\mathbb{Z})$ of index 2 which is the inverse image of A_3 by $\Gamma_1 \to SL_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3$. Then we have:

(1) $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ belongs to $\Gamma_2 - \Gamma_{2,e}$, and it maps $\begin{pmatrix} q_{11} & 1 \\ 1 & q_{22} \end{pmatrix}$ to $\begin{pmatrix} q_{22} & 1 \\ 1 & q_{11} \end{pmatrix}$,

(2) $\psi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\psi_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}$ give homomorphisms $\Gamma_1 \to \Gamma_2$ satisfying that $\alpha \in \Gamma_{1,e} \Leftrightarrow \psi_i(\alpha) \in \Gamma_{2,e}$ for $\alpha \in \Gamma_1$ and $i = 1, 2$.

Further, for $g = 1, 2$, $\mathcal{H}_g/\Gamma_{g,e}$ has a natural model over $\mathbb{Z}[1/2]$ which we denote by $M_{g,e}$, and the normalization $\pi : \overline{M}_{g,e} \to \overline{M}_g$ of $M_{g,e}$ is ramified only along the divisor lying above the (unique) 0-dimensional cusp on M_g^*.

For $g = 1, 2$, let $S_{g,h}^-(*)$ denote the space of Siegel modular forms over a $\mathbb{Z}[1/2]$-algebra of degree g and weight h with odd character for $\Gamma_g \to \Gamma_g/\Gamma_{g,e} \cong \{\pm 1\}$. Then it is known that a square root of $\chi_{10} \in S_{2,10}(\mathbb{Z})$ is given as the product θ_5 of the even theta constants and of 2^{-6} which is a primitive cusp form and belongs to $S_{2,5}^-(\mathbb{Z}[1/2])$. Denote by the same symbol the Siegel cusp form over R obtained from θ_5 naturally. Let $f \in S_{2,h}(R)$ have an odd weight h. Then by [Ig2, Lemma 8], $\mathrm{div}(f) \geq 2\Delta_0 + \Delta_1$, hence f is divided by θ_5, and further the quotient $f/\theta_5 \in S_{2,h-5}^-(R)$ has zero on $\pi^{-1}(\Delta_0) \subset \overline{M}_{2,e}$ of order ≥ 3. The Witt operator W can be also defined on $S_{2,*}^-(R)$ as above, and hence by (1) and (2), $W(f/\theta_5) = F_R(W(f/\theta_5))$ is represented as

$$\sum_i \left(g_i(q_{11}) \cdot h_i(q_{22}) - h_i(q_{11}) \cdot g_i(q_{22})\right),$$

where $g_i, h_i \in S_{2,h-5}^-(R)$ has zero of order ≥ 3 at the cusp $\pi^{-1}\left(\overline{M}_1 - M_1\right)$ of $\overline{M}_{1,e}$. Since $\sqrt{\Delta} \in S_{1,6}^-(\mathbb{Z}[1/2])$ has only zero of order 1 at the cusp of $\overline{M}_{1,e}$, $g_i/\sqrt{\Delta}, h_i/\sqrt{\Delta} \in S_{1,h-11}(R)$ are cusp forms, and (we may assume that) are linearly independent by the above representation. Therefore, by the linearlity of W, we may put

$$(g_i,\ h_i) = \left(\Delta^{3/2+a} \cdot \left(E_4^{(1)}\right)^b \cdot \left(E_6^{(1)}\right)^c,\ \Delta^{3/2} \cdot \left(E_4^{(1)}\right)^{3a+b} \cdot \left(E_6^{(1)}\right)^c\right)$$

for some integers $a, b \geq 0$ and $c \in \{0,1\}$. Since

$$W\left(\chi_{35}/\theta_5\right)\Big|_{\sqrt{q_{12}}=\sqrt{q_{21}}=1}$$
$$= 2\Delta(q_{11})^{3/2}\cdot\Delta(q_{22})^{3/2}\cdot\left(\Delta(q_{11})\cdot E_4^{(1)}(q_{22})^3 - E_4^{(1)}(q_{11})^3\cdot\Delta(q_{22})\right),$$

and

$$\frac{\Delta(q_{11})^a\cdot E_4^{(1)}(q_{22})^{3a} - E_4^{(1)}(q_{11})^{3a}\cdot\Delta(q_{22})^a}{\Delta(q_{11})\cdot E_4^{(1)}(q_{22})^3 - E_4^{(1)}(q_{11})^3\cdot\Delta(q_{22})}$$
$$= \sum_{i+j=a-1}\left(\Delta(q_{11})\cdot E_4^{(1)}(q_{22})^3\right)^i\cdot\left(E_4^{(1)}(q_{11})^3\cdot\Delta(q_{22})\right)^j$$

which belongs to $W\left(S_2^{\mathrm{even}}(R)\right)$ as is shown in the proof of [N, Theorem 4.3], there is an element f' of $S_{2,h-5}^{-}(R)$ divided by χ_{35}/θ_5 such that $W(f') = W(f/\theta_5)$. Therefore, $f/\theta_5 - f'$ belongs to $\mathrm{Ker}(W)$, hence is divided by θ_5, and further $\left(f/\theta_5 - f'\right)/\theta_5$ belongs to $S_{2,h-10}(R)$. Repeating this process one can see that f is divided by $\theta_5\cdot(\chi_{35}/\theta_5) = \chi_{35}$. This completes the proof. $\square$

4 Congruence between modular forms

In this section, we prove a generalization for Siegel modular forms of the congruence property on elliptic modular forms (cf. [K], [S], [Sw]).

First, we review the Siegel full modular form h_{p-1} over $\mathbb{F}_p$ of degree g and weight $p-1$ given in Deligne's letter [D] to Nagaoka at 1977, 1979 (see also [N]) as a generalized Hasse's invariant. This is a unique (by the q-expansion principle) element of $S_{g,p-1}(\mathbb{F}_p)$ such that $F_{\mathbb{F}_p}(h_{p-1}) = 1$ and is obtained as the image of 1 by the homomorphism $\mathcal{O}_{M_g\otimes\mathbb{F}_p} \to \lambda^{\otimes(p-1)}$ which comes from the bundle map $\lambda \to \lambda^{(p)} = \lambda^{\otimes p}$ associated with the Verschiebung. Hence the divisor of h_{p-1} is the locus consisting of principally polarized non-ordinary abelian varieties in $M_g \otimes \mathbb{F}_p$. Recently, Böcherer-Nagaoka [BN] proved that if $p > g+2$, then there exists a Siegel full modular form θ_{p-1} over $\mathbb{Z}_p \cap \mathbb{Q}$ of degree g and weight $p-1$ constructed as a theta series which satisfies that $F_{\mathbb{Z}_p}\left(\theta_{p-1}\right)$ is congruent to 1 modulo p, and hence that h_{p-1} is the reduction of θ_{p-1} modulo p by the q-expansion principle. Using this Siegel modular form, we have:

Theorem 2. *Let k be a perfect field of characteristic $p > g+2$, and let n be a positive integer prime to p such that k contains a primitive n-th root of 1. Assume that two Siegel modular forms f_i $(i=1,2)$ over $W_m(k)$ of degree g and level n having the same Fourier expansion not congruent to 0 modulo p at (at least) one 0-dimensional cusp. Then the weights h_i of f_i $(i=1,2)$ satisfy the congruence $h_1 \equiv h_2$ modulo $(p-1)p^{m-1}$, and*

$$f_i \;=\; f_j\cdot\left(\theta_{p-1}\ \mathrm{mod}(p^m)\right)^{(h_i-h_j)/(p-1)} \in S_{g,n}^{*}\left(W_m(k)\right)$$

if $h_i \geq h_j$. When $m=1$, for any prime p, the same statement holds by replacing θ_{p-1} $\mathrm{mod}\,(p^m)$ with h_{p-1}.

Proof. We extend Katz' proof [K, 4.4] on the congruence of elliptic modular forms. We may assume that k is algebraically closed and that $n \geq 3$. Then all the geometric fibers of $M = M_{g,n}$ and $\overline{M} = \overline{M}_{g,n}$ over $\mathbb{Z}\left[1/n, \zeta_n\right]$ are irreducible. Put $\overline{M}_m = \overline{M} \otimes_{\mathbb{Z}[1/n,\zeta_n]} W_m(k)$, and let $\overline{S}_m$ be the open subscheme of $\overline{M}_m$ on which θ_{p-1} is invertible, where θ_{p-1} is regarded as an element of $H^0\left(\overline{M}_m, \overline{\lambda}^{\otimes(p-1)}\right)$. Then over $\overline{S}_m$, the maximal etale quotient $\mathcal{G}[p^m]^{\mathrm{et}}$ of $\mathcal{G}[p^m] = \mathrm{Ker}\,(p^m : \mathcal{G} \to \mathcal{G})$ is isomorphic to $(\mathbb{Z}/p^m\mathbb{Z})^g$, and hence there is a covering $\overline{T}_m$ of $\overline{S}_m$ called the *Igusa tower* which represents

$$\mathrm{Isom}\left((\mathbb{Z}/p^m\mathbb{Z})^g, \mathcal{G}[p^m]^{\mathrm{et}}\right).$$

Then by [FC, Chapter V, Proposition 7.1] (see also [Hi, Theorem 6.27]), $\overline{T}_m$ is a Galois cover of $\overline{S}_m$ with covering group $GL_g(\mathbb{Z}/p^m\mathbb{Z})$. Further, one can see that by the Cartier duality, each isomorphism $(\mathbb{Z}/p^m\mathbb{Z})^g_{/\overline{T}_m} \xrightarrow{\sim} \mathcal{G}[p^m]^{\mathrm{et}}_{/\overline{T}_m}$ gives

$$\iota : \mathcal{G}[p^m]^{\circ}_{/\overline{T}_m} \xrightarrow{\sim} (\boldsymbol{\mu}_{p^m})^g_{/\overline{T}_m} \hookrightarrow \mathbb{G}^g_{m/\overline{T}_m},$$

where $\mathcal{G}[p^m]^{\circ}$ denotes the connected component of $\mathcal{G}[p^m]$ containing 1 and

$$\mathbb{G}_m^g = \mathrm{Spec}\left(\mathbb{Z}\left[X_1^{\pm 1}, ..., X_g^{\pm 1}\right]\right)$$

denotes the g-dimensional split torus, and that $\iota^*(dX_i/X_i)$ $(i = 1, ..., g)$ are uniquely extended to a basis of 1-forms on $\mathcal{G}$. Therefore, there is an isomorphism

$$\mathcal{G}[p^m]^{\mathrm{et}} \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{\overline{T}_m} \cong s^*\left(\Omega_{\mathcal{G}/\overline{T}_m}\right)$$

which is compatible with the action of $\mathrm{Aut}\left(\overline{T}_m/\overline{S}_m\right)$, and hence we have

$$\det\left(\mathcal{G}[p^m]^{\mathrm{et}}\right) \otimes_{\mathbb{Z}/p^m\mathbb{Z}} \mathcal{O}_{\overline{S}_m} \cong \overline{\lambda}_{\overline{S}_m}.$$

For an k-algebra R and a semi-abelian scheme A over $W_m(R)$ corresponding to a morphism $\mathrm{Spec}\,(W_m(R)) \to \overline{S}_m$, $A[p]^{\circ} \otimes R$ is the kernel of the Frobenius map $A \otimes R \to (A \otimes R)^{(p)}$, and $A/A[p]^{\circ} \cong A \otimes_{W_m(R),\sigma} W_m(R)$, where $\sigma((a_1, ..., a_m)) = (a_1^p, ..., a_m^p)$ $(a_i \in R)$. Hence the correspondence $\mathcal{G} \mapsto \mathcal{G}/\mathcal{G}[p]^{\circ}$ gives rise to a morphism $\varphi : \overline{S}_m \to \overline{S}_m \otimes_{W_m(k),\sigma} W_m(k)$, and $\det\left(\mathcal{G}[p^m]^{\mathrm{et}}\right)$ is the invariant subsheaf of $\overline{\lambda}_{\overline{S}_m}$ under φ.

By the assumption, there is a non-empty open subscheme U of $\overline{S}_m$ on which f_1, f_2 are invertible, and the Fourier expansion (at the cusp considered in the theorem) of the ratio f_1/f_2 is 1. Therefore, f_1/f_2 on U is invariant under φ, and hence the $(h_1 - h_2)$-th tensor power of the representation of $\pi_1(U)$ on $\det\left(\mathcal{G}[p^m]^{\mathrm{et}}\right)$ is trivial. On the other hand, by the above result in [FC], the image of $\pi_1(U) \to \mathrm{Aut}\left(\det\left(\mathcal{G}[p^m]^{\mathrm{et}}\right)\right) \cong (\mathbb{Z}/p^m\mathbb{Z})^{\times}$ is surjective. Therefore, $h_1 - h_2 \equiv 0$ modulo $(p-1)p^{m-1}$, and hence by the q-expansion principle,

$$f_i \;=\; f_j \cdot (\theta_{p-1} \bmod (p^m))^{(h_i - h_j)/(p-1)}$$

if $h_i \geq h_j$.

Assume that $m = 1$. Then the above proof works changing θ_{p-1} modulo p by h_{p-1} which exists for any p. This completes the proof. □

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