$\mathrm{SL}_2(\mathbb{Z})$ 上の Maass 波動形式の閉測地線に沿った周期積分に付随するスペクトル型ゼータ函数について

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1 Introduction

Let $\mathfrak{H} = \{\tau = x + iy \in \mathbb{C} \,|\, y > 0\}$ be the upper half plane. Then the special linear group $G = \mathrm{SL}_2(\mathbb{R})$ acts on $\mathfrak{H}$ by the fractional linear transformation:

$$g\langle\tau\rangle \overset{\mathrm{def}}{=} \tfrac{a\tau+b}{c\tau+d}\,, \quad g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G,\, \tau \in \mathfrak{H}$$

preserving the Poincaré metric $\mathrm{d}s^2 = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2)$ of $\mathfrak{H}$. Let $\Gamma \subset G$ be a Fuchsian group of the first kind, i.e., a discrete subgroup of G which admits a measurable fundamental domain $\mathcal{F}_\Gamma$ in G with the finite invariant volume. Define the Hilbert space $L^2(\Gamma\backslash\mathfrak{H})$ to be the space of all the measurable functions $\phi : \mathfrak{H} \to \mathbb{C}$ with the Γ-invariance

$$\phi(\gamma\langle\tau\rangle) = \phi(\tau), \quad (\forall\gamma \in \Gamma) \quad \text{such that} \quad \int_{\mathcal{F}_\Gamma} |\phi(\tau)|^2 \,\tfrac{\mathrm{d}x\mathrm{d}y}{y^2} < +\infty.$$

We consider the hyperbolic Laplacian $\triangle = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ acting on the space $C^\infty(\Gamma\backslash\mathfrak{H})$ consisting of all the Γ-invariant C^∞-functions on $\mathfrak{H}$.

Definition 1. A complex valued function $\phi(\tau)$ on $\mathfrak{H}$ is said to be an L^2-Maass form on Γ of eigenvalue λ if $\phi(\tau)$ is a C^2-function which is square integrable on $\mathcal{F}_\Gamma$ and satisfies the eigenequation of the Laplacian:

$$\triangle\,\phi(\tau) = \lambda\,\phi(\tau).$$

For a given $\lambda \in \mathbb{C}$, set

$$\mathcal{A}(\Gamma;\lambda) \overset{\mathrm{def}}{=} \{\, L^2\text{-Maass forms } \phi(\tau) \text{ on } \Gamma \text{ of eigenvalue } \lambda\,\}.$$

Set $\Lambda_\Gamma \overset{\mathrm{def}}{=} \{\lambda \in \mathbb{C} \,|\, \mathcal{A}(\Gamma;\lambda) \neq \{0\}\,\}$.

As the initial domain of the Laplacian $\triangle$, we choose the space $\mathcal{D}(\Gamma\backslash\mathfrak{H})$ consisting of all the bounded C^∞-functions on $\mathfrak{H}$ such that $\triangle\phi(\tau)$ is bounded as well. Then the operator $(\triangle, \mathcal{D}(\Gamma\backslash\mathfrak{H}))$ is essentially self-adjoint, whose unique selfadjoint extension will be written by $\triangle_\Gamma$ in this note. The set Λ_Γ coincides with the pure point spectrum of $\triangle_\Gamma$ in $L^2(\Gamma\backslash\mathfrak{H})$. It is a basic fact that the space $\mathcal{A}(\Gamma;\lambda)$ is a finite dimensional $\mathbb{C}$-vector space consisting of automorphic forms. Concerning the set Λ_Γ, it is known that $\Lambda_\Gamma \subset [0,+\infty)$ and $\sharp(\Lambda_\Gamma \cap [0,x)) < +\infty$ for all $x > 0$. (*cf.* [2])

Remark 1. (1) The minimal element of Λ_Γ is 0, which corresponds to the eigenfunction $\phi_0 = (\mathrm{vol}(\mathcal{F}_\Gamma))^{-1/2}$ (constant).
(2) The real analytic Eisenstein series comprise another class of Maass forms which are not L^2. An L^2-Maass form $\phi(\tau)$ is said to be a cusp form if the 0-th Fourier coefficient of $\phi(\tau)$ at any cusp of Γ vanishes. Cusp forms form a subspace of the space $\mathcal{A}(\Gamma;\lambda)$, whose orthogonal complement is exhausted by some residues of the real analytic Eisenstein series constructed for each cusp of Γ.

The *spectral zeta function* for $\triangle$ on Γ is defined by

$$Z_\Gamma(s) \stackrel{\text{def}}{=} \sum_{\lambda\in\Lambda_\Gamma-\{0\}} \frac{1}{\lambda^s}, \quad s\in\mathbb{C}. \tag{1}$$

We put here some of the known results about this Dirichlet series and about the distribution of the spectrum Λ_Γ. Suppose Γ is a cocompact, i.e., $\Gamma\backslash\mathfrak{H}$ is compact. Then $Z_\Gamma(s)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s=1$. The location of other possible poles are also known. If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, a meromorphic continuation is also known and it is observed that double poles may occur ([3]). About the set Λ_Γ for arithmetic lattice Γ, we have Weyl's law: As $x\to+\infty$

$$N_\Gamma(x) := \sum_{\lambda\in\Lambda_\Gamma\cap(0,x)} \dim_{\mathbb{C}} \mathcal{A}(\Gamma;\lambda) \sim \frac{1}{4\pi}\mathrm{vol}(\Gamma\backslash\mathfrak{H})\,x.$$

Still, many basic problems remain unanswered. For example, to obtain a sharp bound of the dimension $\dim_{\mathbb{C}} \mathcal{A}(\Gamma;\lambda)$ is as such. For $\Gamma=\mathrm{SL}_2(\mathbb{Z})$, it is even expected that $\dim_{\mathbb{C}} \mathcal{A}(\Gamma;\lambda) = 1$ for each $\lambda\in\Lambda_\Gamma$.

If we abandon the assumption that Γ is cocompact or arithmetic, then less is known. For example, $\sharp\Lambda_\Gamma = +\infty$ is hard to prove and may be even false in general. But Weyl's law

$$N_\Gamma(x) + M_\Gamma(x) \sim \tfrac{1}{4\pi}\mathrm{vol}(\Gamma\backslash\mathfrak{H})\,x$$

with the term $M_\Gamma(x)$ accounting the continuous spectrum of Δ on $L^2(\Gamma\backslash\mathfrak{H})$ is still true for general Γ. What is unknown is to determine which term ($N_\Gamma(x)$ or $M_\Gamma(x)$) is dominant for non arithmetic Γ.

In this note, for a given simple closed geodesic $C\subset\Gamma\backslash\mathfrak{H}$, I introduce yet another new zeta function similar to (1) associated with the period integrals of Maass forms along C and study its analytic properties (such as meromorphicity and location of poles). As an application an analogue of Weyl's law for the norm square of the period integrals of Maass forms along the geodesic C is obtained.

Given that the nature of the Maass wave forms is still enigmatic for us, a new approach may shed some light on a new aspect of the Maass wave forms. Hopefully, our new zeta function will be so.

This is a write up of my talk which I gave at the Fukuoka conference on number theory held at Kyushu University in August 28–30, 2007. I would like to thank the organizers of the conference, Professor Masanobu Kaneko, Professor Yasuhiro Kishi and Professor Yasuro Gon, for having me there as a speaker.
(During the time I was writing [8], I found a mistake in the argument and that some of the the results I announced in Fukuoka do not hold as they are. I apologize the participants and the organizers for this. The corrected theorems are included in this note.)

2 Results

In order to state our result precisely, we need notation. An element $\eta\in G$ is said to be *hyperbolic* if there exist an $R_\eta\in G$ and a real number $\mathrm{N}(\eta)>1$ such that

$$\eta = \pm R_\eta \begin{pmatrix} \mathrm{N}(\eta)^{1/2} & 0 \\ 0 & \mathrm{N}(\eta)^{-1/2} \end{pmatrix} R_\eta^{-1}.$$

The number $\mathrm{N}(\eta)$, uniquely determined by η, is called the norm of η. The fractional linear transformation on $\mathfrak{H} \cup \mathbb{R} \cup \{\infty\}$ defined by η has exactly two fixed points $\theta_{\pm}(\eta)$ in $\mathbb{R}$.

Let us fix a hyperbolic element $\eta \in \Gamma$ once and for all. Let $C(\eta)$ be the geodesic curve in $\mathfrak{H}$ joining the two fixed points $\theta_{\pm}(\eta) \in \mathbb{R}$ of η. The group $\Gamma_\eta \overset{\text{def}}{=} \{\, \gamma \in \Gamma |\, \gamma\eta\gamma^{-1} = \eta \,\}$ when considered by modulo $\Gamma \cap \{\pm 1_2\}$, is an infinite cyclic group stabilizing the curve $C(\eta)$. The quotient $C_\Gamma(\eta) \overset{\text{def}}{=} \Gamma_\eta \backslash C(\eta)$ is a compact cycle in $\Gamma \backslash \mathfrak{H}$. The hyperbolic element η is said to be *primitive* in Γ if the group $\Gamma_\eta / \{\pm 1_2\} \cap \Gamma$ is generated by η itself. Then the curve $C_\Gamma(\eta)$ is simple and $\int_{C_\Gamma(\eta)} \mathrm{d}s = \log N(\eta)$.

Definition 2. For a Γ-invariant continuous function $f : \mathfrak{H} \to \mathbb{C}$, define the period integral along $C_\Gamma(\eta)$ by

$$\int_{C_\Gamma(\eta)} f \,\mathrm{d}s \overset{\text{def}}{=} \int_0^{\log N(\eta)} f(R_\eta \langle e^{2t}\sqrt{-1}\rangle)\,\mathrm{d}t.$$

Since Λ_Γ is a countable subset of positive numbers without finite accumulation points, it can be enumerated in a sequence

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$$

so that each $\lambda \in \Lambda_\Gamma$ is repeated its multiplicity $\dim_{\mathbb{C}} \mathcal{A}(\Gamma; \lambda)$ times. Then fix an orthonormal system $\{\phi_n\}$ of L^2-Maass forms such that $\triangle \phi_n = \lambda_n \phi_n$ for all $n \in \mathbb{N}$.

Let c_j $(1 \leqslant j \leqslant h)$ be a complete set of inequivalent cusps of Γ. For each j, choose $\sigma_j \in \mathrm{SO}(2)$ such that $\sigma_j\langle\infty\rangle = c_j$. Then define the Eisenstein series of Γ at j by

$$E^{(j)}(s;\tau) = \sum_{\gamma \in \Gamma_{c_j} \backslash \Gamma} \mathrm{Im}(\sigma_j^{-1}\gamma\langle\tau\rangle)^{(s+1)/2}, \quad \mathrm{Re}(s) > 1.$$

The series is absolutely convergent on $\mathrm{Re}(s) > 1$; the $E^{(j)}(s)$ has a meromorphic continuation to the whole s-plane, holomorphic on the imaginary axis.

Definition 3. We define the spectral zeta function with periods as

$$Z_\Gamma^\eta(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{{\lambda_n}^s} \left| \int_{C_\Gamma(\eta)} \phi_n \,\mathrm{d}s \right|^2 + \tfrac{1}{4\pi} \int_{\mathbb{R}} \sum_{j=1}^{h} \left| \int_{C_\Gamma(\eta)} E^{(j)}(it)\,\mathrm{d}s \right|^2 \frac{\mathrm{d}t}{\{4^{-1}(1+t^2)\}^s}.$$

Remark 2. Since $\mathcal{A}(\Gamma;\lambda)$ is a finite dimensional Hilbert space for each $\lambda \in \Lambda_\Gamma$, the linear form $\phi \mapsto \int_{C_\Gamma(\eta)} \phi \,\mathrm{d}s$ on it is represented by some (in fact the unique) function $\Phi_\lambda \in \mathcal{A}(\Gamma;\lambda)$, i.e.,

$$\langle \phi | \Phi_\lambda \rangle \left(\overset{\text{def}}{=} \int_{\Gamma\backslash\mathfrak{H}} \bar{\Phi}_\lambda(\tau)\,\phi(\tau)\,\tfrac{dxdy}{y^2} \right) = \int_{C_\Gamma(\eta)} \phi \,\mathrm{d}s, \quad \phi \in \mathcal{A}(\Gamma;\lambda).$$

Then we have

$$\sum_{n=1}^{infty} \frac{1}{{\lambda_n}^s} \left| \int_{C_\Gamma(\eta)} \phi_n \,\mathrm{d}s \right|^2 = \sum_{\lambda \in \Lambda_\Gamma - \{0\}} \frac{\langle \Phi_\lambda | \Phi_\lambda \rangle}{\lambda^s},$$

an expression independent of the choice of a system $\{\phi_n\}$.

Now, we state our main results.

Theorem 4. *Suppose* Γ *is a subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *with finite index. Then,*
• *the series and the integral defining* $Z_\Gamma^\eta(s)$ *converge absolutely and locally uniformly on* $\mathrm{Re}(s) > 2$*, defining a holomorphic function there.*
• $Z_\Gamma^\eta(s)$ *has a meromorphic continuation to the whole* s*-plane. The possible poles of* $Z_\Gamma^\eta(s)$ *are given as follows.*
  ⋆ $s = \frac{1}{2} - n\,(n \in \mathbb{Z}_{\geqslant 0})$*: possible simple poles.*
  ⋆ $s = -m\,(m \in \mathbb{Z}_{\geqslant 0})$*: possible double poles.*
• $Z_\Gamma^\eta(s)$ *has a simple pole at* $s = 1/2$ *with*

$$\mathrm{Res}_{s=1/2} Z_\Gamma^\eta(s) = (2\pi)^{-1} \log \mathrm{N}(\eta).$$

Theorem 5. *Suppose* Γ *is a subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *of finite index. Then*

$$\sum_{\lambda_n \leqslant x} \left| \int_{C_\Gamma(\eta)} \phi_n \, \mathrm{ds} \right|^2 + \tfrac{1}{4\pi} \int_{-x^{1/2}}^{x^{1/2}} \sum_{j=1}^{h} \left| \int_{C_\Gamma(\eta)} E^{(j)}(2\sqrt{-1}t) \, \mathrm{ds} \right|^2 \mathrm{dt} \sim \frac{\log \mathrm{N}(\eta)}{\pi} x^{1/2}, \quad x \to +\infty.$$

3 Automorphic Green's functions and automorphic heat kernels

From now on, for the sake of simplicity of exposition, we suppose $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Let $\eta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a fixed primitive hyperbolic element of $\mathrm{SL}_2(\mathbb{Z})$.

3.1 Arithmetic objects associated to η

Set

$$B_\eta\left(\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right) \stackrel{\mathrm{def}}{=} [u, v] \begin{pmatrix} c & (d-a)/2 \\ (d-a)/2 & -b \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$\begin{aligned} Q_\eta(X, Y) &\stackrel{\mathrm{def}}{=} B_\eta\left(\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right], \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]\right) \\ &= c(X - \theta_+(\eta)Y)(X - \theta_-(\eta)Y). \end{aligned}$$

The number $D = (\mathrm{tr}(\eta))^2 - 4\,(> 0)$ is the discriminant of Q_η. For $n \in \mathbb{Z} - \{0\}$, set

$$\begin{aligned} \Lambda(Q_\eta; n) &\stackrel{\mathrm{def}}{=} \{(x, y) \in \mathbb{Z}^2 |\, Q_\eta(x, y) = n\}/\Gamma_\eta, \\ \Lambda(Q_\eta; n)^0 &\stackrel{\mathrm{def}}{=} \{(x, y) \in \Lambda(Q_\eta; n) |\, \mathrm{g.c.d}(x, y) = 1\,\}. \end{aligned}$$

For A, $n \in \mathbb{Z} - \{0\}$, set

$$S_\eta(A, n) \stackrel{\mathrm{def}}{=} \sum_{(x,y) \in \Lambda(Q_\eta; n)^0} \exp\left(\tfrac{-2\pi\sqrt{-1}A}{n}\, B_\eta\left(\left[\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]\right)\right),$$

where for $(x, y) \in \Lambda(Q_\eta; n)^0$ the vector $(x_1, y_1) \in \mathbb{Z}^2$ is chosen so that $xy_1 - x_1 y = 1$. (The above expression is independent of the choice of (x_1, y_1)).

Let us define the zeta function of Q_η by

$$\zeta(Q_\eta; s) \stackrel{\mathrm{def}}{=} \sum_{n \in \mathbb{Z} - \{0\}} \frac{\sharp\Lambda(Q_\eta; n)}{|n|^s}, \quad s \in \mathbb{C}.$$

3.2 Eisenstein series for Γ

The modular group $\mathrm{SL}_2(\mathbb{Z})$ has the unique cusp $i\infty$ up to $\mathrm{SL}_2(\mathbb{Z})$-equivalence. The Eisenstein series of $\mathrm{SL}_2(\mathbb{Z})$ at $i\infty$ is defined by

$$E(\nu;\tau) \stackrel{\text{def}}{=} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma\langle\tau\rangle)^{(\nu+1)/2}, \quad \mathrm{Re}(\nu)>1,$$

where $y(\tau)$ means the y-coordinate of $\tau\in\mathbb{C}$. Then the series converges absolutely on $\mathrm{Re}(\nu)>1$, where it satisfies the eigenequation:

$$\triangle E(\nu) = \tfrac{1-\nu^2}{4}\, E(\nu).$$

By the spectral theory, the function $\nu\mapsto E(\nu;\tau)$ has a meromorphic extension to $\mathbb{C}$ satisfying the functional equation

$$E(-\nu;\tau) = \frac{\hat{\zeta}(\nu+1)}{\hat{\zeta}(\nu)}\, E(\nu;\tau) \tag{2}$$

with $\hat{\zeta}(s)=\Gamma_{\mathbb{R}}(s)\,\zeta(s)$.

The computation of the period integral of $E(\nu)$ along $C_\Gamma(\eta)$ is due to Hecke. In our case, the formula is

$$\int_{C_\Gamma(\eta)} E(s)\,\mathrm{ds} = \frac{1}{8}\hat{\zeta}\left(Q_\eta;\frac{s+1}{2}\right)\hat{\zeta}(s+1)^{-1} \tag{3}$$

where $\hat{\zeta}(Q_\eta;s)=D^{s/4}\Gamma_{\mathbb{R}}(s)^2\,\zeta(Q_\eta;s)$.

Remark 3. By the properties of $E(\nu)$ recalled above, we can deduce the meromorphicity of $\zeta(Q_\eta;s)$ and the functional equation: $\hat{\zeta}(Q_\eta;1-s)=\hat{\zeta}(Q_\eta;s)$.

3.3 Green's function associated to η (*cf.* [2], [4], [5], [6])

Let us define the free space (relative) Green's function associated with η by

$$\psi_\eta(s;\tau) \stackrel{\text{def}}{=} \tfrac{-1}{8\pi}\frac{\Gamma\left(\frac{s+1}{4}\right)^2}{\Gamma\left(\frac{s+2}{2}\right)}\left(\tfrac{\sqrt{D}\mathrm{Im}(\tau)}{Q_\eta(\tau,1)}\right)^{(s+1)/2} {}_2F_1\left(\tfrac{s+1}{4},\tfrac{s+1}{4};\tfrac{s+2}{2};\left(\tfrac{\sqrt{D}\mathrm{Im}(\tau)}{Q_\eta(\tau,1)}\right)^2\right)$$

with two variables $\tau\in\mathfrak{H}$ and $s\in\mathbb{C}$. It turns out that for a fixed $s\in\mathbb{C}$ the function $\tau\mapsto\psi_\eta(\tau;s)$ is continuous on $\mathfrak{H}$ and is C^∞ on $\mathfrak{H}-C(\eta)$. Having this function, we define the automorphic (relative) Green's function associated with η by the series

$$\Psi^\Gamma_\eta(s;\tau) \stackrel{\text{def}}{=} \sum_{\gamma\in\Gamma_\eta\backslash\Gamma}\psi_\eta(s;\gamma\langle\tau\rangle), \quad \tau\in\mathfrak{H},\ s\in\mathbb{C}.$$

Proposition 6. (1) *The series* $\Psi^\Gamma_\eta(s;\tau)$ *is absolutely convergent when* $\mathrm{Re}(s)>1$*; the convergence is locally uniform with respect to* (τ,s).
(2) *For a fixed* s *with* $\mathrm{Re}(s)>1$*, the function* $\Gamma\tau\mapsto\Psi^\Gamma_\eta(s;\tau)$ *is an* L^1*-function on* $\Gamma\backslash\mathfrak{H}$*. As a distribution on* $\Gamma\backslash\mathfrak{H}$*, it satisfies an analogue of the Poisson's equation*

$$(\triangle+\frac{s^2-1}{4})\,\Psi^\Gamma_\eta(s) = \delta_{C_\Gamma(\eta)}, \quad \mathrm{Re}(s)>1,$$

where $\delta_{C_\eta(\eta)}$ *is the distribution on* $\Gamma\backslash\mathfrak{H}$ *defined by the linear functional* $f \mapsto \int_{C_\Gamma(\eta)} f\,\mathrm{d}s$.
(3) *If* $y > \frac{\sqrt{D}}{2}$, *we have the Fourier series expansion of* $\Psi_\eta(s)$*:*

$$\frac{-4}{\sqrt{2\pi}}\frac{\Gamma((s+3)/4)}{\Gamma((s+1)/4)}\,\Psi_\eta^\Gamma(s;\tau) = 2^{-s/2}D^{(s+1)/4}\,\frac{\zeta(Q_\eta;\frac{s+1}{2})}{s\,\zeta(s+1)}\,y^{(1-s)/2} + \sum_{A\in\mathbb{Z}-\{0\}}\left\{\sum_{n\neq 0}\frac{S_\eta(A,n)}{|n|^{1/2}}J_{s/2}\left(\tfrac{\pi\sqrt{D}|A|}{|n|}\right)\right\}(\sqrt{D}y)^{1/2}\,K_{s/2}(2\pi|A|y)\,e^{-2\pi iAx},$$

where $J_s(z)$ *is the Bessel function and* $K_s(z)$ *the modified one.*
(4) $\Psi_\eta^\Gamma(s) \in L^p(\Gamma\backslash\mathfrak{H})$ *if* $\mathrm{Re}(s) > \frac{p-2}{2p}$.

3.4 The heat kernel associated to η (*cf.* [7])

Let us define the free space (relative) heat kernel associated with η by

$$\hat{\psi}_\eta(T;\tau) \stackrel{\text{def}}{=} \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \psi_\eta(s:\tau)\,e^{(-1+s^2)T/4}\,s\mathrm{d}s$$

with two variables $T > 0$ and $\tau \in \mathfrak{H}$. Here $c > 1$ is fixed. It turns out that the integral is convergent and independent of the choice of c. Having this function, we define the automorphic (relative) heat kernel associated with η by the series

$$\hat{\Psi}_\eta^\Gamma(T;\tau) \stackrel{\text{def}}{=} \sum_{\gamma\in\Gamma_\eta\backslash\Gamma} \hat{\psi}_\eta(T;\gamma\langle\tau\rangle).$$

Proposition 7. (1) *The series* $\hat{\Psi}_\eta^\Gamma(T;\tau)$ *is absolutely convergent for* $T > 0$ *and* $\tau \in \mathfrak{H}$. *The convergence is locally uniform with respect to* (T,τ).
(2) *The automorphic heat kernel* $\hat{\Psi}_\eta^\Gamma(T)$ *is related to the automorphic Green's function* $\Psi_\eta^\Gamma(s)$ *by*

$$\hat{\Psi}_\eta^\Gamma(T;\tau) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \Psi_\eta^\Gamma(s;\tau)e^{(-1+s^2)T/4}\,s\mathrm{d}s \qquad (4)$$

whenever $c > 1$.
(3) *The function* $\hat{\Psi}_\eta^\Gamma(T;\tau)$ *is* C^∞ *on* $(0,\infty)\times\Gamma\backslash\mathfrak{H}$ *and satisfies the 'heat equation':*

$$-\frac{\partial}{\partial T}\hat{\Psi}_\eta^\Gamma(T;\tau) = \triangle\hat{\Psi}_\eta^\Gamma(T;\tau).$$

(4) $\hat{\Psi}_\eta^\Gamma(T) \in L^p(\Gamma\backslash\mathfrak{H})$ *for* $T > 0$, $p > 0$.
(5) *The spectral expansion:*

$$\hat{\Psi}_\eta^\Gamma(T) = \sum_{n=0}^{\infty} e^{-\lambda_n T}\left(\int_{C_\Gamma(\eta)}\overline{\phi_n}\,\mathrm{d}s\right)\phi_n(\tau) + \frac{1}{4\pi i}\int_{i\mathbb{R}} e^{\frac{-1+\nu^2}{4}T}\left(\int_{C_\Gamma(\eta)}\overline{E(\nu)}\,\mathrm{d}s\right)E(\nu;\tau)\,\mathrm{d}\nu. \qquad (5)$$

The convergence is uniform on any compact subset of $\Gamma\backslash\mathfrak{H}$.

4 Sketch of the proof of main results

The basic idea is this: consider the period integral of the automorphic heat kernel $\hat{\Psi}^{\Gamma}_{\eta}(T)$ along $C_{\Gamma}(\eta)$

$$\hat{\mathbb{P}}^{\Gamma}_{\eta}(T) \stackrel{\mathrm{def}}{=} \int_{C_{\Gamma}(\eta)} \hat{\Psi}^{\Gamma}_{\eta}(T;\tau)\,\mathrm{ds}.$$

Then we compute this integral in two different ways. One by invoking the spectral expansion (5) of $\Psi^{\Gamma}_{\eta}(T)$, and the other by putting the definition (4) of $\hat{\Psi}^{\Gamma}_{\eta}(T)$ and unfolding the integral.

4.1 The spectral side

Since $C_{\Gamma}(\eta)$ is compact and since the spectral expansion (5) of $\hat{\Psi}^{\Gamma}_{\eta}(T;\tau)$ converges uniformly on a compact set of $\Gamma\backslash\mathfrak{H}$, the integration can be taken by term by term. Consequently, we arrive at the expression:

$$\hat{\mathbb{P}}^{\Gamma}_{\eta}(T) = \sum_{n=1}^{\infty} e^{-\lambda_n T}\left|\int_{C_{\Gamma}(\eta)} \phi_n\,\mathrm{ds}\right|^2 + \mathbb{E}(T) + \tfrac{(\log \mathrm{N}(\eta))^2}{\mathrm{vol}(\mathcal{F}_{\Gamma})},$$

$$\mathbb{E}(T) = \tfrac{1}{4\pi i}\int_{i\mathbb{R}} e^{\frac{-1+\nu^2}{4}T}\left|\int_{C_{\Gamma}(\eta)} E(\nu)\,\mathrm{d}s\right|^2 \mathrm{d}\nu.$$

At least formally, we have

$$\Gamma(s)\, Z^{\eta}_{\Gamma}(s) = \int_0^{\infty}\{\hat{\mathbb{P}}^{\Gamma}_{\eta}(T) - \tfrac{(\log \mathrm{N}(\eta))^2}{\mathrm{vol}(\mathcal{F}_{\Gamma})}\}\, T^{s-1}\mathrm{d}T.$$

Therefore, Theorem 4 and Theorem 5 follow from the following Proposition.

Proposition 8. *We have the large time estimate:*

$$\hat{\mathbb{P}}^{\eta}(T) - \tfrac{(\log \mathrm{N}(\eta))^2}{\mathrm{vol}(\mathcal{F}_{\Gamma})} = O(e^{-NT}), \quad T \to +\infty.$$

with some $N > 0$. We have the small time asymptotic expansion:

$$\hat{\mathbb{P}}^{\Gamma}_{\eta}(T) \sim T^{-1/2}\left(\tfrac{\log \mathrm{N}(\eta)}{2\sqrt{\pi}} + \sum_{n=1}^{\infty} a_n T^{n/2}\right) + \log T\,(\sum_{n=0}^{\infty} b_n T^n),$$

The large time asymptotic is rather easy to establish; to show the small time asymptotic expansion, we analyze the 'geometric side'of the period $\hat{\mathbb{P}}^{\Gamma}_{\eta}(T)$.

4.2 The geometric side

For unfolding the integral, it is more enlightening to lift the function $\hat{\psi}_{\eta}(T;\tau)$ on $\mathfrak{H}$ up to $G = \mathrm{SL}_2(\mathbb{R})$ by the diffeomorphism $G/K \ni gK \mapsto g\langle i\rangle \in \mathfrak{H}$, where $K = \mathrm{SO}(2)$. Let G_{η} be the centralizer of η in G. Then the lifted function $g \mapsto \hat{\psi}_{\eta}(T; g\langle i\rangle)$ on G is left G_{η}-invariant as well as right K-invariant. The following computation is quite standard.

$$\begin{aligned}
\hat{\mathbb{P}}_\eta^\Gamma(T) &= \int_{\Gamma_\eta\backslash G_\eta} \hat{\Psi}_\eta^\Gamma(T; h\langle i\rangle)\,\mathrm{d}h \\
&= \int_{\Gamma_\eta\backslash G_\eta} \sum_{\gamma\in\Gamma_\eta\backslash\Gamma} \hat{\psi}_\eta(T;\gamma h\langle i\rangle)\,\mathrm{d}h \\
&= \int_{\Gamma_\eta\backslash G_\eta} \sum_{\gamma\in\Gamma_\eta\backslash\Gamma/\Gamma_\eta} \sum_{\delta\in(\gamma^{-1}\Gamma_\eta\gamma\cap\Gamma_\eta)\backslash\Gamma_\eta} \hat{\psi}_\eta(T;\gamma\delta h\langle i\rangle)\,\mathrm{d}h \\
&= \sum_{\gamma\in\Gamma_\eta\backslash\Gamma/\Gamma_\eta} \int_{\Gamma_\eta\backslash G_\eta} \sum_{\delta\in(\gamma^{-1}\Gamma_\eta\gamma\cap\Gamma_\eta)\backslash\Gamma_\eta} \hat{\psi}_\eta(T;\gamma\delta h\langle i\rangle)\,\mathrm{d}h \\
&= \sum_{\gamma\in\Gamma_\eta\backslash\Gamma/\Gamma_\eta} \int_{\gamma^{-1}\Gamma_\eta\gamma\cap\Gamma_\eta\backslash G_\eta} \hat{\psi}_\eta(T;\gamma h\langle i\rangle)\,\mathrm{d}h \\
&= \sum_{\gamma\in\Gamma_\eta\backslash\Gamma/\Gamma_\eta} v(\gamma)\, I(T;\gamma)
\end{aligned}$$

with

$$\begin{aligned}
v(\gamma) &\stackrel{\text{def}}{=} \mathrm{vol}((\gamma^{-1}\Gamma_\eta\gamma\cap\Gamma_\eta)\backslash(\gamma^{-1}G_\eta\gamma\cap G_\eta)), \\
I(T;\gamma) &\stackrel{\text{def}}{=} \int_{(\gamma^{-1}G_\eta\gamma\cap G_\eta)\backslash G_\eta} \hat{\psi}_\eta(T;\gamma h\langle i\rangle)\,dh.
\end{aligned}$$

4.2.1 Classification of double cosets

Let V_2 be the space of all the binary quadratic forms with coefficients in $\mathbb{R}$, i.e.,

$$V_2 \stackrel{\text{def}}{=} \{P = AX^2 + BXY + CY^2 \mid A, B, C \in \mathbb{R}\}.$$

Then the group $G = \mathrm{SL}_2(\mathbb{R})$ acts on V_2 by the rule

$$(g^{-1}P)(X,Y) = P(aX + bY, cX + dY), \quad g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G,\ P(X,Y) \in V_2,$$

preserving the discriminant function $V_2 \ni P = AX^2 + BXY + CY^2 \mapsto \Delta(P) \stackrel{\text{def}}{=} B^2 - 4AC$. Let $\langle P, Q\rangle$ be the bilinear form on V_2 associated with $\Delta(P)$, i.e.,

$$\langle P, Q\rangle \stackrel{\text{def}}{=} \frac{1}{2}\{\Delta(P+Q) - \Delta(P) - \Delta(Q)\}.$$

For $g \in G$, set

$$B_g \stackrel{\text{def}}{=} \frac{\langle gQ_\eta, Q_\eta\rangle}{\Delta(Q_\eta)}.$$

Then the map $g \mapsto |B_g|$ yields a bijection $G_\eta\backslash G/G_\eta \cong \mathbb{R}_{\geqslant 0}$. The double cosets $\Gamma_\eta\gamma\Gamma_\eta$ are divided to 3 classes according to the value of $|B_\gamma|$:

⋆ The coset Γ_η, which is the one with $|B_\gamma| = 1$.
  $(v(e) = \mathrm{vol}(\Gamma_\eta\backslash G_\eta) = \log \mathrm{N}(\eta))$.

⋆ ('Pseudo-elliptic cosets') Finite number of $\Gamma_\eta\gamma\Gamma_\eta$ with $|B_\gamma| < 1$.
  $(\gamma^{-1}G_\eta\gamma \cap G_\eta = \{\pm 1_2\}$ and $v(\gamma) = 1)$

⋆ ('Pseudo-hyperbolic cosets') Infinite number of $\Gamma_\eta\gamma\Gamma_\eta$ with $|B_\gamma| > 1$.
  $(\gamma^{-1}G_\eta\gamma \cap G_\eta = \{\pm 1_2\}$ and $v(\gamma) = 1.)$

According to this classification, we can write the integral $\hat{\mathbb{P}}_{\eta}^{\Gamma}(T)$ as a sum of the three terms:

$$\hat{\mathbb{P}}_{\eta}^{\Gamma}(T) = \mathbb{I}(T)^{=1} + \mathbb{I}(T)^{>1} + \mathbb{I}(T)^{<1}$$

with

$$\begin{aligned}
\mathbb{I}(T)^{=1} &= \log \mathrm{N}(\eta)\, I(T;e), && \text{(Single term)},\\
\mathbb{I}(T)^{>1} &= \sum_{\gamma \in \Gamma_\eta \backslash \Gamma / \Gamma_\eta ; |B_\gamma|>1} I(T;\gamma), && \text{(Infinite sum)},\\
\mathbb{I}(T)^{<1} &= \sum_{\gamma \in \Gamma_\eta \backslash \Gamma / \Gamma_\eta ; |B_\gamma|<1} I(T;\gamma), && \text{(Finite sum)}.
\end{aligned}$$

(This expression affords the 'geometric side'of the period $\hat{\mathbb{P}}_{\eta}^{\Gamma}(T)$.) Therefore, it suffices to study the asymptotic of these terms separately.

4.2.2 Asymptotic of $\mathbb{I}(T)^{=1}$

Lemma 9. *We have*

$$I(T;e) = \tfrac{1}{8\pi} \int_0^\infty \left| \frac{\Gamma\left(\frac{1+it}{4}\right)}{\Gamma\left(\frac{3+4it}{4}\right)} \right|^2 t \tanh\left(\tfrac{\pi t}{2}\right) e^{-(1+t^2)T/4}\, \mathrm{d}t.$$

Lemma 10. *There exists constants a_n, b_n such that for any $N \in \mathbb{N}$ the estimation*

$$I(T;e) = T^{-1/2}\left(\tfrac{1}{2\sqrt{\pi}} + \sum_{n=1}^{2N+1} a_n T^{n/2} \right) + \log T \,(\sum_{n=0}^{N} b_n T^n) + O(T^{N+1/2}) \quad \text{near} \ \ T=0$$

holds.

This affords the leading term of the small-time asymptotic of $\hat{\mathbb{P}}_{\eta}^{\Gamma}(T)$.

4.2.3 Asymptotic of $\mathbb{I}(T)^{>1}$

Lemma 11. *The integral $I(T;\gamma)$ with $|B_\gamma|>1$ is given as*

$$I(T;\gamma) = \tfrac{-1}{\sqrt{2}\pi i} \int_{L_c} \frac{\Gamma\left(\frac{s+1}{4}\right)\Gamma\left(\frac{s+2}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+3}{4}\right)} f_s(B_\gamma^2)\, e^{(-1+s^2)T/4}\, \mathrm{d}s,$$

with

$$f_s(z) = \frac{\Gamma\left(\frac{s+1}{4}\right)^2}{\Gamma\left(\frac{s+2}{2}\right)} (z-1)^{-\frac{s+1}{4}}\, {}_2F_1\left(\tfrac{s+1}{4}, \tfrac{s+1}{4}, ; \tfrac{s+2}{2}, \tfrac{-1}{z-1}\right)$$

Here $L_c : s = c + it\, (-\infty < t < +\infty)$ with $c>1$.

Lemma 12. *For any $N \in \mathbb{N}$,*

$$\frac{\partial^N}{\partial T^N} I(T;\gamma) = O\left(b_\gamma^{-1/4} \left(\tfrac{\log b_\gamma}{T}\right)^{2N} \exp\left(\tfrac{-(\log b_\gamma)^2}{16T}\right)\right), \quad T>0,\ b_\gamma = B_\gamma^2 - 1.$$

Remark 4. Since $B_\gamma \in \mathbb{Q}$, $b_\gamma = B_\gamma^2 - 1 \neq 1$.

This yields

Lemma 13. *We have*

$$\lim_{T \to +0} \frac{\partial^N}{\partial T^N} \mathbb{I}(T)^{>1} = 0$$

for any $N \in \mathbb{N}$.

4.2.4 Asymptotic of $\mathbb{I}(T)^{<1}$

Lemma 14. *The integral* $I(T;\gamma)$ *with* $|B_\gamma| < 1$ *is given as*

$$I(T;\gamma) = \tfrac{-1}{\sqrt{2\pi i}}\tfrac{1}{2\pi}\int_{i\mathbb{R}} \left|\frac{\Gamma\left(\frac{s+1}{4}\right)}{\Gamma\left(\frac{s}{2}\right)}\right|^2 F_s(B_\gamma^2)\, e^{(-1+s^2)T/4}\mathrm{d}s,$$

with

$$F_s(z) = \tfrac{1}{2\pi i}\int_{L_\sigma} \Gamma\left(\tfrac{s+2}{4}+\zeta\right)\Gamma\left(\tfrac{-s+2}{4}+\zeta\right)\Gamma\left(\tfrac{-1}{4}-\zeta\right)^2(1-z)^{\zeta+\frac{1}{4}}\,\mathrm{d}\zeta.$$

Here L_σ *is the contour* $\zeta = \sigma + it\,(-\infty < t < +\infty)$ *with* $-1/2 < \sigma < -1/4$.

By shifting the contour L_σ to the negative direction, after an involved argument, we obtain

Lemma 15. *For any* $N \in \mathbb{N}$, *the limit* $\lim_{T\to+0}\frac{\partial^N}{\partial T^N}\mathbb{I}(T)^{<1}$ *exists.*

4.2.5 Asymptotic of $\mathbb{E}(T)$

Since $\zeta(Q_\eta;s)$ originally given by a convergent Dirichlet series on $\mathrm{Re}(s) > 1$ has a meromorphic continuation to $\mathbb{C}$ with the functional equation

$$\hat{\zeta}(Q_\eta;s) = \hat{\zeta}(Q_\eta;1-s),$$

by a standard technique, we obtain the convexity bound of $\zeta(Q_\eta;s)$ on the critical line:

$$\zeta\left(Q_\eta;\tfrac{1}{2}+it\right) \prec (1+|t|)^{1/2+\epsilon}, \quad t\in\mathbb{R}$$

for any $\epsilon > 0$. A better bound breaking this is called a subconvexity bound, which is enough for us to eliminate the Eisenstein period from the asymptotic law in Theorem 5:

Theorem 16. *Suppose a bound* $\zeta(Q_\eta;1/2+it) \prec (1+|t|)^{1/2-\epsilon}$ *is true for some* $\epsilon > 0$. *Then*

$$\sum_{\lambda_n\leqslant x}\left|\int_{C_\Gamma(\eta)}\phi_n\,\mathrm{ds}\right|^2 \sim \frac{\log \mathrm{N}(\eta)}{\pi}\,x^{1/2}, \quad x\to+\infty.$$

Remark 5. The following fact is proved by Katok-Sarnak [1]: If ϕ_n is a normalized Hecke eigen Maass cusp form on $\mathrm{SL}_2(\mathbb{Z})$ with $L(\phi_n,\frac{1}{2}) = 0$, then

$$\sum_{[\eta]\in\mathcal{Q}(D)}\int_{C_\Gamma(\eta)}\phi_n\,\mathrm{ds} = 0$$

for each discriminant $D > 0$. $\mathcal{Q}(D)$ the set of Γ-conjugacy classes of primitive hyperbolic elements with $\mathrm{tr}(\gamma)^2 - 4 = D$.

5 Concluding remarks

Let me put some remarks. First, for a cocompact lattice Γ (arising from a indefinite quaternion division algebra over $\mathbb{Q}$), our main result (Theorem 4 and Theorem 5) is true. In this case, it is proved that the convergence region of $Z_\Gamma^\eta(s)$ is $\mathrm{Re}(s) > 1/2$. Second, we have a twisted version also. Let χ be a character of $\mathbb{R}^\times$ such that $\chi(\mathrm{N}(\eta)) = 1$. Then we define the χ-twisted period of ϕ as

$$\int_{C_\Gamma(\eta)} \phi\chi \,\mathrm{ds} \stackrel{\mathrm{def}}{=} \int_0^{\log \mathrm{N}(\eta)} \phi(R_\eta\langle e^{2t}i\rangle)\chi(e^t)\,\mathrm{d}t.$$

About this χ-twisted period and the associated spectral type zeta function we have a similar result. Actually, we generalize this twisted version to real hyperbolic spaces of higher dimension, and have a result which includes Theorem 4 and Theorem 5 as a special case ([8]). In some cases, the Eisenstein contribution is much more difficult to separate in the asymptotic law like Theorem 5; we need a (yet to be proved) quite strong subconvexity bound on the critical line of a certain automorphic L-function of an orthogonal group. Finally, our construction of $Z_\Gamma^\eta(s)$, at least formally, can be generalized for an arbitrary triple of groups (G, H, Γ) such that G and H are semisimple Lie groups with H sitting inside G and Γ is a lattice of G such that $\Gamma \cap H$ is a lattice of H. When H is a symmetric subgroup of G such that the split rank of the symmetric space $H\backslash G$ is one and when Γ is a cocompact lattice in G, we have some guess how the analogue of Weyl's law for the average of the the norm square of periods of Maass forms should look like.

Let me put some problems. It seems interesting to have an error term estimate in Theorem 5. Various generalizations of the spectral zeta function with period $Z_\Gamma^\eta(s)$ are probably possible. For example, for two primitive hyperbolic elements η, $\xi \in \Gamma$, we can consider the series

$$Z_\Gamma^{\eta,\xi}(s) \stackrel{\mathrm{def}}{=} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} \left(\int_{C_\Gamma(\eta)} \phi_n \,\mathrm{ds}\right)\left(\int_{C_\Gamma(\xi)} \overline{\phi_n}\,\mathrm{ds}\right) + \text{Eisenstein part}$$

which reduces our $Z_\Gamma^\eta(s)$ when $\eta = \xi$ and has a non-empty convergence region. Beyond this, we know nothing about $Z_\Gamma^{\eta,\xi}(s)$.

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