Base change lift type spinor L-function of $GSp_2(\mathbb{Q})$

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In [7], we happened to consturct Siegel modular cuspform F and non-cuspform E of degree 2 having the same spinor L-function (of degree 4):

$$L(s, E, spin) = L(s, F, spin).$$

In the theory of automorphic form of $GSp_2(\mathbb{Q})$, it is known that Cuspidal Associated Parabolic representation (denoted by CAP) has a L-function of a non-cuspidal one. However our phenomenon is not the case. In this article, we explain the reason why such a strange phenomenon occures.

By the way, the spinor L-function is equal to the Hasse-Weil zeta function of the jacobian of $C : y^2 = x^5 - x$, that is,

$$L(s, E, spin) = L(s, F, spin) = L(s, H^1_{et}(j(C), \mathbb{Q})).$$

This coincidence of L-function of Siegel modular form and that of Abelian surface is a concrete example of Yoshida's Siegel modularity conjecuture of Abelian surface [14].

1 L-parameter

Now then, we are going to explain the reason of such a strange phenoneon.

1. Determine $L(s, H^1_{et}(C, \mathbb{Q}_l))$.
The hyper-ellitpic curve C has a complex mutiplication such as

$$C : y^2 = x^5 - x \ni (x, y) \longrightarrow (ix, i^{\frac{1}{2}} y) \in C,$$

that is $End(j(C)) \otimes \mathbb{Q} \simeq K = \mathbb{Q}(i^{\frac{1}{2}})$. Thanks to Shimura-Yoshida's CM-theory [13], we can determine the Großencharakter λ on $K^\times_\mathbb{A}$ so that

$$L(s, \lambda) = L(s, H^1_{et}(C, \mathbb{Q}_l)).$$

Further, we can write

$$\lambda = \mu \circ N_{K/\mathbb{Q}(\sqrt{-2})}$$

for a certain μ on $\mathbb{Q}(\sqrt{-2})^\times_\mathbb{A}$, consequently

$$L(s, \lambda) = L(s, \mu) L(s, \overline{\mu}). \quad (1)$$

2. Two L-homomorphisms ${}^LGL_2(\mathbb{Q}_p) \to {}^LGSp_4(\mathbb{Q}_p)$.
From the above μ, we get an elliptic cuspform θ_μ by

$$\theta_\mu(z) = \sum_{\alpha \subset \mathbb{Z}[\sqrt{-2}]} \mu(\alpha) \exp(2\pi i N(\alpha) z) \in S_2(\Gamma_0(64), \chi_{\mathbb{Q}(\sqrt{2})}), \quad (2)$$

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where $\chi_{\mathbb{Q}(\sqrt{2})}$ is the quadratic character associated to the extension $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$.

Now, we consider the L-parameter associated to θ_μ, homomorphism from $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ to $GL_2(\mathbb{C}) \times \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and consider two L-homomorphisms

$$ {}^L GL_2(\mathbb{Q}_p) \longrightarrow GSp_4(\mathbb{C}) = {}^L GSp_4(\mathbb{Q}_v)^\circ $$

One is

$$ \begin{cases} GL_2(\mathbb{C}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow & \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}) \ni \mathrm{Frob}_p \longrightarrow & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \chi_k(p) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \chi_k(p) \end{pmatrix} \end{cases} \tag{3} $$

with $k = \mathbb{Q}(\sqrt{2})$. This L-homomorphism preserves the central characters of automorphic representations, i.e.,

$$ \omega_\pi = \omega_{\Pi^0} \tag{4} $$

where π is the automorphic representation associated θ_μ, and Π^0 is that to the image of the above L-homomorphism. Indeed, if the conjugacy class of the L-parameter of π is $\begin{pmatrix} \alpha & \\ & \omega_\pi\overline{\alpha} \end{pmatrix}$, the image of the L-homomorphism is

$$ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \chi_k(p)\alpha & 0 & 0 \\ 0 & 0 & \omega_\pi\overline{\alpha} & 0 \\ 0 & 0 & 0 & \chi_k(p)\omega_\pi\overline{\alpha} \end{pmatrix}. \tag{5} $$

Another L-homomorphism is as follows. That is a path to $GSp_4(\mathbb{C})$ through ${}^L GL_2(k_p) = GL_2(\mathbb{C})^2 \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ at p in case of $\chi_k(p) = -1$:

$$ \begin{cases} g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow g \times g \longrightarrow & \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}, \\ \mathrm{Frob}_p \longrightarrow & \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}. \end{cases} \tag{6} $$

Then, the image Π^1 of $\left(\begin{pmatrix} \alpha & \\ & \omega_\pi\overline{\alpha} \end{pmatrix} \times \begin{pmatrix} \alpha & \\ & \omega_\pi\overline{\alpha} \end{pmatrix}\right) \rtimes Frob_p$ is

$$ \begin{pmatrix} 0 & -i\alpha & 0 & 0 \\ i\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & i\overline{\alpha} \\ 0 & 0 & -i\overline{\alpha} & 0 \end{pmatrix} \sim \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & \overline{\alpha} & 0 \\ 0 & 0 & 0 & -\overline{\alpha} \end{pmatrix} \tag{7} $$

as $GSp_4(\mathbb{C})$-conjugacy class. This L-homomorphism doesn't preserve the central character of π, different from (4),

$$ \omega_\pi = \chi_k \omega_{\Pi^1}. \tag{8} $$

Comparing (5) and (7), we find that Π^0 and Π^1 has the same L-function.

But, Zharkovskaya relation (explained in the next section) implies Π^0 should be cuspidal and Kudla-Rallis-Soudry's characterization [6] for standard L-function of cuspidal representation implies Π^1 should be non-cuspidal. And by the Yoshida lift [14], we really constructed the cupsform F and non-cupsform E. This is the explanation of the strange phenomenon. Furhter discussion on the Yoshida conjecture is held in [8], [9]. Now, we consider the problem when such a strange phenomenon occures? In order to answer it, we will classify the L-functions of Siegel non-cuspforms in the next section.

Remark 1.1. Since $S_2(\Gamma_0(64), \chi_2) = \mathbb{C}\theta_\mu \oplus \mathbb{C}\theta_{\overline{\mu}}$, and (1), we find that $j(C)$ is isogeneous to the jacobian obtained from the Shimura curve.

2 Generalization of Zharkovskaya relation

The original 'Zharkovskaya relation' is a relation between L-functions of Siegel non-cuspform and the elliptic modular form which is sended by the Siegel operator. If F is a full modular Hecke eigenform non-cuspform of degree 2 of weight κ, then the Siegel operator sends an elliptic modular eigenform $\Phi(F)$

$$M_\kappa(Sp_2(\mathbb{Z})) \ni F \longrightarrow \Phi(F)(z) = \lim_{t\to\infty} F\begin{pmatrix} z & 0 \\ 0 & it \end{pmatrix} \in M_\kappa(SL_2(\mathbb{Z})) \tag{9}$$

for $z \in \mathfrak{H}$, and it holds

$$L(s, F, spin) = L(s, \Phi(F))L(s - \kappa + 2, \Phi(F)).$$

We will generalize her relation for non-holomorphic and non-full modular cases. Let $U_i, i = 1, 2$ be the unipotent radicals of the two parabolic subgroups.

$$U_1(\mathbb{A}) = \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad U_2(\mathbb{A}) = \begin{pmatrix} 1 & & & * \\ * & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \subset Sp_2(\mathbb{A}).$$

If F is not cuspidal, then

$$\int_{U_i(\mathbb{Q})\backslash U_i(\mathbb{A})} F(ug)du \neq 0$$

for $i = 1$ or 2 with a Haar measure du of U_i. We say the former case (CASE 1), and the latter (CASE 2). In the both cases, we obtain automorphic forms on $GL_2(\mathbb{A})$ by

$$\int_{U_i(\mathbb{Q})\backslash U_i(\mathbb{A})} F(u \cdot e_i(g))du,$$

where we write

$$e_1(g) = \begin{pmatrix} {}^tg^{-1} & \\ & g \end{pmatrix}, e_2(g) = \begin{pmatrix} a & & b & \\ & \det(g) & & \\ c & & d & \\ & & & 1 \end{pmatrix} \in GSp_2(\mathbb{A})$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{A})$. So, after the original (9),

Definition 1. We call 'Siegel operator along U_i'

$$\Phi_i(F)(g) = \int_{U_i(\mathbb{Q})\backslash U_i(\mathbb{A})} F(u \cdot e_i(g))du,$$

where du is the Haar measure so that $vol(U_i(\mathbb{Q})\backslash U_i(\mathbb{A})) = 1$.

Remark that the Siegel operator (9) is equal to Φ_2, and that holomorphic F is cuspidal if and only if $\Phi_2(F) = 0$.

Classical form and Adelic form: Let Γ be a congruence subgroup containing a principal congruence subgroup $\Gamma(N)$ and every element

$$\gamma \equiv \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1} & \\ & & & b^{-1} \end{pmatrix} \pmod N$$

with a, b prime to N. Let χ_1, χ_2 be Dirichlet characters of $\mathbb{Q}^\times$ (the adelized character also denote by χ_i). We say a vector valued function f on the Siegel upper half space $\mathfrak{H}_2$ classical form and write $f \in M_{\kappa_1,\kappa_2}(\Gamma, \chi_1, \chi_2)$, if

$$f(\gamma Z) = \mathrm{sym}(\kappa_1 - \kappa_2) \otimes \det{}^{\kappa_2}(cZ + d)\chi_1(d_1)\chi_2(d_2)f(Z)$$

for every $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma, d = \begin{pmatrix} d_1 & * \\ * & d_2 \end{pmatrix}$. An automorphic form F on $GSp_2(\mathbb{A})$ is associated by

$$F(g_\infty) = \nu(g_\infty)^{2\kappa_2+3}\mathrm{sym}(\kappa_1 - \kappa_2) \otimes \det{}^{\kappa_2}(ci_2 + d)f(g_\infty \cdot i_2) \tag{10}$$

with $i_2 = \begin{pmatrix} i & \\ & i \end{pmatrix} \in \mathfrak{H}_2$ for $\nu(g_\infty)^{-\frac{1}{2}}g_\infty = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in GSp_2(\mathbb{R})$, where ν is the similitude norm of g_∞. By the strong approximion theorem of $GSp_2(\mathbb{A})$, we can derive from (11) the automorphic form on $GSp_2(\mathbb{A})$. If f is holomorphic, we say F is holomorphic. Classical elliptic modular form h of weight κ is also associated to adelized form H by

$$H(g_\infty) = \nu(g_\infty)^{\kappa+1}(ci + d)^\kappa h(g_\infty \cdot i) \tag{11}$$

for $g_\infty \in GL_2(\mathbb{R})$.

***L*-function:** Hecke operator is spherical function of $GSp_n(\mathbb{Q}_p)$ with compact support. The action of Hecke operator η on an automorphic form f is

$$\eta * f(g) = \int_{GSp_n(\mathbb{Q}_p)} f(gh)\eta(h^{-1})dh$$

where dh is the Haar measure such that $\mathrm{vol}(GSp_n(\mathbb{Z}_p)) = 1$. If η is corresponding to the linear combination of right $GSp_n(\mathbb{Z}_p)$-coset such as

$$\sum_{a_i} a_i \begin{pmatrix} {}^tD_i p^{-l_i} & * \\ & D_i^{-1} \end{pmatrix} GSp_n(\mathbb{Z}_p)$$

with $a_i \in \mathbb{C}$ and

$$D_i = \begin{pmatrix} p^{r_{i1}} & * & * \\ & \ddots & * \\ & & p^{r_{in}} \end{pmatrix},$$

Satake isomorphism associates to

$$\sum_i (X_0)^{l_i} \prod_{j=1}^{n} (p^{-j} X_j)^{r_{ij}} \in \mathbb{C}[X_0^{\pm}, X_1^{\pm}, \ldots, X_n^{\pm}]^{W_n}$$

where W_n is the Wile group. The $\mathbb{C}$-algebara homomorphism from the Hecke algebra $\mathcal{H}$ to $\mathbb{C}$ is parametrized by the value α_j of X_j, the Satake parameter. For Hecke eigenform F the spinor L-function is defined by

$$L(s, F, spin) = \prod_{r=0}^{n} \prod_{1 \le i_1 < \cdots < i_r \le n} (1 - \alpha_0 \alpha_{i_1} \ldots \alpha_{i_r} p^{-s})$$

and standard L-function is

$$L(s, F, st) = (1 - p^{-s})^{-1} \prod_{i=1}^{n} (1 - \alpha_i p^{-s})^{-1} (1 - \alpha_i^{-1} p^{-s})^{-1}.$$

We note that, by (10), (11), the definition of spinor L-function coincides with the classical L-function. In GSp_2 case, the classical one was called Andrianov L-function. In $GL_2 = GSp_1$ case, the classical L-function coincides with the L-function of the Galois representation associated to elliptic modular form.

Fourier expansion: We fix the standard additive character ψ on $\mathbb{Q}\backslash\mathbb{A}$ ($\psi_\infty(x) = \exp(2\pi i x)$ for $x \in \mathbb{R}$). For automorphic form F and a symmetric matrix $T \in M_2(\mathbb{Q})$, letting

$$F_T(g) = \int_{U_1(\mathbb{Q})\backslash U_1(\mathbb{A})} \psi(-tr(S \cdot T)) F(\begin{pmatrix} 1 & S \\ & 1 \end{pmatrix} g) dS,$$

the Fourier expansion of F is given by

$$F(g) = \sum_{T \in Sym_2(\mathbb{Q})} F_T(g).$$

The Fourier expansion of f on $GL_2(\mathbb{A})$ is $f(g) = \sum_{a \in \mathbb{Q}} f_a(g)$, similarly.

(CASE 1) In this case,

$$F_0(e_1(g)) = f(g) \neq 0. \tag{12}$$

Suppose that F is an eigenform. Then there exists $\delta \in (\widehat{\mathbb{Q}\backslash\mathbb{A}})^\times$ such as

$$F_0(\begin{pmatrix} {}^t g^{-1} & \\ & g \end{pmatrix} \begin{pmatrix} 1_2 & \\ & t \cdot 1_2 \end{pmatrix}) = \delta(t) F_0(\begin{pmatrix} {}^t g^{-1} & \\ & g \end{pmatrix}). \tag{13}$$

Since F_0 and f have the informations of L-paramerters of themselves, by comparision of the actions of Hecke operators on them, we can obtain:

Theorem 2.1. *Under (12), f is also an eigenform outside of bad primes of F. And it holds*

$$\begin{aligned} L(s, F, \omega_f \delta, spin) &= \zeta(s-3) L(s-3, \omega_f^{-1}) L(s-3, f), \\ L(s, F, st) &= \zeta(s) L(s, f) L(s, f, \omega_f^{-1}) \end{aligned}$$

where $L(s, F, \omega_f \delta, spin)$ is the $\omega_f \delta$-twist of $L(s, F, spin)$, and $L(s, F, st)$ is the standard L-function of F.

If F is holomorphic, then by (12) the followings should hold

- f is a nonzero constant.
- $\kappa_1 = \kappa_2$, i.e., F is scalar valued, of weight κ_1.
- $\chi_1 = \chi_2$.

In particular, the L-functions of F are described as follows:

Corollary 2.2. *Assume (12) and that F is holomorphic. Then, at every good place of F,*

$$\begin{aligned} L(s,F,spin) &= \zeta(s)L(s-1,\chi_1)\zeta(s-\kappa_1+2)L(s-\kappa_1+3,\chi_1), \\ L(s,F,st) &= \zeta(s)L(s,\chi_1)L(s,\chi_1^{-1})L(s-1,\chi_1)L(s-1,\chi_1^{-1}). \end{aligned}$$

(CASE 2) In this case, we exclude the (CASE 1). Then,

$$F_{T_\alpha}(g) \neq 0$$

with $T_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$ for some $\alpha \in \mathbb{Q}^\times$. For $x \in \mathbb{A}$,

$$F_{T_\alpha}(\begin{pmatrix} 1 & & x & * \\ * & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} e_2(g_\infty)) = \psi(\alpha x)F_{T_\alpha}(e_2(g_\infty)). \tag{14}$$

The Fourier expansion of $f = \Phi_2(F)$ is

$$f(g) = \sum_{\alpha\in\mathbb{Q}} F_{T_\alpha}(e_2(g)).$$

Suppose that F is an eigenform. Then, it holds that, for some $\chi_2, \delta \in \widehat{\mathbb{Q}^\times\backslash\mathbb{A}^\times}$,

$$F_{T_\alpha}(\begin{pmatrix} a & & & \\ & b & & \\ & & a & \\ & & & b^{-1}a^2 \end{pmatrix} e_2(g_\infty)) = \omega_f(a)\chi_2(b)\delta(a)^2 F_{T_\alpha}(e_2(g_\infty)) \tag{15}$$

for every $a, b \in \mathbb{A}^\times$. As well as (CASE 1), by the comparison of the actions of Hecke operators on F_{T_α} and f, we can obtain:

Theorem 2.3. *Under the above assumptions,*
i) If $\chi_{2\infty}(z) \neq z^2$ or $\chi_{2p}(p) \neq -p^{-2}$, then f is also an eigenform at p with

$$L(s,F,spin) = L(s,f,\chi_2^{-1})L(s-m_2+2,f), \tag{16}$$
$$L(s,F,st) = \zeta(s)^2 L(s+m_2,\chi_2^{-1})L(s,f\otimes(f\times\chi_2^{-1})) \tag{17}$$

at p. Here m_2 is the index of $\chi_{2\infty}$, and $\otimes$ means the Rankin-Selberg convolution.
ii) Ohterwise, although $f|_{SL_2(\mathbb{Q}_p)}$ is still an eigenform at p, but, not an eigenform on $GL_2(\mathbb{Q}_p)$ in general. However, there exists an eigenform f' which satsifies

- $\omega_{f'} = \omega_f$,
- $f'(g) + f'({}^tg^{-1}) = f(g)$.

- f' *keeps the relation* (16), (17)*, in stead of* f*.*

Remark 2.4. In the case that F is holomorpchic and its highest weight is (κ_1, κ_2), the index m_2 in Theorem 2.3 is κ_2, and f is of weight κ_1.

Summing up the above results,

Theorem 2.5. *If a cuspform and a non-cuspform have the same spinor L-function, then the L-function is one of the following form*

- *(CAP type)* $\zeta(s-\frac{\kappa}{2}+1)L(s+\frac{\kappa}{2},\omega_f^{-1})L(s,f)$ *for some automorphic form* f *on* $GL_2(\mathbb{A})$ *of weight* κ*, or*
- *(Base change lift type)* $L(s,f)L(s,f,\chi_k)$ *for some automorphic form* f *on* $GL_2(\mathbb{A})$ *and the quadratic character* χ *associated a quadratic extension* $k/\mathbb{Q}$*.*

Remark 2.6. Conversely, we give many pairs of cuspform and non-cuspform having the same spinor L-function in [9].

In the $GL(2)$-case, the cuspidality of automorphic form is characterized by the entireness of L-function. Combining the above results and Kudla-Rallis [5], in the $GSp(2)$-case, we can characterize, similarly.

Theorem 2.7. *For an irreducible tempered* Π *on* $Sp_2(\mathbb{A})$ *which is not CAP,* Π *is cuspidal, if and only if the following two conditions are satisfied.*

- $\mathrm{ord}_{s=1}L(s,F,\eta,st) \geq -1$ *for any* $\eta \in \widehat{\mathbb{Q}^\times\backslash\mathbb{A}^\times}$ *such that* $\eta^2 = 1$*.*
- $\mathrm{ord}_{s=1}L(s,F,\eta,st) \geq 0$ *for any* $\eta^2 \neq 1$*.*

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