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A Headache-Causing Problem

Conway (J.H.), Paterson (M.S.), and Moscow (U.S.S.R.)

Abstract. After disproving the celebrated Conway–Paterson–Moscow theorem [1], we prove that theorem and make an application to a well-known number-theoretic problem.

Setting. When just N men have been gathered together in the room shown in Figure 1, the blind lady umpire makes the following (true) announcement:

"We are all, as we know, infinitely intelligent and honourable people. Now the janitor, acting on my instructions, has attached to your foreheads small discs bearing the usual notations for various non-negative integers, in such a way that each of you can see the number on everyone else's head, but not that on your own. The sum of all these numbers is one of the numbers you can now see me writing on the blackboard."

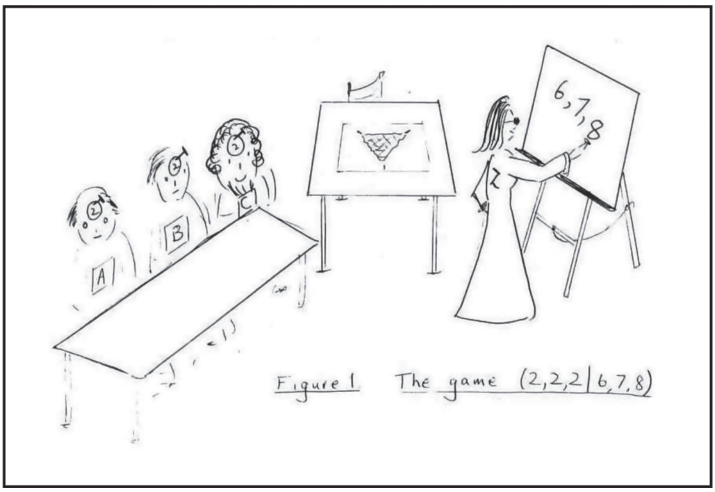

Figure 1. The game $(2, 2, 2 \,|\, 6, 7, 8)$.

"I regret the slight discomfort this proceeding must have caused you—fortunately the theorem of [1] assures us that it will only last for a few more moments. I will now question each of you in turn, and at the first 'Yes' answer we can all go out and enjoy what is left of this lovely afternoon."

She now asks:

"Arthur, can you deduce solely from this information what number is written on your disc?"

If Arthur's reply is "No", she will turn to the next man, and ask:

"Bertram, can you deduce from the above information together with Arthur's reply what number must be written on *your* disk?"

If Bertram in turn says "No", she will question Charles, Duncan, and so on, perhaps reaching the Nth man:

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“Engelbert, can you now deduce your number from the given information together with all the replies you have now heard?”

If even Engelbert says “No”, she will return to Arthur, and continue cyclically, always asking the same question:

“Are you now able to deduce your number solely from the given information and the replies you have heard so far?”
until the game is terminated by a “Yes” reply.

Statement of the theorem. *If the number of numbers written on the blackboard is less than or equal to the number N of men, the game will terminate after a finite number of questions.*

The disproof (commenced). It has become customary (see, e.g., [1]) to present the disproof of this theorem before its proof. The disproof runs as follows.

To demolish this perfectly preposterous proposition, it will suffice of course to disprove any particular case. We shall take the case when $N = 3$, the number on every head is 2, and the numbers on the blackboard are 6, 7, and 8, and establish conclusively that it will never end. We shall refer to this as the case $(2, 2, 2 \mid 6, 7, 8)$.

It will help us to imagine Charles at his breakfast table that morning.

“Oh dear, yet another invitation from Zoe. She’s a lovely girl, and intelligent too, but I do wish she wouldn’t keep dragooning us into playing those ridiculous party games.”

“Wonder what it’ll be this time? I’d better just run through the infinitely many possibilities so we’ll be able to get it over with as quickly as possible. If it’s Charades I’ll repeat exactly the one I did last time—they’re *bound* to get that first go. If it’s Hunt-the-Slipper I’ll . . .

. . .   .

. . . But then again, she might just be thinking of playing the game described so wittily in [1]. In that case, she’s as likely as not to use the particular case called $(2, 2, 2 \mid 6, 7, 8)$ in the handy notation of that reference. What should my reactions be?”

Charles’s Argument. “Let me think now. Since I see two heads numbered 2, I will know from the start that my number will be 2, 3, or 4. Let’s consider these cases.”

“*If my number’s* 2, Arthur will have concluded that *his* number was 2, 3, or 4, and since each of these is consistent with all he was told, he’ll have to say ‘No’.”

“Bertram’s then in a similar position. He’ll think ‘If I have 2, Arthur, by Charles’s argument of the previous paragraph, will say ‘No’. If 3, Arthur will instead have been able to conclude only that his number was 1, 2, or 3, and to say ‘No’ because he’ll only know that his number is 0, 1, or 2. Must remember that she said *non-negative* numbers so that 0 is allowed.’ ”

“Since in this case Bertram won’t be able to eliminate even one of his three possibilities 2, 3, 4, he’ll be forced to say ‘No’. That disposes of the case when my number is 2.”

“*If my number is* 3, Arthur fairly obviously still says ‘No’. Bertram will know probably in condensed form:

‘I can see $A = 2$, $C = 3$, so I know $B = 1, 2$, or 3.

If $B = 1$, A’ll’ve been torn between 2, 3, 4, so’ll’ve said ‘No’.

If $B = 2$, A’ll’ve been torn between 1, 2, 3, so’ll’ve said ‘No’.

If $B = 3$, he’ll’ve been torn between 0, 1, 2, so’ll still’ve said ‘No’. I must therefore say ‘No’ myself, since all three cases are consistent with A’s ‘No’ answer.’ ”

"Bertram and Arthur will also both say 'No' when my number is 3. I think I can prove along the same lines that they'll both say 'No' even when it's 4. But I don't need to check this—my first answer must be 'No' because both 2 and 3 are consistent with the two 'No' answers I'm sure to hear."

"I plainly don't need to consider much more of this stuff—I reckon we'll all go home after saying 'No' half-a-dozen times, and I still won't know what my number is."

The disproof completed. Charles's argument, and various portions of it, can be used to complete with absolute rigor that each of the three players knows from the start that each of the first three answers is going to be "No". *So if they all know what those answers are going to be what information can they possibly gain by hearing them ritually intoned?* At the start of the second round, they will have learned nothing that they did not already know, and so the game will obviously go on forever.

The proof (commenced). We might as well make it clear now that Zoe, the blind lady umpire, is herself ignorant of the numbers fixed to those heads, although she knows, of course, what numbers she wrote on the blackboard. In the interests of good order she will naturally list all situations that are compatible with the numbers she has heard up to any given time, and will strike a situation off her list when and only when she knows that the corresponding game would terminate at the current question. Of course she knows just when this will be, for being infinitely intelligent she can perfectly well imagine herself in the position of the player she is currently addressing in any possible situation.

By a *possible situation* we mean of course an N-tuple of numbers

$$(a, b, c, \ldots, n)$$

which might be the numbers on the respective heads of

$$(A, B, C, \ldots, N)$$

and would have caused 'No' answers to all questions before the current one. We shall call such a situation *ongoing* only if the answer to the current question will also be 'No'.

We claim that Zoe can work out exactly which situations are ongoing by the following argument:

"Let me suppose that the current question is addressed to B. Then certainly

i) I *cannot* strike off $(a, b, c, \ldots, n)$ if there's an *accompanying* situation $(a, b', c, \ldots, n)$ still present with the same values of $a, c, \ldots, n$ but with b' differing from b, for then B cannot eliminate either of the numbers b and b'.

ii) I *can* strike off $(a, b, c, \ldots, n)$ if there's no such accompanying situation, for then B, who can see the numbers $a, c, \ldots, n$ will know that b is the only possible value for his number."

"So when I receive a 'No' answer from B (say), I must strike off those and only those points $(a, b, c, \ldots, n)$ from my N-dimensional record that are unaccompanied by any other point

$$(a, b', c, \ldots, n)$$

in the B-direction."

Since Zoe's argument covers all cases, we can now follow her algorithm to discover exactly which situations will cause the game to end at any given time.

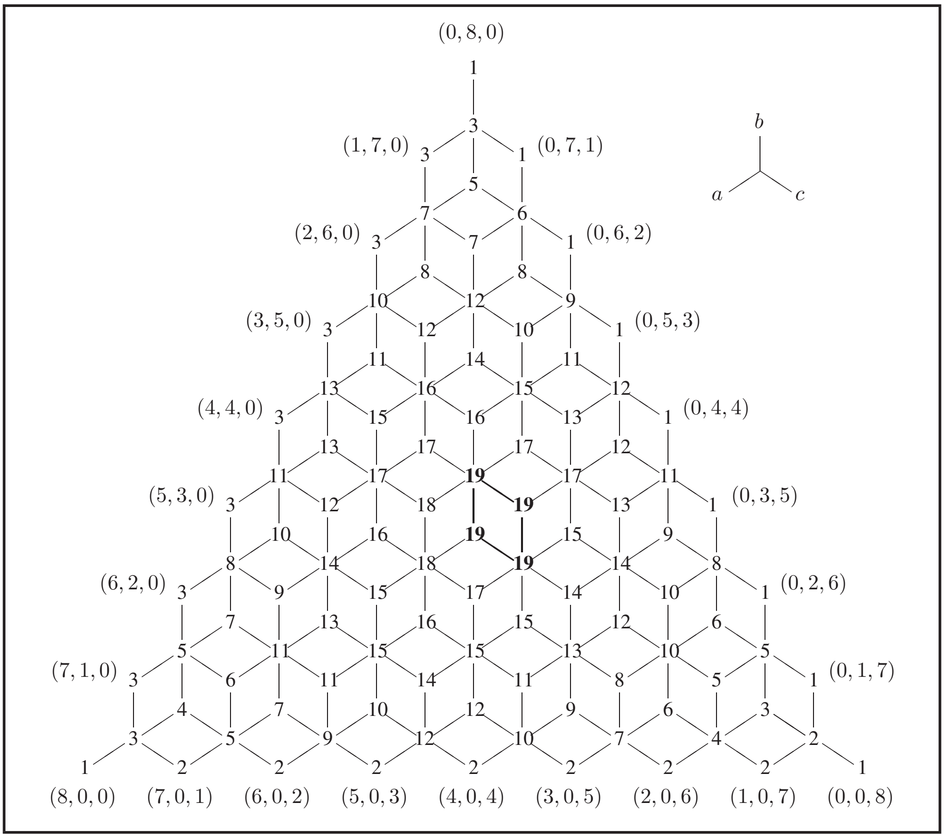

Figure 2. Zoe's notes for all games $(a, b, c \mid 6, 7, 8)$. The coordinate directions for (a, b, c) are indicated in the upper right.

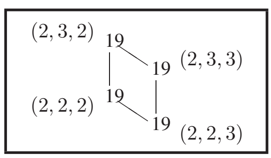

Figure 3. At the end of the game (this detail is highlighted in the center of Figure 2).

The case (2, 2, 2 | 6, 7, 8). Before we resume the proof for the general case, we illuminate the fate of all games of the form $(a, b, c \mid 6, 7, 8)$ in Figure 2. This figure shows an orthogonal projection of the set of all points in (A, B, C) space that yield sums of 6, 7, or 8, together with Zoe's notes as to the number of questions whose 'Yes' answer terminates the game.

The four entries '19', singled out in Figure 3, enable us to verify both of Charles's predictions about the game (2, 2, 2 | 6, 7, 8).

The proof completed. We return now to the discussion of the general case. No matter what numbers are written on the blackboard, provided that there are at most N of them the total number of initial situations will surely be finite. We shall show that no one of these situations can remain ongoing at every possible moment in time.

For otherwise the set of permanently ongoing situations would be non-empty and have the property that every one of its points would be accompanied in every coordinate-direction. It will suffice to show that any such set of points in N dimensions has at least $N + 1$ distinct sums, and we can verify this by induction.

Let a_0 be the least value of the A-coordinate of any permanently ongoing situation in an N-man game. Then the tuples of $N - 1$ numbers

$$(b, c, \ldots, n)$$

for which $(a_0, b, c, \ldots, n)$ is permanently ongoing in this game will themselves form a permanently ongoing set in an $N - 1$ man game, and so will have at least $(N - 1) + 1 = N$ distinct sums. Let

$$s_0 = a_0 + b_0 + c_0 + \cdots + n_0$$

be the greatest of these, arising from the permanently ongoing situation

$$(a_0, b_0, c_0, \ldots, n_0).$$

Then there is a permanently ongoing situation

$$(a, b_0, c_0, \ldots, n_0)$$

accompanying this in the A-direction with $a \neq a_0$, and so $a > a_0$ since a_0 was minimal. The accompanying situation therefore has coordinate-sum greater than any of those already found, and establishes that there must be at least $N + 1$ distinct coordinate-sums in all.

Application to a problem of Fermat. The problem referred to is Fermat's famous assertion that

$$a \geq 1, \quad b \geq 1, \quad c \geq 1, \quad n \geq 3 \qquad \Longrightarrow \qquad a^n \neq b^n + c^n \qquad (\star)$$

for rational integers a, b, c, n.

Now it is well known that for every proposition P, we have

$$(\ P \text{ and not-}P\ ) \Longrightarrow (\ 0 = 1\ )\ .$$

Taking P to be the proposition discussed so disarmingly in [1], and applying *modus ponens*, we deduce that

$$0 = 1\ .$$

Now adding 1 to both sides of this, we obtain

$$1 = 2\ ,$$

which we prefer to write in the more revealing form

$$1^3 = 1^3 + 1^3\ .$$

Thus the lexicographically first case of ($\star$) is disproved. The authors cannot resist the remark that this would surely have been noticed earlier had not modern teaching methods preferred the elaboration of grandiose general theories to the inculcation of elementary arithmetical skills.

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This article was originally presented in a "Festschrift" published on the occasion of Hendrik Lenstra's Ph.D. defense in Amsterdam on May 18, 1977. The privately published book was entitled Een Pak met een Korte Broek: Papers presented to H. W. Lenstra, Jr. on the Occasion of the Publication of his "Eulidische Getallenlichamen." It was edited by P. van Emde Boas, J. K. Lenstra, F. Oort, A. H. G. Rinnooy Kan, and T. J. Wansbeek who have approved the publication of this article in the MONTHLY. *We are excited to present this paper to you and, as P. van Emde Boas stated, "this paper has had a major impact on the early developments of epistemic logic in Amsterdam." The text is unaltered. The* MONTHLY *would like to express its gratitude to Dierk Schleicher for all of his efforts in getting this article to publication.*

REFERENCES

[1] Conway, J.H., Paterson, M.S., Moscow, U.S.S.R. (1977). *Een Pak met een Korte Broek: Papers presented to H. W. Lenstra, Jr. on the Occasion of the Publication of his "Eulidische Getallenlichamen."* Amsterdam: privately published.

CONWAY, J.H. is the John von Neumann Distinguished Professor of Mathematics at Princeton University, and embodies the playful spirit in mathematics. He has worked in many areas of mathematics, notably group theory, knot theory, game theory, and geometry. He is best known for his work on the Game of Life and the Free Will Theorem, but the contribution of which he is most proud is the discovery of the surreal numbers. He is always happy to engage in any kind of mathematical game, including Go, Phutball (his own invention) or even dots-and-boxes.
Department of Mathematics, Princeton University, Princeton, NJ 08544

PATERSON, M.S. (Ph.D., FRS) took degrees in mathematics at Cambridge University and rose to fame as the co-inventor with John Conway of Sprouts. His lifelong fascination with computer science began in the mid-60s with the arrival in Cambridge of a massive new computer with up to 1 megabyte of memory. He evolved from president of the Trinity Mathematical Society to president of the European Association for Theoretical Computer Science, and migrated from MIT to the University of Warwick, where he has been in the Computer Science department for 48 years.
Department of Computer Science, University of Warwick, Coventry CV4 7AL, UK
m.s.paterson@warwick.ac.uk

MOSCOW, U.S.S.R is the capital of Russia, a major urban center with population of about 15 million residents, and a city with nearly 900 years of history. It is the northernmost and the coldest megacity on Earth. Moscow is the largest science center in Russia, the home of the Russian Academy of Sciences and of its numerous research institutes. It has roughly 200 institutions of higher education, including the flagship Lomonosov Moscow State University, founded in 1755.
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