A Straightforward Proof of Descartes's Circle Theorem

PAUL LEVRIE

There are probably not many formulas in mathematics in the discovery of which a princess was instrumental and that are described in a poem written by a Nobel laureate (in Chemistry!). Yet such is the case for what is now known as Descartes's circle formula.

Let C_1, C_2, and C_3 be mutually tangent circles with radii r_1, r_2, and r_3.

Let us assume that the radius of a fourth circle tangent to the other three, the red circle in Figure 1, is r_4.[1] Descartes's circle theorem asserts the following:

THEOREM. *The radii r_1, r_2, r_3, r_4 of four mutually tangent circles satisfy*

$$2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2. \qquad (1)$$

The problem of finding the radius of the fourth circle is a special case of a problem of Apollonius [2]: given three circles, construct the circles tangent to all three circles.[2]

We will give a straightforward proof of Descartes's theorem, using only elementary algebra and Heron's formula for the area of a triangle.

A Short History of Descartes's Circle Theorem

Descartes's circle theorem was first described by Descartes in 1643 in his correspondence with Princess Elisabeth of Bohemia, one of his pupils [5]. In a letter to her, Descartes posed the following problem [4]:

« A M. la Princesse Elizabeth, etc. Touchant le Probleme : trois cercles estant donnez, trouuer le quatriéme qui touche les trois »

which is Apollonius's problem. Descartes soon realized that this might be too difficult, and he reduced the problem to the case that the three given circles are mutually tangent. He also gave the following solution [4]:

$$\begin{array}{lcl} ddeeff & \infty & 2deffxx + 2deeffx \\ +\, ddeexx & & +\, 2deefxx + 2ddeffx \\ +\, ddffxx & & +\, 2ddefxx + 2ddeefx \\ +\, eeffxx, & & \end{array}$$

Here e, f, and g are the given radii, and x is the radius to be found.

Exercise for the reader: Show that this solution is equivalent to (1).

The formula was rediscovered and proved by Jakob Steiner in 1826, and again by Philip Beecroft in 1842. In 1936, Frederick Soddy, who received the Nobel Prize in chemistry in 1921, rediscovered the result, and wrote a poem about it [6] (reprinted here by permission):

[1] All the figures were produced by the author.

[2] This is a problem at which a countess tried her hand, Countess Skorzewska [1, p. 308]. Lambert describes her in his correspondence as a learned Polish lady and a great lover (''Liebhaber'') of the mathematical sciences.

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The Kiss Precise

For pairs of lips to kiss maybe
Involves no trigonometry.
'Tis not so when four circles kiss
Each one the other three.
To bring this off the four must be
As three in one or one in three.
If one in three, beyond a doubt
Each gets three kisses from without.
If three in one, then is that one
Thrice kissed internally.

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

F. SODDY

After the publication of the poem another verse was added to it by Thorold Gosset [7], the generalization of the result to n dimensions.

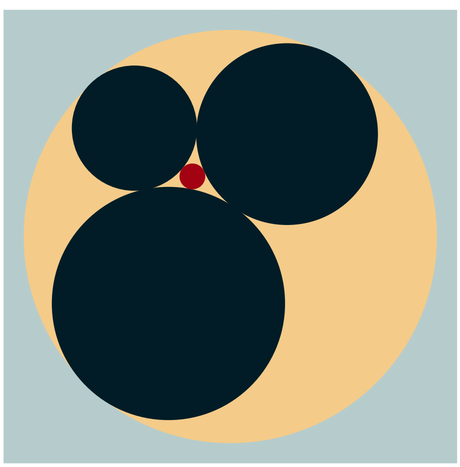

Figure 1. Four mutually tangent circles for Descartes's circle theorem.

More recently, Lagarias et al. [8] proved that a relation very similar to (1) relates the centers of the tangent circles in the complex plane.

The proof of (1) by Steiner [9] uses a result about Pappus chains and a generalization of Viviani's theorem to general triangles. Beecroft's proof makes use of four other mutually tangent circles through the points where the four circles meet. It was later simplified by Coxeter [2]. Coxeter himself gave a new proof, based on inversion with respect to a circle [3]. Pedoe [10] lists some other proofs, one based on a symmetry argument, another using Grassmann calculus. None of these proofs is particularly straightforward. It is not known what path Descartes and Elisabeth followed to derive their result.

The Proof (That Descartes Missed?)

The proof given here is based on Heron's formula [11] for the area of a triangle with sides a, b, and c:

$$\text{area} = \sqrt{o(o-a)(o-b)(o-c)}, \quad \text{where } o = \frac{a+b+c}{2}.$$

This formula was known to Descartes and Princess Elisabeth, and they both probably used it in trying to solve the problem of the touching circles.

If we connect the centers of these four circles (see Figure 2), four triangles are formed, the area of the largest triangle being the sum of the areas of the other three. We can write this out using Heron's formula:

$$\begin{aligned}\sqrt{r_1r_2r_3(r_1+r_2+r_3)} &= \sqrt{r_1r_2r_4(r_1+r_2+r_4)}\\ &\quad+\sqrt{r_1r_3r_4(r_1+r_3+r_4)}+\sqrt{r_2r_3r_4(r_2+r_3+r_4)}.\end{aligned} \tag{2}$$

Note that there is also a circle touching the three given circles externally. If we assume that the radius of this circle is taken to be a negative number, then in the configuration of Figure 2, the same equation (2) is satisfied, as can be seen in the right-hand figure.

Solving equation (2) in the traditional way by repeatedly squaring leads to enormous calculations.[3] However, by carefully simplifying at each step, it is possible to get the result in one page.

In the sequel we will use the following notation:

$$s = r_1+r_2+r_3+r_4, \quad p = r_1r_2r_3r_4,$$
$$t = \frac{p}{s}, \quad u = \frac{1}{s}.$$

Furthermore, let

[3] As Descartes writes in his letter to Elisabeth dated November 29, 1643 [5], "But this route seems to me to lead to so many superfluous multiplications that I would not want to undertake to solve them in three months."

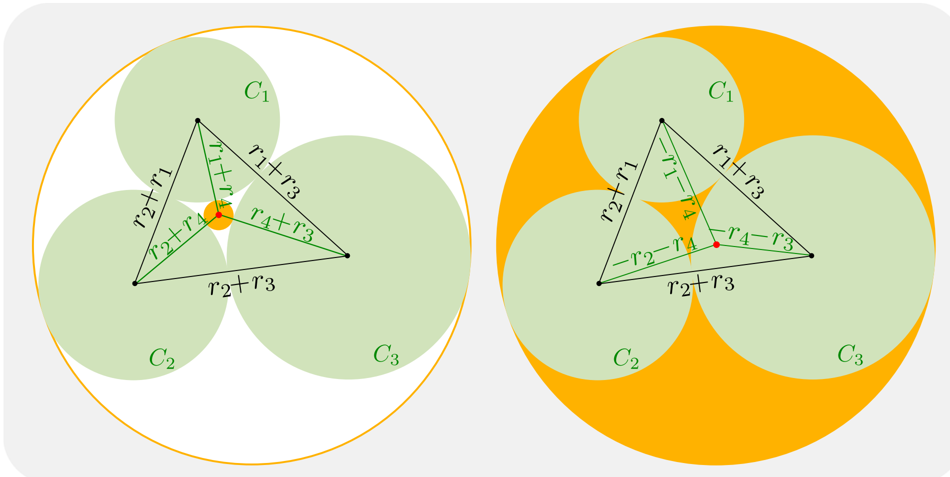

Figure 2. Three mutually touching circles C_1, C_2, and C_3 and the two solutions of Apollonius's problem.

$$\alpha = \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4},$$
$$\beta = \frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}.$$

Using this notation, we can rewrite (2) as

$$\sqrt{r_1r_2r_3s-p} = \sqrt{r_1r_2r_4s-p}+\sqrt{r_1r_3r_4s-p}+\sqrt{r_2r_3r_4s-p}.$$

We now divide by $\sqrt{s}$ and rearrange:

$$\sqrt{r_1r_2r_3-t}-\sqrt{r_1r_2r_4-t} = \sqrt{r_1r_3r_4-t}+\sqrt{r_2r_3r_4-t}.$$

Squaring both sides and rearranging leads to

$$r_1r_2r_3+r_1r_2r_4-r_1r_3r_4-r_2r_3r_4 = 2\left(\sqrt{r_1r_2r_3-t}\sqrt{r_1r_2r_4-t}+\sqrt{r_1r_3r_4-t}\sqrt{r_2r_3r_4-t}\right).$$

We divide this result by p:

$$\frac{1}{r_4}+\frac{1}{r_3}-\frac{1}{r_2}-\frac{1}{r_1}=2\left(\sqrt{\frac{1}{r_4}-u}\cdot\sqrt{\frac{1}{r_3}-u}+\sqrt{\frac{1}{r_2}-u}\cdot\sqrt{\frac{1}{r_1}-u}\right).$$

Again we square both sides:

$$\beta+\frac{2}{r_3r_4}-\frac{2}{r_2r_4}-\frac{2}{r_1r_4}-\frac{2}{r_2r_3}-\frac{2}{r_1r_3}+\frac{2}{r_1r_2} = \frac{4}{r_3r_4}+\frac{4}{r_1r_2}-4\alpha u+8u^2 + 8\sqrt{\frac{1}{r_4}-u}\cdot\sqrt{\frac{1}{r_3}-u}\cdot\sqrt{\frac{1}{r_2}-u}\sqrt{\frac{1}{r_1}-u}.$$

which after rearranging becomes

$$\beta-\frac{2}{r_3r_4}-\frac{2}{r_2r_4}-\frac{2}{r_1r_4}-\frac{2}{r_2r_3}-\frac{2}{r_1r_3}-\frac{2}{r_1r_2}+4\alpha u-8u^2 = 8\sqrt{\frac{1}{r_4}-u}\cdot\sqrt{\frac{1}{r_3}-u}\cdot\sqrt{\frac{1}{r_2}-u}\cdot\sqrt{\frac{1}{r_1}-u}.$$

Note that since

$$\alpha^2-\beta=\frac{2}{r_3r_4}+\frac{2}{r_2r_4}+\frac{2}{r_1r_4}+\frac{2}{r_2r_3}+\frac{2}{r_1r_3}+\frac{2}{r_1r_2},$$

we can rewrite this equation as

$$(2\beta-\alpha^2)+4\alpha u-8u^2=8\sqrt{\frac{1}{r_4}-u}\cdot\sqrt{\frac{1}{r_3}-u}\cdot\sqrt{\frac{1}{r_2}-u}\cdot\sqrt{\frac{1}{r_1}-u}. \tag{3}$$

Squaring both sides (again), we find for the left-hand side,

$$(2\beta-\alpha^2)^2+8(2\beta-\alpha^2)\alpha u-16(2\beta-\alpha^2)u^2+16\alpha^2u^2 -64\alpha u^3+64u^4.$$

The right-hand side is given by

$$\frac{64}{r_1r_2r_3r_4}-64\left(\frac{1}{r_1r_2r_3}+\frac{1}{r_1r_2r_4}+\frac{1}{r_1r_3r_4}+\frac{1}{r_2r_3r_4}\right)u +32(\alpha^2-\beta)u^2-64\alpha u^3+64u^4.$$

Note that the first two terms cancel. The terms containing u^2, u^3, and u^4 are the same on both the left- and right-hand sides. Hence after rearranging, we get

$$(2\beta-\alpha^2)^2+8(2\beta-\alpha^2)\alpha u=0,$$

or equivalently,

$$(2\beta-\alpha^2)\cdot(2\beta-\alpha^2+8\alpha u)=0.$$

The second factor cannot be zero, since in that case, we would have $2\beta-\alpha^2=-8\alpha u$, which would result in a negative left-hand side in (3). Hence we have $2\beta=\alpha^2$, or

$$2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2.$$

Note that given r_1, r_2, and r_3, this is a quadratic equation in r_4 with two solutions: the radii of the two tangent circles in Figure 2, called the inner and outer Soddy circles.

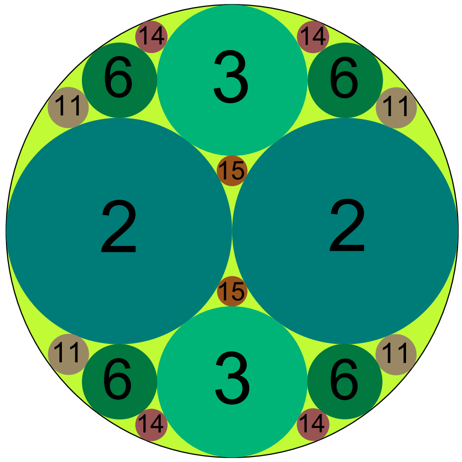

Figure 3. The largest circles in an integral Apollonian circle packing.

A Remark on Circle Packings

The problem of the "kissing circles" and Descartes's circle theorem are as current today as they were some four hundred years ago. To give but one example, Descartes's formula plays an important role in the theory of circle packings in the plane. A circle packing is an arrangement of circles that all touch one another. A special case is that of Apollonian circle packings, which are constructed by starting with three mutually touching circles and adding the two circles tangent to the first three. Taking one of those two and combining it with two of the original circles leads to a similar situation in which we can find two new circles tangent to them. And we can continue in the same way. The first steps in such a process can be seen in Figure 3.

The numbers in the circles denote their curvatures (1/radius). It is a nice consequence of the form of Descartes's circle formula that if we start with four circles with integral curvature, one of them being negative, all the other curvatures will be integers too. Such a packing is called an integral Apollonian circle packing [12].

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Faculty of Applied Engineering
UAntwerpen
Groenenborgerlaan 171
2020 Antwerp, Belgium
and
Department of Computer Science
KU Leuven, Celestijnenlaan 200A
P.O. Box 2402
3001 Heverlee (Leuven), Belgium
e-mail: paul.levrie@cs.kuleuven.be

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