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Bar Bets and Generating Functions: The Distribution of the Separation of Two Distinct Card Ranks

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Abstract. What is the probability two distinct ranks (say, a king and a five) will appear within a card of one another in a shuffled deck? An old bar bet is predicated on it being nearly guaranteed. By modeling the outcome using a finite automaton, we construct the probability generating function for card rank separation. We then derive the corresponding probability mass function, mean, and variance. We conclude with an unimpressive modification to make the scam a virtual lock.

1. INTRODUCTION. The *rank* of a playing card is its numeric or face value. The five of hearts has rank "five," the king of diamonds has rank "king," and so on. An old bar bet works as follows. You hand a deck of cards to a mark and ask him or her to shuffle it and then to name any two distinct ranks. Next, you claim you can magically force at least one occurrence of the two ranks to appear within one card of each another. If successful, the mark buys you a drink. Lounge lore claims this works "99 percent of the time [2]."

Several authors in the literature of recreational mathematics [4, 8, 9] and in online forums [3, 7, 10] have worked out the probability that two distinct ranks are adjacent. A few others have even considered the probability that the two ranks are separated by at most one card [5, 6]. We aim to take this a step further and derive the probability mass function for separation by s cards, as well as compute the mean separation and its variance.

For clarity of discussion, we say two cards in a deck are at *separation* s if there are s other cards between them. For example, if the subsequence

$$\ldots, \mathrm{K}\heartsuit, 9\clubsuit, \mathrm{J}\spadesuit, \ldots$$

appears in the deck, we say that the king of hearts and the jack of spades are separated by one card, or at separation 1. We say two distinct ranks are at separation s if that is the smallest separation between all instances of the two ranks in the deck. For example, to find the separation of the king and jack ranks, we note the separation of all 16 distinct king-jack pairs in the deck and take the smallest. If we let S be the separation of two designated ranks, our goal is to work out the distribution of S.

2. AN ELEMENTARY APPROACH. Without loss of generality, let us continue to use the king and jack as the designated ranks. Consider the event $\{S \geq s\}$. In a well-shuffled deck, all 52! permutations of the cards are equally likely, which provides a denominator for the desired probability. For the event in question to occur, we can have a favorable configuration by the following construction:

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- Choose four locations out of 52 for the jacks in $\binom{52}{4}$ ways. Call the four chosen locations i, j, k, ℓ, in increasing order. Note that the highest values for i, j, k, ℓ are respectively 49, 50, 51, 52.
- Permute the four jacks of the deck over these locations in 4! ways.
- There are $i-1$ locations to the left of position i. If i is large enough (namely *larger* than $s+1$), then we have positions to the left of i to place kings at separation at least s. Precisely, there are $i-1-s$ positions to the left of i to place kings at separation s, if i is large enough, and none if i is not large enough. We can say there are $\max(i-s-1, 0)$ positions where we can place kings at separation at least s from the jack at position i.
- Likewise, we can place kings at separation at least s at any of $\max(j-i-1-2s, 0)$ positions between i and j.
- Likewise, we can place kings at separation at least s at any of $\max(k-j-1-2s, 0)$ positions between j and k.
- Likewise, we can place kings at separation at least s at any of $\max(\ell-k-1-2s, 0)$ positions between k and ℓ.
- Likewise, we can place kings at separation at least s at any of $\max(52-\ell-s, 0)$ positions above ℓ.
- From all the candidate positions for the kings at separation at least s, choose four positions and permute the four kings over them in 4! ways.
- Now that the jacks and kings are placed at locations at least s apart, we can permute the remaining 44 cards in 44! ways over the remaining places.

Let $y(s)$ be the number of all candidate positions for the four kings. So, we have

$$\begin{aligned} y(i, j, k, \ell; s) = \max(i-s-1, 0) &+ \max(j-i-1-2s, 0) \\ &+ \max(k-j-1-2s, 0) + \max(\ell-k-1-2s, 0) \\ &+ \max(52-\ell-s, 0). \end{aligned}$$

Now we have the probability

$$\mathbb{P}(S \geq s) = \frac{4! \times 4! \times 44!}{52!} \sum_{i=1}^{49} \sum_{j=i+1}^{50} \sum_{k=j+1}^{51} \sum_{\ell=k+1}^{52} \binom{y(i, j, k, \ell; s)}{4}.$$

From this expression we can find the distribution of S, namely we have

$$\mathbb{P}(S = s) = \mathbb{P}(S \geq s) - \mathbb{P}(S \geq s+1).$$

While this leads to a solution that can easily be coded into a program, we had to very carefully work through multiple cases to arrive at the correct construction. Moreover, to obtain a closed-form expression for $\mathbb{P}(S = s)$, we will have to delicately break apart the nested summations according the behavior of $y(s)$. We propose an approach that frees us from this tedious bookkeeping.

3. A GENERATING FUNCTION APPROACH. We begin by noting that once we have selected two ranks, the ranks of the remaining cards are unimportant. Thus, we can recast this problem as working with a deck of 52 colored cards: 4 red, 4 green, and the remaining 44 blue.

Next, let us consider the alphabet $\mathcal{C} = \{r, g, b\}$—the letters standing in for the three colors in the obvious way—and $\mathcal{Q}_0$, the subset of words over $\mathcal{C}$ containing at least one occurrence of the letters r and g separated by at most s occurrences of the letter b. The finite automaton (FA) in Figure 1 accepts precisely those words that are members of $\mathcal{Q}_0$. Although FAs are typically depicted with one token per arc, for clarity and economy of size, we occasionally connect two states in our diagram with an arc labeled by the regular expression corresponding to the substring recognized by the (now hidden) intermediary states and arcs. For example, our FA transitions from state q_1 to q_0 whenever it encounters $s+1$ consecutive b symbols.

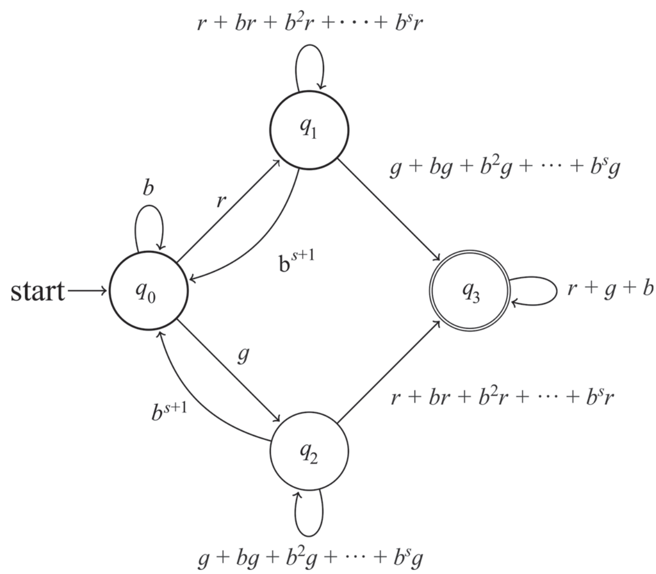

Figure 1. Subset $\mathcal{Q}_0$ of words over alphabet $\mathcal{C}$.

We seek the generating function $Q_0(z, u, v)$ where the coefficient of the term $z^n u^j v^k$, denoted by $[z^n u^j v^k]\, Q_0(z, u, v)$, is the number of words in $\mathcal{Q}_0$ of length n containing j occurrences of letter r and k occurrences of letter g.

Let $\mathcal{Q}_i$, $i \in \{1, 2, 3\}$, denote the subset of words accepted when starting from state q_i and $Q_i(z, u, v)$ be the associated counting generating function so that coefficient $[z^n u^j v^k]\, Q_i(z, u, v)$ is the number of words in $\mathcal{Q}_i$ of length n containing j occurrences of letter r and k occurrences of letter g. Then from Figure 1, we obtain the following system of generating functions:

$$\begin{aligned}
Q_0(z,u,v) &= zQ_0(z,u,v) + uzQ_1(z,u,v) + vzQ_2(z,u,v),\\
Q_1(z,u,v) &= z^{s+1}Q_0(z,u,v) + uz\left(\frac{1-z^{s+1}}{1-z}\right)Q_1(z,u,v)\\
&\qquad + vz\left(\frac{1-z^{s+1}}{1-z}\right)Q_3(z,u,v),\\
Q_2(z,u,v) &= z^{s+1}Q_0(z,u,v) + vz\left(\frac{1-z^{s+1}}{1-z}\right)Q_2(z,u,v)\\
&\qquad + uz\left(\frac{1-z^{s+1}}{1-z}\right)Q_3(z,u,v),\\
Q_3(z,u,v) &= 1 + (1+u+v)zQ_3(z,u,v).
\end{aligned}$$

Backward substitution gives us the desired expression for $Q_0(z, u, v)$ in terms of z, u, and v alone [1, pp. 56–58].

We are only interested in those members of $\mathcal{Q}_0$ with four r's and four g's. Hence we define the generating function

$$F^{\langle s \rangle}(z) \equiv [u^4 v^4]\, Q_0(z, u, v),$$

so that $[z^n]\, F^{\langle s \rangle}(z)$ is the number of words over alphabet $\mathcal{C}$ of length n containing four r's and four g's and at least one occurrence of the letters r and g separated by at most s occurrences of letter b. Using a symbolic manipulation system, we find

$$F^{\langle s \rangle}(z) = \frac{2z^8(35 - z^{s+1} - 3z^{2s+2} - 9z^{3s+3} - 9z^{4s+4} - 9z^{5s+5} - 3z^{6s+6} - z^{7s+7})}{(1-z)^9}. \tag{1}$$

Let us next define the generating function $G(z, u)$ so that $[z^n u^s]\, G(z, u)$ denotes the number of words over alphabet $\mathcal{C}$ of length n containing four r's and four g's and the minimum number of b's between any pair of letters r and g is s. This generating function is defined implicitly in terms of $F^{\langle s \rangle}(z)$ by

$$\frac{G(z, u)}{1-u} = \sum_{s \geq 0} F^{\langle s \rangle}(z) u^s.$$

Substitution of (1) into this relation and solving for $G(z, u)$ gives us

$$G(z, u) = \frac{2(1-u)z^8}{(1-z)^9} \left[\frac{35}{1-u} - \frac{z}{1-uz} - \frac{3z^2}{1-uz^2} - \frac{9z^3}{1-uz^3} \right.$$
$$\left. - \frac{9z^4}{1-uz^4} - \frac{9z^5}{1-uz^5} - \frac{3z^6}{1-uz^6} - \frac{z^7}{1-uz^7} \right]. \tag{2}$$

With generating functions (1) and (2) in hand, we have the means to address several questions about the distribution of S. For example, to find the expected value of S, let S_n denote the quantity S for a deck with four red, four green, and $n - 8$ blue cards. Then we have

$$\mathbb{E}\,[S_n] = \frac{[z^n]\ \frac{\partial}{\partial u} G(z, u)\big|_{u=1}}{[z^n] G(z, 1)},$$

and so in our case where $n = 52$ we obtain

$$\mathbb{E}\,[S] = \frac{[z^{52}]\ \frac{\partial}{\partial u} G(z, u)\big|_{u=1}}{[z^{52}]\, G(z, 1)} = \frac{59005603980}{\binom{52}{4}\binom{48}{4}} \approx 1.1201.$$

For the variance of S, we first find the second factorial moment of S_n, namely

$$\mathbb{E}\,[S_n(S_n - 1)] = \frac{[z^n]\ \frac{\partial^2}{\partial u^2} G(z, u)\Big|_{u=1}}{[z^n]\, G(z, 1)},$$

yielding

$$\mathbb{E}\,[S(S-1)] = \frac{[z^{52}]\ \frac{\partial^2}{\partial u^2} G(z, u)\Big|_{u=1}}{[z^{52}]\, G(z, 1)} = \frac{168872169804}{\binom{52}{4}\binom{48}{4}} \approx 3.2058.$$

Thus, the variance is given by

$$\mathbb{V}\mathrm{ar}(S) = \mathbb{E}[S(S-1)] + \mathbb{E}[S] - (\mathbb{E}[S])^2 = \frac{789111942650231327}{256938608269126875} \approx 3.0712.$$

To derive a closed-form expression for $\mathbb{P}(S = s)$, rewrite $G(z, u)$ in (2) as

$$G(z,u) = \frac{2(1-u)z^8}{(1-z)^9} \sum_{j=0}^{7} \frac{\alpha_j z^j}{1 - uz^j},$$

so that the probability generating function of S is given by

$$[z^{52}] \frac{G(z,u)}{\binom{52}{4}\binom{48}{4}} = \left[\binom{52}{4}\binom{48}{4}\right]^{-1} 2(1-u) \sum_{j=0}^{7} \alpha_j [z^{44-j}] \frac{1}{(1-z)^9(1-uz^j)}. \tag{3}$$

Let us consider the term

$$[z^{44-j}] \frac{1}{(1-z)^9(1-uz^j)}$$

in (3) for each j. When $j = 0$, we obtain

$$\frac{1}{1-u} [z^{44}] \frac{1}{(1-z)^9} = \frac{1}{1-u}\binom{52}{8}.$$

On the other hand, when $j \geq 1$, we find

$$\begin{aligned}
&[z^{44-j}] \frac{1}{(1-z)^9(1-uz^j)} \\
&= [z^{44-j}] \sum_{n\geq 0} \binom{n+8}{8} z^n \sum_{n \geq 0} [\![n \equiv 0 \pmod{j}]\!] u^{\frac{n}{j}} z^n \\
&= [z^{44-j}] \sum_{n\geq 0} \sum_{i=0}^{n} \binom{i+8}{8} [\![i \equiv n \pmod{j}]\!] u^{\frac{n-i}{j}} z^n \\
&= \sum_{i=0}^{44-j} \binom{i+8}{8} [\![i \equiv 44 \pmod{j}]\!] u^{\frac{44-i-j}{j}} \\
&= \sum_{i=0}^{\left\lfloor \frac{44-j}{j} \right\rfloor} \binom{52-j-ij}{j} u^i,
\end{aligned}$$

where $[\![\cdot]\!]$ is Iverson bracket notation for an indicator function.

Thus, (3) becomes

$$2\left[\binom{52}{4}\binom{48}{4}\right]^{-1} \left\{ 35\binom{52}{8} - (1-u)\left[\sum_{i=0}^{43} \binom{51-i}{8} u^i - 3\sum_{i=0}^{21} \binom{50-2i}{8} u^i \right.\right.$$

$$-9\sum_{i=0}^{13}\binom{49-3i}{8}u^i-9\sum_{i=0}^{10}\binom{48-4i}{8}u^i-9\sum_{i=0}^{7}\binom{47-5i}{8}u^i$$
$$-3\sum_{i=0}^{6}\binom{46-6i}{8}u^i-\sum_{i=0}^{5}\binom{45-7i}{8}u^i\Bigg]\Bigg\}. \quad (4)$$

Extracting the coefficient of the u^s term of (4) yields

$$\left[\binom{52}{4}\binom{48}{4}\right]^{-1}\left\{70\binom{52}{8}[\![s=0]\!]-2\left[\binom{51-s}{8}-\binom{52-s}{8}[\![s>0]\!]\right]\right.$$
$$-6\left[\binom{50-2s}{8}-\binom{52-2s}{8}[\![s>0]\!]\right]-18\left[\binom{49-3s}{8}-\binom{52-3s}{8}[\![s>0]\!]\right]$$
$$-18\left[\binom{48-4s}{8}-\binom{52-4s}{8}[\![s>0]\!]\right]-18\left[\binom{47-5s}{8}-\binom{52-5s}{8}[\![s>0]\!]\right]$$
$$\left.-6\left[\binom{46-6s}{8}-\binom{52-6s}{8}[\![s>0]\!]\right]-2\left[\binom{45-7s}{8}-\binom{52-7s}{8}[\![s>0]\!]\right]\right\}, \quad (5)$$

where we set $\binom{n}{k}\equiv 0$ when $k>n$. Since $70=2+6+18+18+18+6+2$, we can consolidate the indicators in (5), further simplify and obtain the form

$$\mathbb{P}(S=s)=\left\{2\left[\binom{52-s}{8}+\binom{52-7s}{8}-\binom{51-s}{8}-\binom{45-7s}{8}\right]\right.$$
$$+6\left[\binom{52-2s}{8}+\binom{52-6s}{8}-\binom{50-2s}{8}-\binom{46-6s}{8}\right]$$
$$+18\left[\binom{52-3s}{8}+\binom{52-4s}{8}+\binom{52-5s}{8}\right.$$
$$\left.\left.-\binom{49-3s}{8}-\binom{48-4s}{8}-\binom{47-5s}{8}\right]\right\}\Big/\left[\binom{52}{4}\binom{48}{4}\right].$$

Let us conclude by returning to the original bar bet. Will it work "99 percent of the time?" To answer this, we need to find $\mathbb{P}(S\le 1)$. From (1), this is simply,

$$\frac{[z^{52}]\,F^{\langle 1\rangle}(z)}{\binom{52}{4}\binom{48}{4}}=\frac{38769062856}{\binom{52}{4}\binom{48}{4}}\approx 0.7360.$$

So no, far from it. What is the smallest s such that $\mathbb{P}(S\le s)\ge 0.99$? Well noting

$$\mathbb{P}(S\le 7)=\frac{[z^{52}]\,F^{\langle 7\rangle}(z)}{\binom{52}{4}\binom{48}{4}}=\frac{52083420946}{\binom{52}{4}\binom{48}{4}}\approx 0.9887$$

and

$$\mathbb{P}(S\le 8)=\frac{[z^{52}]\,F^{\langle 8\rangle}(z)}{\binom{52}{4}\binom{48}{4}}=\frac{52259016240}{\binom{52}{4}\binom{48}{4}}\approx 0.9921,$$

we see $s=8$ is the desired quantity. Good luck getting someone to take that bet.

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