Mathematics Magazine

ISSN: (Print) (Online) Journal homepage: https://maa.tandfonline.com/loi/umma20

Cycloids Generated by Rolling Regular Polygons

Terence H. Perciante

To cite this article: Terence H. Perciante (2021) Cycloids Generated by Rolling Regular Polygons, Mathematics Magazine, 94:4, 267-272, DOI: 10.1080/0025570X.2021.1951536

To link to this article: https://doi.org/10.1080/0025570X.2021.1951536

Published online: 10 Sep 2021.

Cycloids Generated by Rolling Regular Polygons

TERENCE H. PERCIANTE
Wheaton College
Wheaton, IL 60187
terry.p@wheaton.edu

Imagine rolling a wheel with a point marked on it. Visualize the marked point as continuously leaving a trace of its location. The shape of the curve that is drawn depends upon (1) the shape of the wheel, (2) the placement of the tracing point on the wheel, and (3) the curvature of the path traversed by the wheel. For example, a point placed on the perimeter of a circular wheel when rolled along a straight line path traces the well known *cycloid*, as shown in Figure 1.

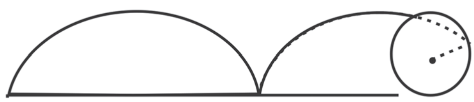

Figure 1 The marked point on the rolling wheel traces out a cycloid.

While it is not known when, in antiquity, the cycloid was first studied, Christiaan Huygens proved in 1673 that the cycloid solved the famous tautochrone problem. Newton, Leibnitz, and others demonstrated in 1696 that the cycloid also solved the equally famous brachistochrone problem. Much of this history is recounted in the excellent article by Martin [**2**].

Instead of rolling circular wheels along straight lines or circular paths, an interesting variation arises from "rolling" regular n-sided polygons along straight lines. Placing a tracing point at the midpoint of one polygonal side results in a traced path composed of joined circular arcs that form the arches of a curve that we shall call a *polycycloid*, as shown in Figure 2.

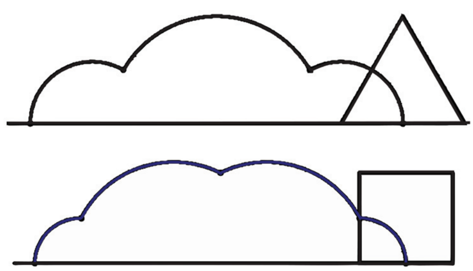

Figure 2 Examples of polycycloids.

In a noteworthy 1992 article by Leon Hall and Stan Wagon in this MAGAZINE, the authors investigated the shape required of a road to allow a polygonal wheel to

Math. Mag. **94** (2021) 267–272. doi:10.1080/0025570X.2021.1951536
MSC: 26A06

roll smoothly [1]. Subsequently, in a widely read 1999 *Math Horizons* piece, Wagon showed that a bicycle with square wheels would roll smoothly along a road constructed as a sequence of inverted catenary curves [3].

Whereas the focus of these prior articles was on the required road shapes for a wheel to roll smoothly, this present piece investigates not the road, but (1) the length of traced polycycloidal paths, and (2) the area bounded by a polycycloid above and by the x-axis below. The analysis involves limit techniques central to the calculus and it inspires a number of related challenge problems for interested students, as presented at the end of this article.

Arc Length

For a cycloid created by rolling a circle of radius $r = 1$, with tracing point located on the perimeter of the circle, the horizontal displacement D of the tracing point defined by one revolution of the circle equals the circles circumference. Specifically, $D = 2\pi(r) = 2\pi(1) = 2\pi$

Using the parametric equations for a cycloid, $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$, the following integral shows that the length L of the arc generated by rolling the circle through one revolution is $8r$:

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} r\sqrt{2 - 2\cos\theta}\, d\theta$$
$$= \int_0^{2\pi} 2r \sin\frac{\theta}{2}\, d\theta = 8r.$$

When $r = 1$, the arc length L for one arch of the cycloid formed by rolling a circle is $L = 8$.

In order to compute the length L_n of one arch of a polycycloid, we need the radius of each of the component circular arcs. For convenience, place a regular n-gon and a circumscribing circle of radius r so that the circle and n-gon are both centered at the origin (0, 0), as shown in Figure 3.

The rotation angle θ defined by rolling the n-gon about a vertex as the n-gon rolls from one side onto its adjacent side is $\theta = \frac{2\pi}{n}$

The radius r of the circumscribing circle must be determined from the size of the n-gon with side length s_n, prescribed by the requirement that the perimeter of the polygon must equal the circumference of a unit circle if the resulting polycycloids are to start and end at the same locations.

The distance formula provides expressions for lengths r_n and s_n identified in Figure 3.

$$r_n = r\sqrt{\left(\frac{\cos(\frac{2k\pi}{n}) + \cos(\frac{2(k-1)\pi}{n}) - 2)}{2}\right)^2 + \left(\frac{\sin(\frac{2k\pi}{n}) + \sin(\frac{2(k-1)\pi}{n})}{2}\right)^2}$$

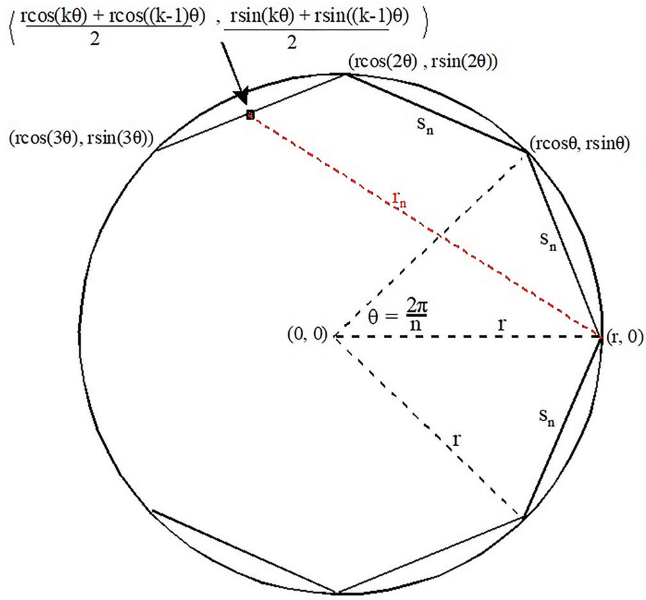

Figure 3 A regular n-gon with a circumscribed circle, centered at the origin.

$$
\begin{aligned}
s_n &= \sqrt{(r\cos\theta - r)^2 + (r\sin\theta)^2} = \sqrt{r^2\cos^2\theta - 2r^2\cos\theta + r^2 + r^2\sin^2\theta} \\
&= r\sqrt{\cos^2\theta - 2\cos\theta + 1 + \sin^2\theta} = r\sqrt{2 - 2\cos\theta} \\
&= r\sqrt{4\sin^2\left(\frac{\theta}{2}\right)} = 2r\sin\left(\frac{\theta}{2}\right).
\end{aligned}
$$

Letting $\theta = 2\pi/n$, the relationship between s_n and r simplifies to $s_n = 2r\sin(\pi/n)$, and therefore, $r = s_n/(2\sin(\pi/n))$.

Since $s_n = D/n$, it follows that $r = D/(2n\sin(\pi/n))$, and the total polycycloid arc length L_n is given by

$$
\begin{aligned}
L_n &= \sum_{k=1}^{n} \theta r_k = \sum_{k=1}^{n} \frac{2\pi r_k}{n} = \frac{2\pi}{n}\sum_{k=1}^{n} r_k = \left(\frac{2\pi}{n}\right)\left(\frac{D}{2n\sin(\frac{\pi}{n})}\right) \times \\
&\quad \sum_{k=1}^{n}\sqrt{\left(\frac{\cos(\frac{2k\pi}{n}) + \cos(\frac{2(k-1)\pi}{n}) - 2)}{2}\right)^2 + \left(\frac{\sin(\frac{2k\pi}{n}) + \sin(\frac{2(k-1)\pi}{n})}{2}\right)^2} \\
&= \left(\frac{1}{2}\right)\left(\frac{2\pi}{n}\right)\left(\frac{2\pi}{2n\sin(\frac{\pi}{n})}\right)\sum_{k=1}^{n}\sqrt{A+B},
\end{aligned}
$$

where

$$
A = \left(\cos\left(\frac{2k\pi}{n}\right) + \cos\left(\frac{2(k-1)\pi}{n}\right) - 2\right)^2
$$

$$
B = \left(\sin\left(\frac{2k\pi}{n}\right) + \sin\left(\frac{2(k-1)\pi}{n}\right)\right)^2.
$$

Since regular n-gons become increasingly circular as n approaches infinity, the length L_n of one arch of a polycycloid must converge to $L = 8$, which is the length of one arch of a cycloid generated by rolling a circle of radius $r = 1$. See Figure 4.

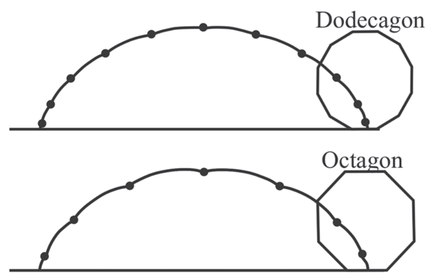

Figure 4 Polycycloids with $n = 8$ and $n = 12$.

Using the L_n formula, values of L_n for regular n-gon polycycloids can be seen to converge to $L = 8$ as n increases, just as expected. This is shown in Table 1.

TABLE 1: Arc length L_n of one arch of a polycycloid for different values of n.

n sides	L_N
3	8.185303
4	7.984678
5	7.952617
6	7.951943
7	7.957557
8	7.963763
12	7.980583
30	7.996458
100	7.999672
1000	7.999997

Despite the apparent pattern revealed in Table 1, many mathematics students would find it challenging to prove that $\lim_{n\to\infty} L_n = 8$ even with the powerful methods of calculus at their disposal.

Area

As the number of sides n of a polygon increases, the area A_n under one arch of a polycycloid traced by the rolling n-sided regular polygon approaches the area A under a cycloid traced by rolling a circle. In order to compute the area A under one arch of a cycloid traced by rolling a circle of radius $r = 1$, we again use the parametric equations of the cycloid, $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$ with $dx = r(1 - \cos\theta)\,d\theta$, to write an integral that computes the required area:

$$A = \int_{\theta=0}^{\theta=2\pi} y\,dx = \int_0^{2\pi} r^2(1-\cos\theta)^2\,d\theta = 3\pi r^2$$

For a circle of radius $r = 1$, the area A under the cycloid reduces to $A = 3\pi(1)^2 = 3\pi \approx 9.424778$.

Now we compute the area A_n under a polycycloid generated by a regular n-gon. In contrast to the cycloid formed by rolling a circle, a pattern of successive circular arcs join to form the polycycloidal curve generated by the n-gon. In Figure 5, each arc is translated to the n-gon's original starting position.

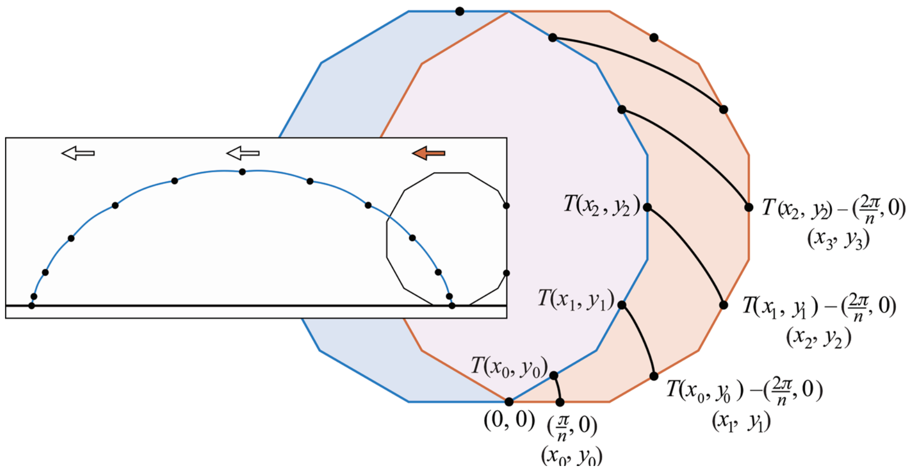

Figure 5 Endpoint coordinates associated with successive arcs of a polycycloid generated by an n-gon.

We find the successive integral limits by rotating the n-gon through counterclockwise rotations of $2\pi/n$ radians starting with $(x_0, y_0) = (\pi/n, 0)$. For $1 \le k \le n$, we define a point $T(x_k, y_k)$ according to the recursive formulas:

$$x_{k+1} = x_k \cos\frac{2\pi}{n} - y_k \sin\frac{2\pi}{n} \qquad \text{and} \qquad y_{k+1} = x_k \sin\frac{2\pi}{n} + y_k \cos\frac{2\pi}{n}.$$

We then apply a horizontal translation of $2\pi/n$ to the point $T(x_k, y_k)$ as shown in Figure 5.

The area A_n under one complete cycle of a polycycloid can be determined by summing the areas bounded above by each of the circular arcs (that together form the polycycloid) and below by the horizontal axis.

$$A_n = 2\left(\int_{T(x_0,y_0)}^{x_0} \sqrt{r_1^2 - x^2}\,dx + \sum_{k=1}^{n/2} \int_{T(x_k,y_k)}^{T(x_{k-1},y_{k-1})+(\frac{2\pi}{n},0)} \sqrt{r_{k+1}^2 - x^2}\,dx\right),$$

where the integral limits represent the x-coordinates of the aforementioned points. (See Figure 5).

Again, as the number of sides of a regular n-gon increases, the shape of the n-gon converges to that of a circle. Accordingly, the area A_n generated by rolling a regular polygon must approach that of the cycloid traced by rolling a circle.

Using the area formula for A_n with successively larger values of n, the polycloid area A_n increases as expected to $A = 9.424778$, the area defined by the cycloid (obtained by rolling a circle). The areas for a few values of n are shown in Table 2.

TABLE 2: The area A_n for a polycycloid generated by a polygon with n sides.

n Sides	A_n
3	7.641309
4	8.280178
8	9.1103217
12	9.2828896
Circle	9.424778

Possibilities for further investigation

Students wishing to further explore aspects of polycycloids might choose to attempt some of the following problems:

1. As $n \to \infty$, the number of cusps that occur on a polycycloid increases (that is, the number of points where the path is non-differentiable increases) suggesting that the final curve is everywhere non-differentiable. On the other hand, as $n \to \infty$, polycycloids converge to the smooth path of the everywhere differentiable cycloid. Does this suggest that, as $n \to \infty$, the polycycloids yield a curve that is both everywhere differentiable and simultaneously everywhere non-differentiable?
2. Investigate curves produced by rolling regular n-gons around circular path. For comparison, curves formed by rolling a circle around the outside of another circle are called epicycloids. If a circle is rolled around the inside of a containing circle, then the resulting curve is called a hypocycloid.
3. Evaluate $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} A_n$.

REFERENCES

[1] Hall, L., Wagon, S. (1992). Roads and wheels. *Math. Mag.* 65(5): 283–301. doi:10.1080/0025570X.1992.11996043

[2] Martin, J. (2010). The Helen of geometry. *College Math. J.* 41(1): 17–28. doi:10.4169/074683410X475083.

[3] Wagon, S. (1999). The ultimate flat tire. *Math. Horizons* 6(3): 14–17. doi:10.1080/10724117.1999.11975090

Summary. Historical problems related to cycloids form the background for an investigation of paths traced by rolling regular polygons. Using trigonometry, geometry, and calculus, the lengths of and areas under the generated paths are shown to converge to values associated with the classical cycloid as the number of polygonal sides increases.

TERENCE H. PERCIANTE (MR Author ID: 323472) earned his Ed.D. from the State University of New York at Buffalo in 1972. For the next 40 years, he was immeasurably blessed to serve in the Wheaton College, Department of Mathematics until his retirement in 2012. He maintains an active interest in dynamical systems, fractal geometry, and chaos theory.