Mathematics Magazine

ISSN: (Print) (Online) Journal homepage: https://maa.tandfonline.com/loi/umma20

Harriot’s Observation of Resisted Trajectories

C. W. Groetsch

To cite this article: C. W. Groetsch (2021) Harriot’s Observation of Resisted Trajectories, Mathematics Magazine, 94:5, 361-368, DOI: 10.1080/0025570X.2021.1983390

To link to this article: https://doi.org/10.1080/0025570X.2021.1983390

Published online: 27 Oct 2021.

Harriot's Observation of Resisted Trajectories

C. W. GROETSCH
The Citadel
Charlestown, SC 29409
charles.groetsch@citadel.edu

For nearly two millennia, Aristotle's theory of violent and natural local motion held sway. Such was the influence of the great philosopher that the notion of a hyper-angular ballistic trajectory, reminiscent of a Wile E. Coyote cartoon, consisting of a violent linear motion, followed by a vertical natural motion, was taken seriously well into the sixteenth century (see Figure 1). Early in the sixteenth century, mathematicians, notably Niccolo Tartaglia (1499-1557), endeavored to formulate theories to better describe actual trajectories observed by gunners for several centuries.

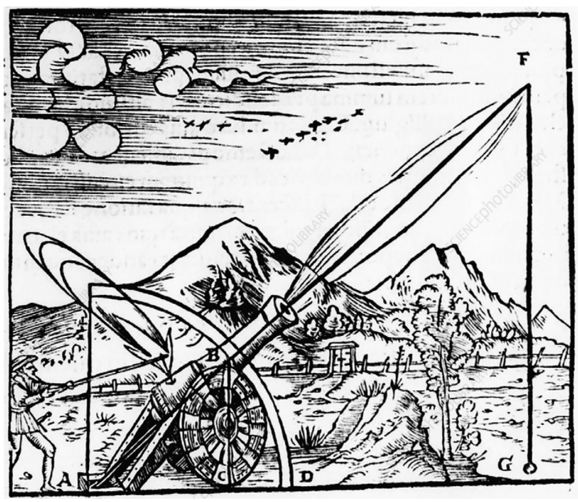

Figure 1 An Aristotelian Ttrajectory. The image is from Santbech [8].

Tartaglia's proposed trajectory modified the Aristotelian trajectory by blending a transitional arc between straight line violent and natural motions, see Figure 2 [12, p. 40]. In his words (translation from [12]; [1, p. 84]), "Every violent trajectory or motion . . . will always be partly straight and partly curved, and the curved part will form part of the circumference of a circle. Yet in the very next sentence he continued, "Truly no violent trajectory or motion . . . can have any part that is perfectly straight . . . , indicating that he realized his trajectory was an idealization.

The work of Thomas Harriot (1560–1621) on trajectories is not well-known today because he published nothing during his lifetime, other than his *A Briefe and True Report of the New Found Land of Virginia* (1588). His nachlass, comprising thousands of folio sheets containing notes on algebra, astronomy, navigation, mechanics, optics and other subjects, resided in private hands for more than three centuries. Around 1784, the bulk of these papers came under the control of Franz Zach who, in the employ of Lord Egremont

Math. Mag. **94** (2021) 361–368. doi:10.1080/0025570X.2021.1983390 © Mathematical Association of America
MSC: 26A06, 01A40

> . . . had gone through them so hurriedly, seeking sensational or exciting materials, and had been careless about retaining their original order. Much of the time he had turned the pages as he read them; at other times he piled the pages without turning them over, and in some instances he had gathered bundles without regard for top or bottom or had put together sections which did not belong together. As a result . . . Harriot's papers were in an almost chaotic condition. [10, p. 20]

Thanks largely to the intrepid scholars Johannes Lohne [6] and Martin Schemmel [9], Harriot's notes have been disentangled and interpreted.

In his unpublished notes, a sketch in Harriot's hand can be seen of a typical trajectory familiar to contemporary practitioners of "great artillery." This sketch indicates that Harriot turned away from the tripartite violent-transitional-natural Tartaglian trajectory in favor of a bipartite trajectory consisting of two branches, neither of which contain any straight part. The ascending branch of the trajectory rises to a "culmination point" (apex), followed by a descending branch ending at ground level. Schemmel's comment that the ascending branch is "longer and flatter than the path from the culmination point down to the ground" [9, p. 28] is the inspiration for this note.

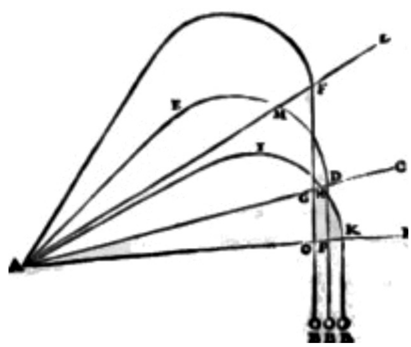

Figure 2 Tartaglian Trajectories.

Studies of projectile motion subjected to resistance proportional to velocity are a rich source of analytical topics accessible to undergraduates. Several recent works on this theme have dealt with quantitative aspects of trajectory optimization and some have reintroduced and popularized the Lambert W-function (for example, Kantrowitz and Neumann [5], and Packel and Yuen [7]). Other articles have made connections with the history of science and have emphasized qualitative aspects of trajectories ([2], [3], [4], [11]). This note, which is of the latter sort, is an entirely elementary treatment validating Schemmel's remark on the relative length and flatness of the two branches of a linearly resisted trajectory. We interpret and validate the flatness and length observation, from both a macro and micro perspective, for a simple model of projectile motion in a resisting medium. Our analysis originated in discussions in an honors calculus course and uses only basic concepts and techniques that are accessible to first-year calculus students.

The model

Newton's laws of motion for a point particle of unit mass launched from the origin into a medium offering resistance proportional to velocity, with initial speed v, at an angle

$\theta \in [0, \pi/2)$ with respect to the positive x-axis, are expressed by the system of initial value problems

$$\ddot{x} = -k\dot{x},\, x(0) = 0,\, \dot{x}(0) = v\cos\theta,$$
$$\ddot{y} = -g - k\dot{y},\, y(0) = 0,\, \dot{y}(0) = v\sin\theta,$$

where $k > 0$ is a drag coefficient and g is the constant acceleration of gravity. The dots signify derivatives with respect to time t. Integrating this un-coupled system reveals the parameterized trajectory

$$x(t) = \frac{v\cos\theta}{k}\left(1 - e^{-kt}\right),$$
$$y(t) = \frac{g}{k^2}\left(1 + \frac{vk\sin\theta}{g}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t. \tag{1}$$

On setting

$$z = \frac{k}{v\cos\theta}x = 1 - e^{-kt}$$

and eliminating the parameter t, we find that the shape of the trajectory may be expressed by the shorter expression

$$u(z) = cz + \ln(1 - z), \quad 0 \le z < 1, \tag{2}$$

where $u = (k^2/g)y$ and the parameter

$$c = 1 + \frac{kv\sin\theta}{g} > 1$$

encapsulates all physical parameters (g, k) and control parameters (v, θ) of the model. Furthermore, since

$$\frac{dz}{dx} = \frac{k}{v\cos\theta},$$

we have

$$\frac{du}{dz} = \frac{k^2}{g}\frac{dy}{dz} = \frac{k^2}{g}\frac{dy}{dx}\frac{dx}{dz} = \frac{kv\cos\theta}{g}\frac{dy}{dx},$$

showing that the derivatives $u'(z)$ and $y'(x)$ are, for fixed values of the parameters, positively proportional. Therefore, qualitative notions of "flatness" are the same for either of the descriptions in equations (1) and (2).

Basic features of the graph of function (2) are easily verified: it achieves its maximum at the point

$$(z^*, u^*) = (1 - 1/c,\ c - 1 - \ln c), \tag{3}$$

it is strictly increasing on the interval $(0, z^*)$ and strictly decreasing on $(z^*, 1)$, and it is asymptotic to the line $z = 1$. So, in addition to the root at the origin, u has a unique root R in the interval $(z^*, 1)$. Also, $u'(z) = c - 1/(1 - z)$ is not constant on any subinterval of $(0, R)$, reflecting Tartaglia's comment that his trajectory "can have no part that is perfectly straight," in sharp contrast to the Aristotelian trajectory of Figure 1. The graph of a typical function in the class (2) appears in Figure 3.

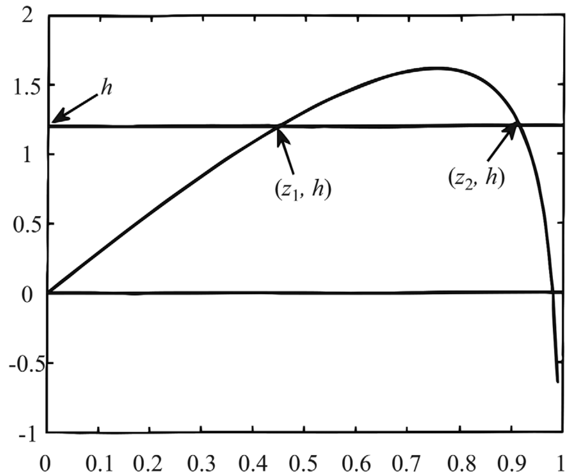

Figure 3 A generic trajectory.

It is apparent from this graph that resistance causes a certain asymmetry: z^* lies beyond the mid-range $R/2$. This fact, proved in Groetsch [3] by a timing-related argument, is a consequence of the more general result given in equation (5) below.

The intermediate value theorem and the strict monotonicity of the ascending, respectively descending, branches of equation (2) guarantee that for a given number $h \in [0, u^*)$ there is a unique $z_1 = z_1(h) \in [0, z^*)$ and a unique $z_2 = z_2(h) \in (z^*, R)$ satisfying

$$u(z_1) = h = u(z_2). \tag{4}$$

By extension we define $z_1(u^*) = z_2(u^*) = z^*$.

As h traverses the interval $[0, u^*)$ of the vertical axis, the horizontal lines through $(0, h)$ rise, the pre-image $z_1(h)$ advances, and $z_2(h)$ retreats. It is less apparent that as h increases the average value of the pre-images remains strictly less than z^*.

Proposition 1. *If $h \in [0, u^*)$, then*

$$\frac{z_1(h) + z_2(h)}{2} < z^*. \tag{5}$$

Proof. The proof is accomplished by sequential reformulation of equation (5) and a change of variable. To streamline notation, $z_1(h)$ will be represented by z_1 and similarly $z_2 = z_2(h)$.

First, note that equation (5) is equivalent to $z_1 < p$, where $p = 2z^* - z_2 < z^*$. Also, since $u(z)$ is strictly increasing for $z < z^*$ and strictly decreasing for $z > z^*$, and since

$$u(z_2) = h = u(z_1),$$

we have that $z_1 < p$ is equivalent to $u(z_2) < u(p)$. That is,

$$c(z_2 - p) + \ln(1 - z_2) < \ln(1 - p),$$

or equivalently,

$$c(1 - z_2)e^{c(z_2 - p)} < c(1 - p). \tag{6}$$

However, by equation (3),

$$c(1 - p) = c(1 + z_2 - 2z^*) = 2c(1 - z^*) - c(1 - z_2)$$

$$= 1 + (1 - c(1 - z_2)),$$

and hence

$$c(z_2 - p) = c(z_2 - 1 + 1 - p) = 2(1 - c(1 - z_2)).$$

Setting $w = 1 - c(1 - z_2)$, we note that $w < 1$ and, since $z_2 > z^*$, we see that $w > 1 + c(z^* - 1) = 0$. Making this substitution and rearranging a bit, inequality (6) becomes:

$$f(w) = (1 - w)[1 + e^{2w}] < 2, \quad 0 < w < 1.$$

But note that $f(0) = 2$, and for $w > 0$ we have

$$f'(w) = e^{2w}(1 - 2w) - 1 < e^{2w}e^{-2w} - 1 = 0.$$

Hence, $f(w) < 2$, implying f is strictly deceasing, proving (6) and thereby establishing equation (5). ■

This result is key to comparing the relative flatness of the ascending and descending branches of the trajectory. In particular we note that

$$\frac{z_1(0) + z_2(0)}{2} = \frac{R}{2} < z^*. \tag{7}$$

Flatness and length

How is the "flatness" of a branch to be interpreted? A reasonable macroscopic gauge is the average flatness—the average slope of the tangent line. We now show that the average value of $u'(z)$ on the ascending branch, $u(z)$, $0 \le z \le z^*$, is strictly less than the average value of $-u'(z)$ on the descending branch. That is, for $z^* \le z \le R$. This gives a global interpretation of the observation that the trajectory is *flatter* to the left of the "culmination point" (z^*, u^*) than it is to the right of that point.

The average value of $u'(z)$ on the ascending branch is

$$A = \frac{1}{z^*}\int_0^{z^*} u'(z)\,dz = \frac{u^*}{z^*},$$

while the average value of $-u'(z)$ on the descending branch is

$$D = \frac{-1}{R - z^*}\int_{z^*}^{R} u'(z)\,dz = \frac{u^*}{R - z^*}.$$

Therefore, $A < D$ if and only if $R < 2z^*$, a fact established in equation (7) of the previous section.

A natural microscopic, or pointwise, interpretation of relative flatness is given in terms of the h-level sets

$$\{z : u(z) = h \in [0, u^*)\},$$

each of which consists of exactly two points, one on the ascending branch, namely $z_1(h)$, and the other, $z_2(h)$, on the descending branch.

We show that

$$\frac{du}{dz}(z_1) < -\frac{du}{dz}(z_2),$$

expressing the point-wise relative "flatness" of the ascending branch. To this end, it is helpful to note the following:

Lemma 1. *The function*

$$g(h) = \frac{1}{1 - z_1(h)} + \frac{1}{1 - z_2(h)}$$

is strictly decreasing on the interval $(0, u^*)$.

Proof. This is a consequence of equation (5).

In fact, from equation (4) we find that

$$\left(c - \frac{1}{1 - z_i}\right) \frac{dz_i}{dh} = 1, \quad i = 1, 2,$$

and hence that

$$\begin{aligned} g'(h) &= (1 - z_1)^{-2} \frac{dz_1}{dh} + (1 - z_2)^{-2} \frac{dz_2}{dh} \\ &= \frac{1}{(1 - z_1)(c(1 - z_1) - 1)} + \frac{1}{(1 - z_2)(c(1 - z_2) - 1)}. \end{aligned}$$

Since $1 - z_1 > 1 - z^*$ and $0 < 1 - z_2 < 1 - z^*$, we see that

$$c(1 - z_1) - 1 > c(1 - z^*) - 1 = 0$$

and

$$c(1 - z_2) - 1 < c(1 - z^*) - 1 = 0,$$

respectively. From this and equation (3), we obtain

$$\begin{aligned} g'(h) &< \frac{1}{(1 - z^*)(c(1 - z_1) - 1)} + \frac{1}{(1 - z^*)(c(1 - z_2) - 1)} \\ &= \frac{c}{c(1 - z_1) - 1} + \frac{c}{c(1 - z_2) - 1} = \frac{1}{z^* - z_1} + \frac{1}{z^* - z_2}. \end{aligned}$$

Finally, equation (5) gives $z_2 - z^* < z^* - z_1$ and hence $1/(z_2 - z^*) > 1/(z^* - z_1)$. Therefore,

$$g'(h) < \frac{1}{z^* - z_2} + \frac{1}{z_2 - z^*} = 0,$$

proving that $g(h)$ is strictly decreasing. ∎

Proposition 2. *For each* $h \in (0, u^*)$, *we have that*

$$\frac{du}{dz}(z_1) < -\frac{du}{dz}(z_2).$$

Proof. By equation (2), the desired inequality is the same as

$$c - \frac{1}{1 - z_1(h)} < -c + \frac{1}{1 - z_2(h)},$$

or equivalently,

$$2c < \frac{1}{1 - z_1(h)} + \frac{1}{1 - z_2(h)},$$

that is, $2c < g(h)$. But $g(h)$ is strictly decreasing on $(0, u^*)$, as established in the lemma. Note that

$$g(u^*) = \frac{2}{1 - z^*} = 2c.$$

We claim that

$$g(0) = \frac{1}{1 - 0} + \frac{1}{1 - R} > 2c,$$

or equivalently,

$$\frac{2c - 2}{2c - 1} < R. \tag{8}$$

Since $0 < (2c - 2)/(2c - 1) < 1$, we see that equation (8) holds if and only if

$$u\left(\frac{2c - 2}{2c - 1}\right) > u(R) = 0.$$

Setting $x = 2c - 1$, we find that $x > 1$ and

$$u\left(\frac{2c - 2}{2c - 1}\right) = u\left(\frac{x - 1}{x}\right) = \frac{x + 1}{2}\frac{x - 1}{x} - \ln x.$$

Hence, the required condition (8) is equivalent to

$$x \ln x < \frac{x^2 - 1}{2} \quad \text{for} \quad x > 1. \tag{9}$$

Since $1/t < 1$ for $t > 1$, we see that for $x > 1$,

$$\int_1^x \ln s \, ds < \int_1^x \int_1^s \frac{1}{t} \, dt \, ds < \int_1^x \int_1^s 1 \, dt \, ds = \int_1^x s - 1 \, ds.$$

That is,

$$x \ln x - x + 1 < x^2/2 - x + 1/2,$$

proving equation (9) and verifying that $g(0) > 2c$. As $g(h)$ is strictly decreasing on $(0, u^*)$ and $g(u^*) = 2c$, we find that $g(h) > 2c$ for all $h \in [0, u^*)$.

Therefore, at each level $h \in [0, u^*)$

$$\frac{du}{dz}(z_1) < -\frac{du}{dz}(z_2). \qquad \blacksquare$$

That the ascending branch is longer than the descending branch is an easy consequence of this "flatness" result. In fact, given $z \in [0, z^*]$ there is a unique $y \in [z^*, R]$ with $u(z) = u(y)$ (in the former notation $y = z_2(u(z))$). Differentiating with respect to z, we obtain

$$\frac{dy}{dz} = \frac{u'(z)}{u'(y)} \quad \text{for} \quad z \in [0, z^*).$$

Therefore,

$$\left(\frac{dy}{dz}\right)^2 < 1. \tag{10}$$

Noting that y decreases with increasing z, and changing variable in the integral representing arclength of the descending branch, we find that:

$$\int_{z^*}^{R} \sqrt{1+u'(y)^2}\,dy = \int_{z^*}^{0} \sqrt{1+u'(y)^2}\,\frac{dy}{dz}\,dz = \int_0^{z^*} \sqrt{1+u'(y)^2}\,(-\frac{dy}{dz})\,dz$$
$$= \int_0^{z^*} \sqrt{1+u'(y)^2}\,|\frac{dy}{dz}|\,dz = \int_0^{z^*} \sqrt{(\frac{dy}{dz})^2 + (\frac{du}{dy}\frac{dy}{dz})^2}\,dz.$$

Finally, by equation (10),

$$\int_{z^*}^{R} \sqrt{1+u'(y)^2}\,dy = \int_0^{z^*} \sqrt{\left(\frac{dy}{dz}\right)^2 + \left(\frac{du}{dy}\frac{dy}{dz}\right)^2}\,dz < \int_0^{z^*} \sqrt{1+u'(z)^2}\,dz.$$

That is, the ascending branch is longer than the descending branch.

REFERENCES

[1] Drake, S., Drabkin, I. E, (Eds and Translators). (1969). *Mechanics in Sixteenth-Century Italy: Selections from Tartaglia, Benedetti, Guido Ubaldo and Galileo*. Madison: Univ. of Wisconsin Press.

[2] Groetsch, C. W. (1996). Tartaglia's inverse problem in a resistive medium. *Am. Math. Monthly*. 103(7): 227-236. doi.org/10.1080/00029890.1996.12004784

[3] Groetsch, C. W. (2003). Timing is everything: the French connection. *Am. Math. Monthly*. 110(10): 950-955. doi.org/10.1080/00029890.2003.11920038

[4] Groetsch, C.W. (2015). Three views of a comment by Mersenne. *Math. Sci.* 40(1): 7-12.

[5] Kantrowitz, R., Neumann, M. (2015). Optimization of projectile motion under linear air resistance. *Rend. Circ. Mat. Palermo*. 64(3): 365-382. doi.org/10.1007/s12215-015-0205-y

[6] Lohne, J.A. (1979). Essays on Thomas Harriot. *Arch. Hist. Exact Sci.* 20(3/4): 189-312. doi.org/10.1007/BF00327737

[7] Packel, E.W., Yuen, D.S. (2004). Projectile motion with resistance and the Lambert W function. *Coll. Math. J.* 35(5): 337-350. DOI: 10.1080/07468342.2004.11922095

[8] Santbech, D. (1561). *Problematum Astronomicorum et Geometricorum*. Basel: H. Petri et P. Pernum.

[9] Schemmel, M. (2008). *The English Galileo: Thomas Harriot's Work on Motion as an Example of Preclassical Mechanics, Vol. 1: Interpretation*. Berlin: Springer.

[10] Shirley, J.W. (1983). *Thomas Harriot: A Biography*. Oxford: Clarendon Press.

[11] Stewart, S. (2012). On trajectories of projectiles depicted in early ballistic woodcuts. *Eur. J. Phys.* 33(1): 149–166. doi.org/10.1088/0143-0807/33/1/013

[12] Tartaglia, Niccolo. (1537) *Nova Scientia*. Venice.

Summary. Features of Thomas Harriot's sketch of the shape of cannon ball trajectories, commonly observed by late sixteenth century gunners, are validated for a model of linearly resisted projectiles. The analysis uses only basic concepts and familiar techniques from first-year calculus.

C. W. GROETSCH (MR Author ID: 77275) is Distinguished Professor of Mathematical Science at The Citadel in Charleston, SC, where he was the founding dean of the School of Science and Mathematics. Currently one of his interests is the history of the mathematization of physical science in the sixteenth and seventeenth centuries.