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Projectile Motion: Resistance Is Fertile

William W. Hackborn

1. INTRODUCTION. The nature of projectile motion, occupying as it does an important place in the work of Galileo and Newton, lies near the heart of the scientific revolution. The topic itself and its application to ballistics have attracted the interest of many mathematicians, but not all of them: G. H. Hardy [**9**, p. 140] wrote that ballistics is "repulsively ugly and intolerably dull; even Littlewood could not make ballistics respectable." Hardy's views notwithstanding, one goal of this work is to show that the motion of a projectile subject to air resistance is a fertile source of elegant and accessible mathematics.

This paper deals with the motion of a projectile in a uniform, downward gravitational field and resisted by a drag force due to the air through which it moves. The drag force is assumed to act in a direction opposite to that of the projectile's velocity and to depend only on its magnitude. Coriolis forces and Earth's curvature are ignored. Hence, by Newton's second law,

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{du}{dt} = -\frac{f(s)}{s}u, \qquad \frac{dv}{dt} = -g - \frac{f(s)}{s}v, \tag{1}$$

where (x, y) is the position vector of the projectile, (u, v) is its velocity vector, $s = \sqrt{u^2 + v^2}$ is its speed, $f(s)$ is the magnitude of the drag force on it per unit mass, and g is the acceleration of gravity. The initial conditions imposed on (1) are

$$x = 0, \quad y = 0, \quad u = a = s_I \cos\phi, \quad v = b = s_I \sin\phi, \quad \text{at } t = 0, \tag{2}$$

where a and b are the initial components of velocity, $s_I = \sqrt{a^2 + b^2}$ is the initial speed, ϕ is the launch angle, and $a > 0$ is assumed to simplify notation.

The case of air resistance varying as the speed of the projectile, i.e., $f(s) = ks$ with k a positive constant, has been widely studied, e.g., [**3, 5, 17**]. System (1) is linear in this case; solving for x and y and eliminating t from the result produces

$$y = \left(b + \frac{g}{k}\right)\frac{x}{a} + \frac{g}{k^2}\log\left(1 - \frac{kx}{a}\right). \tag{3}$$

An encounter with (3) in a video [**2**] on the scientific revolution led me to become curious about its origins and accuracy. I eventually learned that Newton [**14**, pp. 636–640], although he did not state a Cartesian equation like (3), provided a geometric construction for the solution curve described by (3). Newton was dissatisfied, however, with this solution, remarking that the hypothesis that resistance is proportional to speed "belongs more to mathematics than to nature," and he went on to make a plausible physical argument that resistance varies as the square of the speed in "mediums wholly lacking in rigidity": a body moving at a greater speed through such a medium produces proportionally greater velocities in proportionally more particles of the medium per unit time, and so the change per unit time in the total momentum of the medium, and hence its resisting force (by Newton's second and third laws of motion), is greater in proportion to the square of the body's speed [**14**, p. 641]. *The Principia* describes extensive experiments to measure fluid drag, including some in which inflated hogs' bladders and glass balls filled with air or mercury were dropped from the top of St. Paul's Cathedral in London [**14**, pp. 756–759]; Newton found the results of these experiments to accord well with resistance varying as the speed squared. Modern research on fluid drag confirms this for bodies moving at subsonic speeds in air (see [**7, 12, 15**], for example).

Projectile motion under *quadratic drag* was at the center of a fascinating episode in the calculus priority dispute: Newton failed to construct the exact solution for a projectile in a fluid whose resistance is proportional to the square of the projectile's speed; in 1718, this failure and the fallout from an error found by Johann Bernoulli in Newton's approximate analysis of this problem in the first edition of *The Principia* prompted Oxford professor John Keill to challenge Bernoulli to solve it; Bernoulli then found an implicit solution involving nonelementary quadratures for projectile motion subject to an even more general resistance law (see [**14**, pp. 168–171] and [**8**, pp. 152–156]). Bernoulli's solution became the foundation of many approximations over the next two centuries, including Euler's approximation for the case of quadratic drag, which was still used for trench mortars and other subsonic guns as recently as World War II [**13**, p. 258].

This paper emerged from a desire to find a formula analogous to (3) when drag varies as the square of the projectile's speed. Although I was unable to find an elementary expression for the trajectory of the projectile in this realistic case, I have rederived an elementary first integral of (1), rediscovered an elementary expression that approximates the trajectory quite well for launch angles up to moderate size, developed a proof that this approximate trajectory lies between the exact trajectory and Galileo's parabolic trajectory, and investigated other elementary expressions for approximate trajectories. These results appear below.

2. RESISTANCE VARIES AS SQUARE OF THE SPEED. The motion of a projectile under quadratic drag is governed by (1) with $f(s) = cs^2$, c a positive constant. The last two equations in (1) determine solutions in the (u, v) plane, and $(u, v) = (0, -s_T)$ with $s_T = \sqrt{g/c}$ is the only fixed point solution, representing a vertical trajectory downward at terminal speed s_T. A local analysis near this fixed point and the Hartman-Grobman Theorem [**6**, p. 13] reveal that

$$u \sim C_1 e^{-\sqrt{cg}\,t}, \qquad v \sim C_2 e^{-2\sqrt{cg}\,t} - s_T, \qquad \text{as } t \to \infty, \tag{4}$$

where C_1 and C_2 are constants. Hence, this fixed point is asymptotically stable and, since there are no periodic solutions by Bendixson's criterion [**6**, p. 44], a global attractor in the (u, v) plane. Letting $w = v/u$, the last two equations in (1) give

$$\frac{dw}{dt} = -\frac{g}{u}, \qquad \frac{du}{dw} = \frac{f(s)}{gs}u^2 = \frac{c}{g}u^3\sqrt{1+w^2}, \qquad w = \frac{v}{u}. \tag{5}$$

Solving the second equation in (5) and employing (2) yields the first integral

$$\frac{g}{cu^2} + h(w) = \frac{g}{ca^2} + h\left(\frac{b}{a}\right), \qquad h(w) = w\sqrt{1+w^2} + \log(w + \sqrt{1+w^2}). \tag{6}$$

The result in (6), generalized to $f(s) = cs^n$ for arbitrary n, was derived in a different way by Johann Bernoulli, who used this result to express t, x, and y as indefinite integrals involving w as a parameter (see [**16**, pp. 95–96]). Now, (5) and (6) imply

$$\frac{dw}{dt} = -\sqrt{cg\,p(w)}, \qquad p(w) = \frac{g}{ca^2} + h\left(\frac{b}{a}\right) - h(w). \tag{7}$$

Let $w_e(t)$ be the exact solution of (7) satisfying $w = b/a$ at $t = 0$. I was unable to find an elementary expression for $w_e(t)$. However, $h(w) = 2w + O(w^3)$ as $w \to 0$, and replacing $h(w)$ by $2w$ in (7) yields the approximate equation

$$\frac{dw}{dt} = -\sqrt{cg\,q(w)}, \qquad q(w) = \frac{g}{ca^2} + \frac{2b}{a} - 2w. \tag{8}$$

Thus, letting $w_s(t)$ be the solution of (8) having $w = b/a$ at $t = 0$,

$$w \approx w_s(t) = b/a - gt/a - cgt^2/2. \tag{9}$$

Note that $w = v/u = \tan\theta$, where θ is the inclination angle of the velocity vector. It is expected that $w_e(t) \approx w_s(t)$ on trajectories truncated at a finite value of t for which θ, and thus w, is uniformly small. Since (6) and (7) imply $u = [p(w)]^{-1/2} \approx [q(w)]^{-1/2}$ for small θ, and $v = uw$, it follows from (1), (8), and (9) that

$$x \approx x_s(t) = c^{-1}\log(act + 1), \tag{10}$$

$$y \approx y_s(t) = (b + g/2ac)(ac)^{-1}\log(act + 1) - gt^2/4 - gt/2ac. \tag{11}$$

Finally, using (10) to eliminate t from (11) produces

$$y \approx Y_s(x) = \left(b + \frac{g}{2ac}\right)\frac{x}{a} + \frac{g(1 - e^{2cx})}{4a^2c^2}, \tag{12}$$

which is analagous, for quadratic drag and small inclinations, to (3) for linear drag. Formula (12) also appears in [**10**, pp. 294–297] and [**15**] but is derived differently.

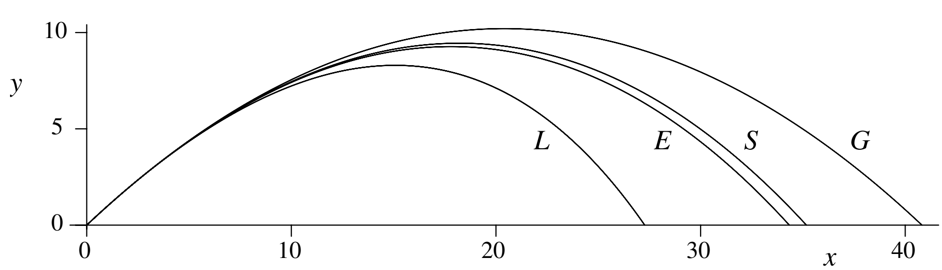

Figure 1. Trajectories for linear drag (L), exact quadratic drag (E), the small inclination approximation (S), and no drag, Galileo's parabola (G), for terminal speed 40 m/s and initial speed 20 m/s at launch angle 45°.

Let $Y_e(x)$ be the exact solution for y in terms of x. Figure 1 shows trajectories given by (3), $Y_e(x)$ computed (by MAPLE) using a Runge-Kutta method, $Y_s(x)$, and Galileo's parabola $Y_g(x) = bx/a - gx^2/2a^2$. These trajectories were calculated using $g = 9.8$ m/s^2, $c = g/s_T^2$, and (since $s_T = g/k$ for linear drag) $k = g/s_T$ in (3), with $s_T = 40$ m/s (89.48 miles/hour, roughly the terminal speed of a baseball [**12**] and the speed of a typical pitch in the major leagues). Evidently, the trajectory for linear drag is very different from that for quadratic drag in this figure, and $Y_s(x)$ is quite close to $Y_e(x)$ though the launch angle (45°) is not small.

Figure 1 suggests a relationship: $Y_e(x) < Y_s(x) < Y_g(x)$ for $x > 0$. Actually, this cannot quite be true since $Y_e(x)$, unlike $Y_s(x)$ and $Y_g(x)$, has a vertical asymptote. Why? Note that (4) implies that the total change, $\int_0^\infty u\,dt$, in x over an entire trajectory is finite, while the total change, $\int_0^\infty v\,dt$, in y is negatively infinite. Hence, $Y_e(x)$ has a vertical asymptote at, say, $x = x_\infty$. See [**16**, p. 99] for another proof of this asymptote's existence. Theorems 1 and 2 below address relationships between the exact solution for quadratic drag, the small inclination approximation, and Galileo's solution. These theorems seem to originate with the author, unlike the main results above which were obtained by the author and later found to be published elsewhere. Theorems 1 and 2 continue something of a MONTHLY theory series (see [**1, 3, 4**]) on motion subject to air resistance. Care has been taken to make their proofs as elementary as possibile. Both proofs use a corollary to the Mean Value Theorem (CMVT) which implies that a differentiable function exceeds another on an interval when its derivative is greater (less) than that of the other on the interior of the interval and the functions are equal at the interval's left (right) endpoint.

Theorem 1. (a) $Y_e(x) < Y_s(x)$ *for* $0 < x < x_\infty$. (b) $Y_s(x) < Y_g(x)$ *for* $x > 0$.

Proof. (a) From (1) and (5), $w = dy/dx$, so $Y_e'(x) = W_e(x)$, where $W_e(x)$ is the exact solution for w in terms of x; furthermore, $dw/dx = -g/u^2$. Hence, from (6), $W_e(x)$ satisfies the initial value problem

$$\frac{dw}{dx} = -c\,p(w), \qquad w = b/a \quad \text{at } t = 0, \tag{13}$$

and similarly, $Y_s'(x) = W_s(x)$, where $W_s(x)$ satisfies

$$\frac{dw}{dx} = -c\,q(w), \qquad w = b/a \quad \text{at } t = 0, \tag{14}$$

with $p(w)$ and $q(w)$ as in (7) and (8). Also, $p(b/a) = q(b/a) = g/ca^2$ and

$$p'(w) = -2\sqrt{1+w^2} < -2 = q'(w) \quad \text{for } w \neq 0.$$

Using CMVT, it follows that

$$p(w) > q(w) > 0 \quad \text{for } w < b/a. \tag{15}$$

Assume $X_e(w)$ satisfies $X_e'(w) = -[c\,p(w)]^{-1}$ with $X_e(b/a) = 0$, and $X_s(w)$ satisfies $X_s'(w) = -[c\,q(w)]^{-1}$ with $X_s(b/a) = 0$. From (15), $X_e(w)$ and $X_s(w)$ are decreasing on the domain $(-\infty, b/a]$, and so, with respect to this domain, these functions have inverses. From (13) and (14), it is clear that these inverses are $W_e(x)$ and $W_s(x)$, which must also be decreasing on their corresponding domains, $[0, x_\infty)$ and $[0, \infty)$, respectively. But (15) implies $X_s'(w) < X_e'(w)$ for $w < b/a$. Thus, by CMVT,

$$X_s(w) > X_e(w) > 0 \quad \text{for } w < b/a. \tag{16}$$

Let $\hat{x}$ satisfy $0 < \hat{x} < x_\infty$, and let $\hat{w} = W_e(\hat{x})$: $\hat{w} < b/a$, since $W_e(0) = b/a$ and $W_e(x)$ is decreasing on $[0, x_\infty)$. Now, $X_s(\hat{w}) > X_e(\hat{w})$, from (16). This implies that $W_s(X_s(\hat{w})) < W_s(X_e(\hat{w}))$, since $W_s(x)$ is decreasing on $[0, \infty)$. Consequently, $W_e(\hat{x}) < W_s(\hat{x})$, using properties of inverses and the definition of $\hat{w}$. Therefore,

$W_e(x) < W_s(x)$ for $0 < x < x_\infty$. But $W_e(x) = Y_e'(x)$, $W_s(x) = Y_s'(x)$, and $Y_e(0) = Y_s(0) = 0$. Hence, using CMVT again, $Y_e(x) < Y_s(x)$ for $0 < x < x_\infty$.

(b) Expanding $(1 - e^{2cx})$ in the expression for $Y_s(x)$ in (12) produces

$$Y_s(x) = \frac{bx}{a} + \frac{gx}{2a^2c} - \frac{g}{4a^2c^2}\sum_{n=1}^{\infty}\frac{(2cx)^n}{n!} = \frac{bx}{a} - \frac{gx^2}{2a^2} - \frac{g}{a^2}\sum_{n=3}^{\infty}\frac{(2c)^{n-2}x^n}{n!}. \tag{17}$$

Since $Y_g(x) = bx/a - gx^2/2a^2$, it follows that $Y_s(x) < Y_g(x)$ for $x > 0$. ■

Let $y_e(t)$ be the exact solution for y in terms of t and $y_g(t) = bt - gt^2/2$, Galileo's solution. A theorem similar to Theorem 1 can be proven for these functions.

Theorem 2. (a) $y_e(t) < y_s(t)$ *for* $t > 0$. (b) $y_s(t) < y_g(t)$ *for* $0 < t \le 2b/g$, $b > 0$.

Proof. (a) Recall that $w_e(t)$ and $w_s(t)$ satisfy (7) and (8), respectively, subject to $w = b/a$ at $t = 0$. This together with (15) shows that $w_e(t) < w_s(t)$ for $t > 0$, using an argument analogous to that used in the proof of Theorem 1(a) to infer that $W_e(x) < W_s(x)$ for $0 < x < x_\infty$. Thus, $q(w_e(t)) > q(w_s(t))$ for $t > 0$, since $q(w)$ is decreasing on $\Re$, and it follows from (15) that $p(w_e(t)) > q(w_s(t))$ for $t > 0$. Also, (6) and (7) imply $u = [p(w)]^{-1/2}$. Therefore,

$$u_e(t) = [p(w_e(t))]^{-1/2} < [q(w_s(t))]^{-1/2} = u_s(t) \quad \text{for } t > 0, \tag{18}$$

where $u_e(t)$ is the exact solution for u in terms of t and $u_s(t) = x_s'(t)$, derivable from (10). But (1) and (5) imply $dy/dt = v = uw$, and so, using (11) and (18),

$$y_e'(t) = u_e(t)w_e(t) < u_s(t)w_s(t) = y_s'(t) \quad \text{for } t > 0. \tag{19}$$

Since $y_e(0) = y_s(0) = 0$, (19) and CMVT imply $y_e(t) < y_s(t)$ for $t > 0$.

(b) Using $y_s(t)$, as given in (11), and $y_g(t) = bt - gt^2/2$ with $b > 0$ yields

$$y_s'(t) - y_g'(t) = \left(b + \frac{g}{2ac}\right)\frac{1}{act+1} + \frac{gt}{2} - \frac{g}{2ac} - b = \frac{act}{act+1}\left(\frac{gt}{2} - b\right). \tag{20}$$

Thus, $y_s'(t) < y_g'(t)$ for $0 < t < 2b/g$. Also, $y_s(0) = y_g(0) = 0$. Hence, by CMVT, $y_s(t) < y_g(t)$ for $0 < t \le 2b/g$. *Note:* $y_g(t) \ge 0$ if and only if $0 \le t \le 2b/g$. ■

Some other trajectory approximations will be briefly considered. The Maclaurin expansion of $w_e(t)$, like one in [**15**] of a function equivalent to $-w_e'(t)$, is used here as a source of such approximations. Repeated differentiation of (7) leads to

$$w_e(t) = \frac{b}{a} - \frac{gt}{a} - \frac{cgs_It^2}{2a} + \frac{bcg^2t^3}{6as_I} + \left(\frac{b}{a} - \frac{ag}{cs_I^3}\right)\frac{c^2g^2t^4}{24} + O(t^5), \tag{21}$$

for sufficiently small t, with $s_I = \sqrt{a^2 + b^2}$ as in (2). Keeping only terms of degree 2 or less in (21) and following the steps in the derivation of $Y_s(x)$ in (12) gives

$$y \approx Y_{w2}(x) = \left(b + \frac{g}{2cs_I}\right)\frac{x}{a} + \frac{g(1 - e^{2cxs_I/a})}{4c^2s_I^2}. \tag{22}$$

This approximation appears in [**15**]; it reduces to $Y_s(x)$ when s_I is replaced by a, and it is expected to be accurate when the t^3 term in (21) is small relative to the t^2 term, i.e., when $t \ll 3s_I^2/|b|g$. $Y_s(x)$ and $Y_{w2}(x)$ estimate the projectile's range (i.e., the x value at which the projectile returns to its initial height) about equally well, on average, over a set of test cases for which $s_T = 40$ m/s, $s_I \in \{10, 20, 40, 80\}$ m/s, and launch angle $\phi \in \{15°, 30°, 45°, 60°, 75°\}$. However, $Y_s(x)$ overestimates the range in all test cases, in accordance with Theorem 1(a), while $Y_{w2}(x)$ underestimates it. So a better trajectory approximation seems to be $Y_{\text{av}}(x) = [Y_s(x) + Y_{w2}(x)]/2$, which gives excellent range estimates with errors of less than 1% in all test cases except those having $s_I \in \{40, 80\}$ m/s with $\phi \in \{60°, 75°\}$. A higher-order approximation, not pursued in [**15**], is found by retaining only terms of degree 3 or less in (21); this leads to a messy expression, $Y_{w3}(x)$, omitted here but given in [**7**]. $Y_{w3}(x)$, like $Y_{\text{av}}(x)$, gives more accurate range estimates than both $Y_s(x)$ and $Y_{w2}(x)$ in all test cases, but $Y_{\text{av}}(x)$ generally gives the best estimates. More details on test case results for various approximations are provided in [**7**]. Figure 2 depicts trajectories $Y_s(x)$, $Y_{w2}(x)$, $Y_{w3}(x)$, $Y_{\text{av}}(x)$, and $Y_e(x)$ for one test case.

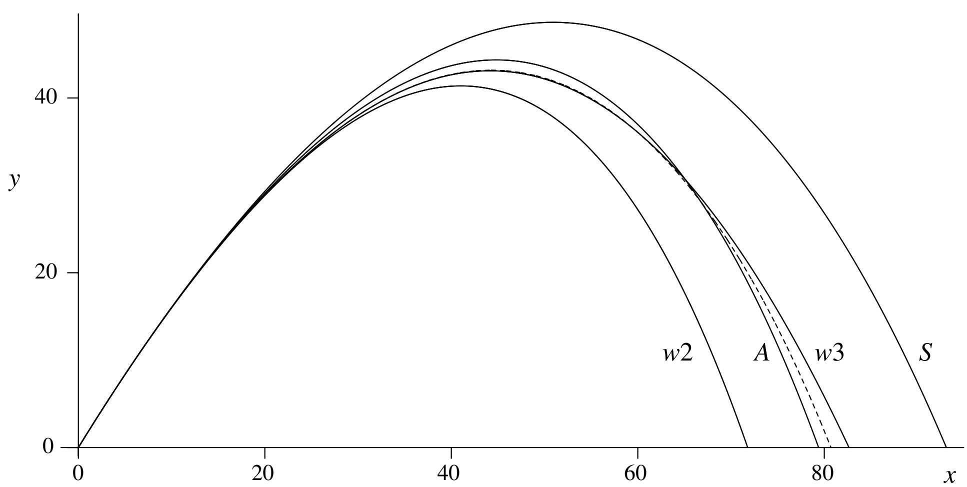

Figure 2. Trajectories $Y_s(x)$, $Y_{w2}(x)$, $Y_{w3}(x)$, $Y_{\text{av}}(x)$ (labeled A), and $Y_e(x)$ (dashed), for terminal speed 40 m/s and initial speed 40 m/s at launch angle 60°.

Rewriting (1) in terms of derivatives of y, s, and θ with respect to x and repeatedly differentiating the result, as in Littlewood [**11**], generates

$$Y_e(x) = \frac{b}{a}x - \frac{g}{2a^2}x^2 - \frac{cgs_I}{3a^3}x^3 - cg\frac{2cs_I^2 - bg/s_I}{12a^4}x^4 + O(x^5), \tag{23}$$

for sufficiently small x. Although (23) has some value as a trajectory approximation (when extended to error $O(x^7)$, its range estimates on the test cases are generally a bit worse, and less consistent, than those of $Y_{\text{av}}(x)$), it is most useful in proofs (as in [**11**]) and for deducing other results, such as this expansion for the range $\overline{x}$:

$$\overline{x} = \frac{2a^2}{g}\phi - \frac{8a^4c}{3g^2}\phi^2 + \frac{40a^6c^2 + 6a^2g^2}{9g^3}\phi^3 + O(\phi^4), \tag{24}$$

for sufficiently small ϕ, which agrees with Galileo's range $2a^2g^{-1}\tan\phi$ when $c = 0$. In [**7**], (24) is derived and shown to be consistent, surprisingly, with expansions of $\overline{x}$ obtained from $Y_s(x)$, $Y_{w2}(x)$, $Y_{w3}(x)$, and $Y_{\text{av}}(x)$. Ranges calculated from these approximate trajectories are all correct to exactly $O(\phi^4)$ as $\phi \to 0$, though the test case results show that some of these range estimates are much better than others.

REFERENCES

1. F. Brauer, What goes up must come down, eventually, this MONTHLY **108** (2001) 437–440.
2. J. Burke, Science Revises the Heavens, part 5 of the British Broadcasting Corporation's 10-part video series *The Day the Universe Changed: A Personal View*, McIntyre Media, Rexdale, Ontario, 1986.
3. C. W. Groetsch, Tartaglia's inverse problem in a resistive medium, this MONTHLY **103** (1996) 546–551.
4. ———, The commandant's corollary, this MONTHLY **107** (2000) 15–23.
5. C. W. Groetsch and B. Cipra, Halley's comment—Projectiles with linear resistance, *Math. Mag.* **70** (1997) 273–280.
6. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.
7. W. W. Hackborn, Motion through air: What a drag, *Can. Appl. Math. Q.* **14** (2006) 285–298.
8. A. R. Hall, *Ballistics in the Seventeenth Century*, Cambridge University Press, Cambridge, 1952.
9. G. H. Hardy, *A Mathematician's Apology*, Cambridge University Press, Cambridge, 1940. Reprinted with a forward by C. P. Snow, 1967.
10. H. Lamb, *Dynamics*, Cambridge University Press, Cambridge, 1929.
11. J. E. Littlewood, Adventures in ballistics, 1915–1918, *Bull. Inst. Math. Appl.* **10** (1974) 323–328.
12. L. N. Long and H. Weiss, The velocity dependence of aerodynamic drag: A primer for mathematicians, this MONTHLY **106** (1999) 127–135.
13. E. J. McShane, J. L. Kelley, and F. V. Reno, *Exterior Ballistics*, University of Denver Press, Denver, CO, 1953.
14. I. Newton, *The Principia* (trans. I. B. Cohen and A. Whitman), 3rd ed., preceded by *A Guide to Newton's Principia* by I. B. Cohen, University of California Press, Berkeley, 1999.
15. G. W. Parker, Projectile motion with air resistance quadratic in the speed, *Amer. J. Phys.* **45** (1977) 606–610.
16. E. J. Routh, *A Treatise on Dynamics of a Particle with Numerous Examples*, Cambridge University Press, Cambridge, 1898.
17. R. D. H. Warburton and J. Wang, Analysis of asymptotic projectile motion with air resistance using the Lambert *W* function, *Amer. J. Phys.* **72** (2004) 1404–1407.

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