The College Mathematics Journal

ISSN: 0746-8342 (Print) 1931-1346 (Online) Journal homepage: https://maa.tandfonline.com/loi/ucmj20

The Oldest Trig in the Book

Harold P. Boas

To cite this article: Harold P. Boas (2019) The Oldest Trig in the Book, The College Mathematics Journal, 50:1, 9-20, DOI: 10.1080/07468342.2019.1535158

To link to this article: https://doi.org/10.1080/07468342.2019.1535158

Published online: 29 Jan 2019.

The Oldest Trig in the Book

Harold P. Boas

Harold P. Boas (boas@tamu.edu) defended his PhD dissertation on functions of several complex variables the year that *Talley's Folly* (by Lanford Wilson) won a Pulitzer Prize. He has been a faculty member at Texas A&M University since the title year of George Orwell's dystopian novel. Fifty years ago, he played chess with the future author of *A Game of Thrones*. When not teaching and learning mathematics, he enjoys literature of all genres.

Do you know the value of $\cos(\pi/5)$ out of your head? When I ask students this question, responses range from apologetic shrugs to nervous chuckles to blank stares, and faculty hardly do better.

Mathematics majors and their instructors are supposed to be familiar with the values of the cosine function at the special angles shown in Table 1. The gap in the table at angle $\pi/5$ stands out like an empty seat in an otherwise full classroom.

Actually, few of my colleagues know all the entries in this table by rote, preferring instead to catch the non-integer cosine values on the fly by visualizing the triangles shown in Figure 1. But the angle $\pi/5$ occurs in no right triangle having two integer sides, so does the value of $\cos(\pi/5)$ admit any simple expression?

I do not recall ever encountering this question during my student days, but recently my attention was drawn to $\cos(\pi/5)$ when an undergraduate who was researching the history of number theory asked for help understanding an article of Cauchy [3] about Fermat's last theorem. A puzzling statement in the paper turned out to be wrong, and the simplest counterexample involved the fifth roots of -1 in the complex plane, hence the angle $\pi/5$.

θ	$\frac{\pi}{1}$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$		$\frac{\pi}{6}$
$\cos(\theta)$	-1	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$		$\frac{1}{2}\sqrt{3}$

Table 1. The standard values of the cosine function.

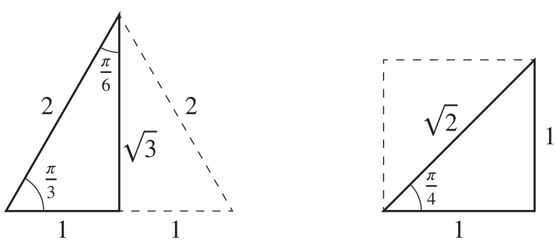

Figure 1. Right triangles for determining the cosine of $\pi/3$, $\pi/6$, and $\pi/4$.

Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/ucmj.

doi.org/10.1080/07468342.2018.1535158

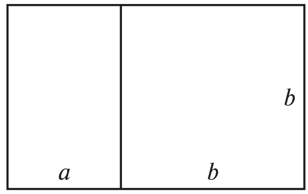

Figure 2. Golden ratio: The small rectangle and the large rectangle are similar.

This special angle not only is a key component of cultural symbols ranging from the talisman of the ancient Pythagoreans to the flag of the modern African nation of Togo but also is a crucial element of applications ranging from Ptolemy's 2nd-century trigonometric table to Penrose's 20th-century aperiodic tiling of the plane. Readers of this journal will have seen in a recent prize-winning article by Travis Kowalski [**18**] a diagram from which $\cos(\pi/5)$ can be determined. Table 1 suggests that the value should be $1/2$ times some special number, and that number turns out to be the golden ratio.

The quotient b/a of two positive real numbers represents the golden ratio if the numbers a and b are related in such a way that b is the geometric average of a and $a+b$. Equivalently,

$$\frac{b}{a} = \frac{b+a}{b}.$$

Figure 2 shows a standard geometric interpretation of this equation. Nowadays, the similarity in the figure is interpreted as the initial stage of a fractal construction.

Representing b/a by x yields that $x = 1 + x^{-1}$, or $x^2 = x + 1$, so the quadratic formula implies that the golden ratio equals $(1+\sqrt{5})/2$. A common notation for the golden ratio is the Greek letter ϕ, but I reserve that letter for a different use later. In summary, here is the missing entry in Table 1:

$$\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4} = \frac{1}{2} \times \text{(golden ratio)}. \tag{$\bigstar$}$$

This formula might be the oldest significant mathematical fact missing from the standard American curriculum. My goal in this article is to popularize the result, which is a legacy of some of the most influential mathematicians of all time. I begin by exhibiting three proofs that first-year undergraduates can readily understand—or even discover, given suitable hints. Then I set the problem into a broader mathematical and historical framework.

Double your fun

The first proof surprisingly sneaks up on the angle $\pi/5$ by using only angle doubling. When θ is an arbitrary angle, two applications of the double-angle formula reveal that

$$\begin{aligned}\cos(4\theta) &= 2\cos^2(2\theta) - 1 = 2(2\cos^2(\theta) - 1)^2 - 1 \\ &= 8\cos^4(\theta) - 8\cos^2(\theta) + 1.\end{aligned}$$

Accordingly, if θ is an angle for which $\cos(4\theta)$ happens to equal $\cos(\theta)$, then $\cos(\theta)$ will be a root of the polynomial $8x^4 - 8x^2 - x + 1$.

Certainly $\cos(4\theta) = \cos(\theta)$ when 4θ and θ represent the same angle: namely, when $\theta = 0$ and also when $\theta = \pm 2\pi/3$. Since $\cos(0) = 1$ and $\cos(\pm 2\pi/3) = -1/2$, the indicated polynomial has both $x - 1$ and $2x + 1$ as factors. Division of polynomials now produces the following identity:

$$8x^4 - 8x^2 - x + 1 = (x - 1)(2x + 1)(4x^2 + 2x - 1).$$

The roots of the quadratic factor $4x^2 + 2x - 1$ must be the values of the cosine arising from the other cases in which $\cos(4\theta) = \cos(\theta)$: namely, when 4θ and $-\theta$ represent the same angle, that is, when $\theta = \pm 2\pi/5$ and also when $\theta = \pm 4\pi/5$. Since the angle $2\pi/5$ lies in the first quadrant (where the cosine has positive values), and $4\pi/5$ lies in the second quadrant (where the cosine has negative values), applying the quadratic formula to the polynomial $4x^2 + 2x - 1$ shows that

$$\cos\left(\frac{2\pi}{5}\right) = \frac{-1 + \sqrt{5}}{4} \qquad \text{and} \qquad \cos\left(\frac{4\pi}{5}\right) = \frac{-1 - \sqrt{5}}{4}.$$

Angles that sum to π have cosines that are negatives of each other, so

$$\cos\left(\frac{3\pi}{5}\right) = \frac{1 - \sqrt{5}}{4} \qquad \text{and} \qquad \cos\left(\frac{\pi}{5}\right) = \frac{1 + \sqrt{5}}{4}.$$

This method yields not only the cosine of $\pi/5$ but also the cosines of the integer multiples of $\pi/5$.

Imagine that

A more direct evaluation of $\cos(\pi/5)$ is available through exploiting the complex exponential function. When θ is an arbitrary angle,

$$\begin{aligned} 2\cos(\theta) &= e^{i\theta} + e^{-i\theta}, \qquad \text{so} \\ 4\cos^2(\theta) &= e^{2i\theta} + 2 + e^{-2i\theta}. \end{aligned}$$

Substitute $\pi/5$ for θ, subtract 1 plus the first line from the second line, rearrange the terms, and use that $e^{i\pi} = -1$:

$$\begin{aligned} 4\cos^2\left(\frac{\pi}{5}\right) - 2\cos\left(\frac{\pi}{5}\right) - 1 &= 1 - e^{i\pi/5} + e^{2\pi i/5} + e^{-2\pi i/5} - e^{-i\pi/5} \\ &= 1 - e^{i\pi/5} + e^{2\pi i/5} - e^{3\pi i/5} + e^{4\pi i/5}. \end{aligned}$$

The final expression vanishes, for $e^{i\pi/5}$ is a root of the polynomial $1 + z^5$, which factors as $(1 + z)(1 - z + z^2 - z^3 + z^4)$. The upshot is that $\cos(\pi/5)$ is a root of the polynomial $4x^2 - 2x - 1$. The quadratic formula yields the required positive value $(1 + \sqrt{5})/4$.

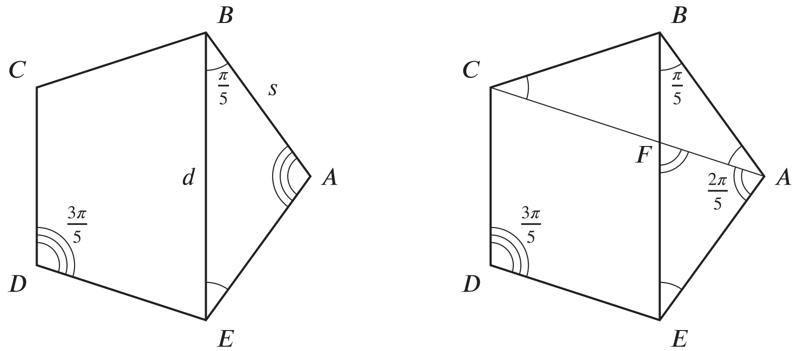

Figure 3. A pentagonal construction.

Hang five

A geometric argument is yet another way to arrive at the value of $\cos(\pi/5)$. Figure 1 shows that the cosine can be obtained for angles $\pi/3$ and $\pi/4$ by bisecting an equilateral triangle and a square, so a natural idea is next to consider a pentagon.

Since the angles in an n-gon sum to $(n-2)\pi$, each angle in a regular pentagon equals $3\pi/5$, as indicated in the left-hand part of Figure 3. The diagonal BE cuts off an isosceles triangle, and since the angle at vertex A is $3\pi/5$, each of the other two angles in triangle ABE equals $\pi/5$. Dropping a perpendicular from vertex A to the midpoint of line BE reveals that $\cos(\pi/5)$ equals half the ratio of the length d of a diagonal to the length s of a side of the pentagon. What remains to show is that this ratio d/s equals the golden ratio.

The right-hand part of Figure 3 shows a second diagonal AC crossing the first at a point F. By symmetry, the angle at A in triangle ABC is $\pi/5$. Since the full angle at A is $3\pi/5$, the angle at A in triangle AEF is $2\pi/5$. The three angles in triangle AEF sum to π, so the angle at F in this triangle is $2\pi/5$ as well. Thus the triangle AEF is isosceles, whence the segment EF has the same length as the segment AE: namely, length s. Moreover, the isosceles triangle BAF is similar to the triangle BEA, so the length ratio BE/AB equals the ratio AB/BF. Since $AB = s = EF$, an equivalent statement is that

$$\frac{d}{s} = \frac{BE}{EF} = \frac{EF}{BF}.$$

In other words, the diagonal BE is divided at F according to the golden ratio, and this ratio equals d/s, as claimed.

Take the next step

Can Table 1 be extended with a nice expression for the cosine of $\pi/7$, the next angle in the sequence? I challenge the reader to adapt either of the preceding analytic methods to show that the numbers $\cos(\pi/7)$, $\cos(3\pi/7)$, and $\cos(5\pi/7)$ are the roots of the cubic polynomial $8x^3 - 4x^2 - 4x + 1$. The formula for solutions of cubic equations then implies that

$$\cos\left(\frac{\pi}{7}\right) = \frac{1}{6}\left(1 + \sqrt[3]{\frac{7}{2}\left(-1 + 3i\sqrt{3}\right)} + \sqrt[3]{\frac{7}{2}\left(-1 - 3i\sqrt{3}\right)}\right),$$

where the cube roots of complex numbers on the right-hand side are chosen to be the roots closest to the positive part of the real axis. This representation for $\cos(\pi/7)$ certainly is not a simple or memorable one!

But the regular heptagon (or 7-gon) cannot be constructed with straightedge and compass, so might $\pi/7$ be an exceptional, unlucky choice of angle? To the contrary, I claim that integer multiples of $\pi/4$, $\pi/5$, and $\pi/6$ are the *only* simple angles that have simple cosines. (Multiples of $\pi/2$ and $\pi/3$ are automatically included, since they are multiples of $\pi/6$.) What I mean by "simple" is that the angle θ is a rational multiple of π radians—equivalently, a rational number of degrees—and that $\cos(\theta)$ is a root of a quadratic polynomial having integer coefficients. For example, the polynomial $2x^2 - 1$ has $\cos(\pi/4)$ as a root.

To verify the claim, start with the easy special case of integer values of the cosine. The realization of $\cos(\theta)$ as the abscissa of a point on the unit circle with angle θ shows that 1, 0, and -1 are the only integers in the range of the cosine function, and these values are realized for angles equivalent to 0, $\pm\pi/2$, and π.

What about non-integer rational values? The rational numbers $\pm 1/2$ are taken by the cosine at angles $\pm\pi/3$ and $\pm 2\pi/3$. Many authors have observed that no additional rational values can be obtained as $\cos(m\pi/n)$ when m and n are natural numbers, a canonical reference being Ivan Niven's Carus Monograph [25]. J. M. H. Olmsted published a wholly elementary proof [26] that uses nothing more than multiple-angle formulas for the cosine.

Here is a shorter but more sophisticated argument. Suppose $\cos(\theta)$ equals a rational number r that is not an integer. Since

$$e^{i\theta} + e^{-i\theta} = 2\cos(\theta) = 2r,$$

multiplying by $e^{i\theta}$ implies that

$$e^{2i\theta} - 2re^{i\theta} + 1 = 0.$$

Accordingly, the complex number $e^{i\theta}$ and its complex conjugate $e^{-i\theta}$ are the two (distinct) roots of the quadratic polynomial $z^2 - 2rz + 1$. If additionally the angle θ is a rational multiple of π, then there is some natural number n larger than 2 such that $e^{i\theta}$ and $e^{-i\theta}$ are roots of the polynomial $z^n - 1$. This polynomial can be factored as a product $(z^2 - 2rz + 1)p(z)$ for some polynomial p with rational coefficients. Since the polynomial $z^n - 1$ has integer coefficients and leading coefficient 1, a standard algebraic proposition known as Gauss's lemma implies that the factors $z^2 - 2rz + 1$ and $p(z)$ have *integer* coefficients. In particular, the rational number $2r$ is an integer. Since $|r| \leq 1$, and r is not an integer, the only possible values that r can have are $\pm 1/2$.

Finally, admit irrational numbers into the mix. The values of $\cos(\theta)$ that are simple—according to my definition—are real numbers of the form $r_1 + r_2\sqrt{m}$, where r_1 and r_2 are rational numbers and m is a positive integer. The same reasoning as in the preceding paragraph shows that

$$(e^{i\theta} + e^{-i\theta} - 2r_1)^2 = 4mr_2^2,$$

and multiplying by $e^{2i\theta}$ reveals that $e^{i\theta}$ satisfies a fourth-degree polynomial equation with rational coefficients. If θ is a rational multiple of π, then θ can be written in the form $2\pi k/n$ for coprime natural numbers k and n. In other words, $e^{i\theta}$ is a primitive nth root of unity. I now use a result of Gauss about cyclotomic polynomials: namely, the minimal polynomial over the rational numbers of a primitive nth root of unity has

n	2	3	4	5	6	7	8	9	10	11	12	≥ 13
$\phi(n)$	1	2	2	4	2	6	4	6	4	10	4	≥ 6

Table 2. Values of the totient function.

degree equal to the total number of primitive nth roots of unity. This quantity is Euler's totient function $\phi(n)$, the number of positive integers up to n that are coprime to n. The preceding discussion can be summarized as saying that $\phi(n) \leq 4$.

This inequality is easily seen to hold for only a few special values of n. Indeed, $\phi(p) = p - 1$ when p is a prime number; more generally $\phi(p^j) = p^{j-1}(p-1)$ when p is prime and j is a positive integer; and ϕ has the multiplicative property that $\phi(ab) = \phi(a)\phi(b)$ when a and b are coprime. Using these properties, you should be able to verify the values of ϕ shown in Table 2, which reveals that if $\phi(n) \leq 4$, then $2\pi/n$ is a multiple of $\pi/4$ or $\pi/5$ or $\pi/6$.

I have proved my claim, but there is more to the story. D. H. Lehmer [**19**] showed that if k and n are coprime natural numbers, then $\cos(2\pi k/n)$ satisfies a polynomial equation with rational coefficients, and when $n > 2$, the minimal degree of such an equation is $\phi(n)/2$.

Reach for the stars

The preceding topics have a long history. I now address the background, a fascinating part of mathematical culture.

All that glisters. Five centuries ago, the sun revolved around the earth, physicians believed in the four humors, and moveable type was cutting-edge printing technology. Would a time traveler from those days recognize any aspect of today's society?

One luminary who would is the Renaissance scholar Luca Pacioli, a Franciscan friar remembered as the father of accounting due to the exposition of double-entry bookkeeping contained in his mathematics textbook [**28**]. Perhaps he would be surprised by the durability of the principles of keeping ledgers that he learned from Venetian merchants. Pacioli devoted another treatise [**27**] to what he called the "divine proportion" (the golden ratio). This opus is remembered not so much for the scientific content as for the drawings of polyhedra contributed by Pacioli's roommate and star mathematics pupil, Leonardo da Vinci. The letter M used as a logo for many years by the Metropolitan Museum of Art in New York City derives from an alphabet constructed geometrically in Pacioli's book (Figure 4).

Since Pacioli is not a household name, you might think that I am exaggerating the significance of his contributions. Indeed, some writers have faulted him for alleged plagiarism [**13**, p. 150]. Yet his influence is undeniable. A century after the publication of Pacioli's book, the words "divine proportion" were standard terminology for the imperial mathematician Johannes Kepler, a creative genius who is a star of the first magnitude on everybody's chart.

Some mathematics groupies maintain that the golden ratio is ubiquitous in art and in nature, but professional mathematicians who have studied the matter regard this notion as a childish fantasy [**5**, **13**, **24**]. For the past century and a half, starting with the pioneering work of Gustav Theodor Fechner [**6**], experimental psychologists have examined whether the golden rectangle shown in Figure 2 is a maximally aesthetically pleasing shape, the current consensus being negative [**20**, pp. 178–179], [**31**, Section 6.5].

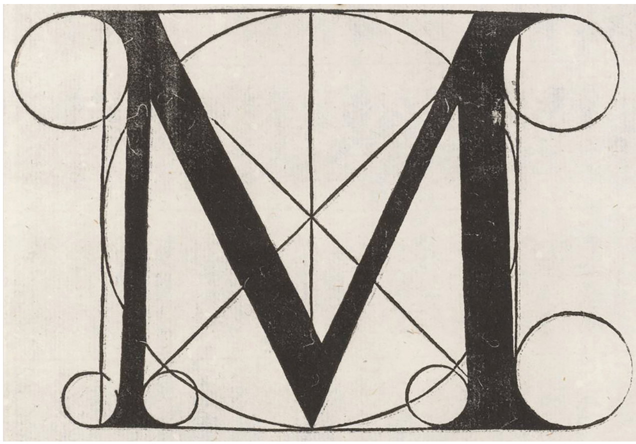

Figure 4. The letter M designed in *Divina proportione*.

The numerical value of $(1+\sqrt{5})/2$ (about 1.618) can be approximated crudely by 3/2 and more closely by 8/5, a rational number that has the appealing representations

$$1+\frac{36}{60} \qquad \text{and} \qquad 1+\frac{1}{2}+\frac{1}{10}$$

in Babylonian sexagesimal style and in Egyptian-fraction style. More accurate rational approximations are available by taking the quotient of any two consecutive Fibonacci numbers, say 144/89. The simple irrational numbers $\sqrt{2}$ and $\sqrt{3}$ roughly approximate the golden ratio, and the transcendental numbers $\pi/2$ and $(4/\pi)^2$ do better. The golden ratio is also close to 1.609, the conversion factor from miles to kilometers. If you encounter in the wild a number in the vicinity of 1.6, why should you believe that the true value is the golden ratio rather than one of the competing approximations?

There is no documentary evidence that the golden ratio was a design element in classical Greek architecture. Nonetheless, the concept of "the extreme and mean ratio" (as the golden ratio is named in Definition 3 of Book VI of Euclid's *Elements*) was familiar to the followers of Pythagoras, according to Sir Thomas Heath [**12**, p. 403]. In Book XIII of the *Elements*, Euclid demonstrates the key step in the geometric proof of equation (★): namely that the ratio of the diagonal of a regular pentagon to the side is equal to the golden ratio. Consequently, there is ample justification for saying that the value of $\cos(\pi/5)$ is implicit in the mathematics of ancient Greece, even though the concept of the cosine function did not yet exist.

Some writers have gone further in attributing modern ideas to the predecessors of Euclid, observing that Figure 5 implicitly contains a proof that the golden ratio is an irrational number. By considering similar triangles in the figure, you should be able to see that if the diagonal and the side of the pentagon were commensurable quantities—that is, integer multiples of some common unit of measurement—then the diagonal and the side of the smaller pentagon in the center of the figure would be multiples of the very same unit. Iterating this argument until the central pentagon shrinks to a size smaller than the supposed common unit of measurement results in a contradiction.

This geometric argument for the irrationality of the golden ratio uses no knowledge about factorization of integers, hence may be viewed as conceptually simpler than the standard proof of the irrationality of $\sqrt{2}$. If the second-century rhetorician Lucian's

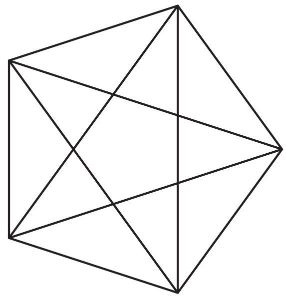

Figure 5. A pentagram inscribed in a regular pentagon.

Figure 6. Flag of Togo.

"Slip of the tongue" essay is to be believed, the pentagram had special significance to the followers of Pythagoras. Accordingly, some authors have proposed [**4**, **9**, **34**] that the diagonal of a regular pentagon of side 1, not the diagonal of a square of side 1, could have been the first number discovered to be irrational. This charming conjecture is a heterodox opinion [**13**], [**16**, pp. 29–31] that is historically undecidable—neither verifiable nor falsifiable. There is, however, compelling evidence [**7**] that everything you think you know about the discovery of irrational numbers is wrong.

The five-pointed star is an iconic emblem found everywhere from the shield of Sir Gawain and the badge of the Texas Rangers to the Hollywood Walk of Fame. The star appears on many world flags, including that of the United States of America, and on 23 of the 50 individual state flags. The flag of the West African republic of Togo (Figure 6), created in 1960 by artist Paul Ahyi, not only displays a five-pointed star but supposedly was designed to have aspect ratio equal to the golden ratio, a feature apparently not implemented in practice. (For that matter, commercial United States flags do not conform to the government specification [4 U. S. C. Section 1] of a 10 by 19 shape.)

General joy of the whole table. In fields ranging from astronomy to aeronautics and from surveying to navigation, "computer" used to be not a programmable device but a job title [**10**] (a job often held by women [**14**, **32**, **33**]). Tables of numerical values of special functions once were an indispensable aid to calculation. Modern digital electronics have made such tables largely obsolete, but there is continuing interest in finding closed-form expressions of values of functions at particular arguments [**18**].

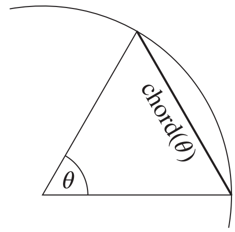

Figure 7. The chord of angle θ.

Special values of functions were more widely known in the 19th century than now in the 21st century. William Hopkins, in an exposition of trigonometry "published under the superintendence of the Society for the Diffusion of Useful Knowledge," presents a table [**15**, pp. 38–39] of exact values of the sine function (equivalently the cosine, which is the sine of the complementary angle) at all multiples of 3° in the first quadrant. His method is first to invoke multiple-angle formulas to evaluate the trigonometric functions at angle $2\pi/5$ (equivalently 72°) and hence at the complementary 18° angle. The half-angle formula applied to the standard 30° angle gives the functions at 15°. Identities for trigonometric functions at the difference of two angles yield the values at 3° from those for 18° and 15°. Multiple-angle formulas then generate exact values of the trigonometric functions at all multiples of 3°. What distinguishes these angles is that they have integer values in degrees and are constructible by straightedge and compass [**8**].

This method of starting with angle 72° and working down to 3° goes back at least to Claudius Ptolemy's magnum opus on mathematical astronomy, the *Almagest* (second century CE), available in a definitive English translation by G. J. Toomer [**29**]. The details differ, of course, since Greek trigonometry is based not on the sine or the cosine but on the chord of the angle (Figure 7), which equals twice the sine of half the angle inscribed in a unit circle. And Ptolemy is interested not in exact values but in numerical approximations useful for calculation. Moreover, Ptolemy's decisive technical device is not the difference formula for sines and cosines but rather his eponymous theorem about a quadrilateral inscribed in a circle: the product of the diagonals equals the sum of the products of opposite sides (Figure 8).

The first trigonometric table for which there is convincing historical evidence is that of Hipparchus in the second century BCE, but his work is lost, supplanted by Ptolemy's masterpiece, "dominant to an extent and for a length of time which is unsurpassed by any scientific work except Euclid's *Elements*" [**29**, p. 2]. The ancient Babylonian clay tablet known as Plimpton 322 has been in the news recently, touted as the oldest trigonometric table [**22**], but this speculative interpretation is well known [**17**, **23**] and has previously been judged doubtful [**1**, **2**, **30**].

Since the geometry of angle $\pi/5$ has been a part of mathematical lore for such a long time, everybody was astonished in the 1970s by Sir Roger Penrose's remarkable discovery that the two rhombuses in Figure 9 can be used to tile the plane (Figure 10), but never in a periodic way [**11**, Section 10.3]. Such tilings have local (but not global) fivefold symmetry. Suddenly mathematicians perceived such symmetries lurking everywhere, even in Islamic art contemporaneous with Fibonacci [**21**]. Then in the 1980s, Dan Shechtman revolutionized materials science by observing an "impossible" fivefold symmetry in a mixture of aluminum and manganese. He subsequently received the 2011 Nobel Prize in Chemistry for his discovery of so-called quasicrystals, a topic of continuing interest both to applied scientists and to pure mathematicians.

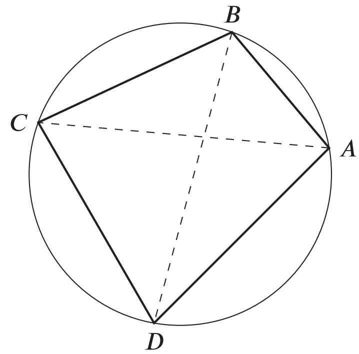

Figure 8. Ptolemy's theorem: $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

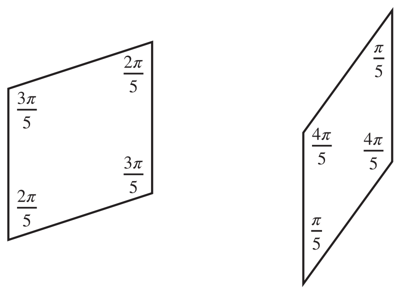

Figure 9. Penrose rhombs.

Figure 10. Tiled floor in the lobby of the Mitchell Physics Building at Texas A&M University in College Station.

Go, and sin( ) no more

The past and the present of $\cos(\pi/5)$ are intertwined with such prominent personages as Pythagoras, Ptolemy, Pacioli, and Penrose. There is yet more to the plot, for every pithy answer contains the seed of a new problem. I pose the following puzzle for you to ponder.

I have demonstrated that multiples of $\pi/4$, $\pi/5$, and $\pi/6$ are the only simple angles with simple cosines. And you learned in trigonometry that the sine function and the cosine function are parallel objects: both functions arise from projecting points of the unit circle onto a line, and the graphs of the two functions are translates of each other.

Yet equation (★) implies, via the basic identity that $\sin^2(\theta) + \cos^2(\theta) = 1$, the following evaluation:

$$\sin\left(\frac{\pi}{5}\right) = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}.$$

Consequently, in contrast to the number $\cos(\pi/5)$, the value $\sin(\pi/5)$ satisfies not a second-degree equation with integer coefficients but instead a fourth-degree equation.

How do you explain that the cosine function is a simpler function than the sine?

Summary. This historical exposition discusses the theory and the applications of the cosine of 36 degrees.

References

[1] Britton, J. P., Proust, C., Shnider, S. (2011). Plimpton 322: A review and a different perspective. *Arch. Hist. Exact Sci.* 65: 519–566. doi.org/10.1007/s00407-011-0083-4
[2] Buck, R. C. (1980). Sherlock Holmes in Babylon. *Amer. Math. Monthly*. 87(5): 335–345. doi.org/10.2307/2321200
[3] Cauchy, A. (1847). Mémoire sur de nouvelles formules relatives à la théorie des polynômes radicaux, et sur le de nier théorème de Fermat. *C. R. Acad. Sci. Paris*. 24: 469–481.
[4] Choike, J. R. (1980). The pentagram and the discovery of an irrational number. *Two-Year College Math. J.* 11(5): 312–316. doi.org/10.2307/3026893
[5] Falbo, C. (2005). The golden ratio—a contrary viewpoint. *College Math. J.* 36(2): 123–134. doi.org/10.2307/30044835
[6] Fechner, G. T. (1865). Über die Frage des goldenen Schnittes. *Arch. zeichn. Künste*. 11: 100–112.
[7] Fowler, D. H. (1994). The story of the discovery of incommensurability, revisited. In: Gavroglu, K., Christianidis, J., Nicolaidis E., eds. *Trends in the Historiography of Science*. Dordrecht: Springer, pp. 221–235. doi.org/10.1007/978-94-017-3596-4_17
[8] Francis, R. L. (1978). A note on angle construction. *Two-Year College Math. J.* 9(2): 75–80. doi.org/10.2307/3026605
[9] von Fritz, K. (1945). The discovery of incommensurability by Hippasus of Metapontum. *Ann. Math., second ser.* 46(2): 242–264. doi.org/10.2307/1969021
[10] Grier, D. A. (2005). *When Computers were Human*. Princeton, NJ: Princeton Univ. Press.
[11] Grünbaum, B., Shephard, G. C. (2016). *Tilings and Patterns*, 2nd ed. Mineola, NY: Dover.
[12] Heath, T. L. (1908). *The Thirteen Books of Euclid's Elements*, Vol. 1. Cambridge: Cambridge University Press.
[13] Herz-Fischler, R. (1998). *A Mathematical History of the Golden Number*. Mineola, NY: Dover.
[14] Holt, N. (2016). *Rise of the Rocket Girls: The Women Who Propelled Us, from Missiles to the Moon to Mars*. Boston, MA: Little, Brown.
[15] Hopkins, W. (1833). *Elements of Trigonometry*. London: Baldwin and Cradock.
[16] Knorr, W. R. (1975). *The Evolution of the Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry*. Dordrecht: Reidel.
[17] Knuth, D. E. (1972). Ancient Babylonian algorithms. *Comm. ACM*. 15(7): 671–677. doi.org/10.1145/361454.361514

[18] Kowalski, T. (2016). The sine of a single degree. *College Math. J.* 47(5): 322–332. doi.org/10.4169/college.math.j.47.5.322

[19] Lehmer, D. H. (1933). A note on trigonometric algebraic numbers. *Amer. Math. Monthly*. 40(3): 165–166. doi.org/10.2307/2301023

[20] Livio, M. (2002). *The Golden Ratio: The Story of Phi, the World's Most Astonishing Number*. New York: Broadway Books.

[21] Makovicky, E. (1992). 800-year-old pentagonal tiling from Marāgha, Iran, and the new varieties of aperiodic tiling it inspired. In: Hargittai, I., ed. *Fivefold Symmetry*. Singapore: World Scientific, pp. 67–86.

[22] Mansfield, D. F., Wildberger, N. J. (2017). Plimpton 322 is Babylonian exact sexagesimal trigonometry. *Hist. Math.* 44(4): 395–419. doi.org/10.1016/j.hm.2017.08.001

[23] Maor, E. (1998). *Trigonometric Delights*. Princeton, NJ: Princeton University Press.

[24] Markowsky, G. (1992). Misconceptions about the golden ratio. *College Math. J.* 23(1): 2–19. doi.org/10.2307/2686193

[25] Niven, I. (1956). *Irrational Numbers*. Washington, DC: Mathematical Association of America.

[26] Olmsted, J. M. H. (1945). Rational values of trigonometric functions. *Amer. Math. Monthly*. 52(9): 507–508. doi.org/10.2307/2304540

[27] Pacioli, L. (1509). *Divina proportione*. Venice: Paganini.

[28] Pacioli, L. (1523). *Summa de arithmetica, geometria, proportioni, et proportionalita*, reprint of the 1494 original. Toscolano: Paganini.

[29] *Ptolemy's Almagest* (1998). Transl. from the Greek, annotated, with a preface, introduction and appendices by G. J. Toomer. Princeton, NJ: Princeton University Press.

[30] Robson, E. (2002). Words and pictures: New light on Plimpton 322. *Amer. Math. Monthly*. 109(2): 105–120. doi.org/10.2307/2695324

[31] van der Schoot, A. (2005). *Die Geschichte des goldenen Schnitts: Aufstieg und Fall der göttlichen Proportion*. (Häring, S., trans.) Stuttgart: Frommann-Holzboog.

[32] Shetterly, M. L. (2016). *Hidden Figures: The American Dream and the Untold Story of the Black Women Mathematicians Who Helped Win the Space Race*. New York: HarperCollins.

[33] Sobel, D. (2016). *The Glass Universe: How the Ladies of the Harvard Observatory Took the Measure of the Stars*. New York: Viking.

[34] Zeuthen, H.-G. (1910). Notes sur l'histoire des mathématiques. VIII. Sur la constitution des livres arithmétiques des Éléments d'Euclide et leur rapport à la question de l'irrationalité. *Overs. Kgl. Danske vidensk. selsk. forh.* 1910(5): 395–435.