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CERTAIN MATHEMATICAL ACHIEVEMENTS OF JAMES GREGORY

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For a long time the light of James Gregory did not shine as brightly as did that of John Wallis, Isaac Barrow and Isaac Newton, the other three great British mathematicians of the seventeenth century. Only recently, through the endeavours of several Scottish mathematicians, especially E. T. Whittaker, G. A. Gibson and H. W. Turnbull, Gregory's genius is revealed and fills with admiration all those interested in the development of modern mathematics.

The "*James Gregory Tercentenary Memorial Volume*," edited by H. W. Turnbull [1], contains Gregory's momentous scientific correspondence, mostly with J. Collins. An extremely important supplement is the large number of Gregory's hitherto unpublished notes, recording his mathematical ideas and calculations. These notes were found in a collection of documents in the University of St. Andrews Library, written on the blank spaces of letters to Gregory. This material affords the possibility of studying his achievements and ideas.

In this paper we shall discuss Gregory's expansions of general and particular functions into series. In addition, we shall exhibit the ideas which are set forth in his first mathematical publication "*Vera circuli et hyperbolae quadratura*" [2]. These ideas are concerned, to some extent, with the associated problem of constructing by certain limiting processes the functions which measure the areas of circles and conics.

1. The "Taylor's series". In a letter of February 15, 1671 to J. Collins (see "*Memorial*" [1], pp. 170 ff.) Gregory gives the power series for seven important functions, each with 5 or 6 terms. These functions are, if for the sake of brevity we may use modern notations,

$$\text{arc tan } x,\ \tan x,\ \sec x,\ \log \sec x,\ \log \tan\left(\frac{x}{2}+\frac{\pi}{4}\right),$$

$$\text{arc sec } (\sqrt{2}\, e^x), \qquad 2 \text{ arc tan}\left(\tanh \frac{x}{2}\right).$$

He mentions without further explanation that he had some knowledge of Newton's "universal method." Hereby, he refers to some series which Newton had discovered and which Collins had but recently communicated to him.

We may surmise that he obtained the arc tangent series in a way analogous to that by which three years earlier N. Mercator [3] had found the series for $\log(1+x)$. He may have considered arc tan x as the area under the curve $y=(1+x^2)^{-1}$, transformed $(1+x^2)^{-1}$ by formal division into a power series and finally integrated this infinite sum. However, there is no possibility of obtaining the other series in a similar way.

On the blank space of a letter to Gregory, dated January 29, 1671, Turnbull found a group of computations about just these seven functions [4]. The comparison of these computations with Gregory's expansions indicates the way of his thoughts. First, they include almost without exception, as many of the successive derivatives of the functions, as would be needed in finding the 5 or 6 numerical coefficients of the series by successive differentiation. Second, all coefficients in Gregory's series are correct with the exception of a single coefficient in both the expansions for tan x and for log sec x. (The second error is a consequence of the first since he obviously obtained the log sec series by integrating the tangent series.) Finally, all derivatives in Gregory's notes are correct with the exception of a single numerical error in the derivatives of tan x, which was probably due to miscopying one number. However, using this erroneous value one finds exactly the erroneous coefficients in the series for tan x and log sec x. From these two facts, Turnbull argues conclusively that Gregory used the tables of the derivatives for the construction of his power series.

We see two possibilities for such a construction. On the one hand, we may imagine that Gregory applied in each particular case something like the "method of undetermined coefficients" together with successive differentiation. That he mentions "Newtons universal method" immediately before giving his series may be considered as supporting this assumption. In fact, if we look upon the whole of Newton's work we are justified in assuming that Gregory thought of this combined method as "Newtons universal method," even though the idea had been sketched as early as 1637 by Descartes in his "géométrie," and had since been applied by many other mathematicians. Nevertheless, Gregory's remark must be considered as a mere guess based upon the few results from Newton's still unpublished investigations which Collins had communicated to him with no hint about Newton's method.

On the other hand, we may suppose that Gregory could have applied the same process for an unspecified function and could have obtained the general expression for the nth coefficient of the expansion. Thus he would have anticipated Taylor's classical expansion by forty-four years. Neither the letters nor the other material, so far as published, substantiate the latter possibility. From all these facts, we may conclude that Gregory possessed a method for finding the Taylor expansion of any *particular* function, but we cannot affirm that he possessed Taylor's formula for an *unspecified* function.

It may be interesting that the second man, C. Maclaurin, whose name is closely associated with this series, deduced it seventy years later, in his "*Treatise of Fluctions*" (1742) by a reasoning similar to that of Gregory. Of course he applied it at once to an unspecified function. He quotes Taylor's book for the formula but could not have known Gregory's discovery then buried in the correspondence.

2. The interpolation formula. For the independent discovery by Gregory of a famous interpolation formula, full evidence is given in a letter of his published long ago. Nevertheless, nobody seems to have realized this fact until E. T. Whittaker brought it to general notice. In the letter to Collins [5] of November 23, 1670 Gregory stated explicitly a formula which interpolates for a function $y=f(x)$ when its values at equidistant points $0, c, 2c, 3c, \cdots$ are given. This formula is identical with the famous formula

$$(1)\ f(x) = f(0) + \frac{x}{c}\Delta f(0) + \frac{x(x-c)}{c\cdot 2c}\Delta^2 f(0) + \frac{x(x-c)(x-2c)}{c\cdot 2c\cdot 3c}\Delta^3 f(0) + \cdots,$$

which Newton made known some years later [6] and which mostly bears his name. It is not essential that Gregory assumes here $f(0)=0$. Further, we may note that, of course, he did not have for the differences the notation $\Delta f(0)$, $\Delta^2 f(0)$, $\Delta^3 f(0), \cdots$. This came into use much later under the influence of Leibniz's symbolism. He takes single letters $d, f, h, \cdots$ for these values, carefully defined by forming the sequences of the 1st, 2nd, 3rd, $\cdots$ differences. Newton uses almost the same notation as Gregory.

In the correspondence on this formula between Collins and Gregory [7], there is mentioned the procedure which Briggs had used in extending his table of logarithms to subintervals. Briggs took differences, generalizing the older method of linear interpolation. His procedure can be considered in some way as the predecessor of the interpolation formula. However, Briggs does not state such a formula nor does he give any motivation of this procedure. Gregory's formula was given in answer to a question raised by Collins for such a motivation.

Of course, Gregory also states his formula without a proper proof, but it is obvious that he could and did verify the formula for polynomials. The same is true for Newton's first publications, although later, in the "*methodus differentialis*," he sketches a way to derive the formula. It is interesting that the interpolation of tables is only *one* aim of Gregory's statement; he emphasizes strongly its use for the problem of approximate quadrature of curves and gives various formulas in this connection. Incidentally Newton [8] makes the same application of the interpolation formula.

The infinite process which is involved in this interpolation formula implies a serious mathematical difficulty which even its discoverers may have felt semiconsciously. The polynomial $P_n(x)$ of the nth degree which is given by the first $n+1$ terms of the formula (1) takes on the values of $f(x)$ at the equidistant points $0, c, 2c, \cdots, nc$, and is determined by this property. This, obviously, is the essential fact which was discovered and communicated by Gregory and Newton. Yet they tacitly assumed that for other unspecified values of x the successive polynomials $P_n(x)$ yield an approximation to $f(x)$ which can be improved by increasing n. Apparently, they thought only of such values of x which are located *between* $0, c, \cdots, nc$, that is to say, they considered only the proper problem of *interpolation*. Here the fact of the steadily improved approximation looks rather evident although a precise formulation and an exact proof was not within the range of these early developments. Things are different if one turns to the problem of *extrapolation*, considering values x *outside* the interval of the multiples of c. The published material gives no evidence that Gregory used his formula for extrapolation. And Newton in the "*Philosophiae Naturalis Prin-*

cipia" [6] applies the interpolation formula not in order to find the place of a comet at any time beyond the range of the observations, but only for intermediate moments.

It is important to realize this situation since the way from the interpolation formula to the Taylor series goes through a sort of extrapolation. Assuming c infinitely small, one concentrates $0, c, 2c, \cdots$ in an arbitrarily small neighborhood of a fixed value and one seeks an expression for $f(x)$ at another fixed value at a finite distance. This can be done formally by applying the usual symbols of the difference and differential calculus. One has only to replace, corresponding to this limiting process, the nth difference quotient $\Delta^n y/\Delta x^n$ in Newton's formula by the nth derivative $d^n y/dx^n$. But in doing so one leaps over a very serious difficulty, using the symbols without regard to their original meaning. In fact, the higher derivatives are defined originally by iteration of the differentiation process (limit of first difference quotient) and their connection with the higher difference quotients is not trivial. And still more difficult for a critical mathematician is the whole limiting process from the interpolation formula to the infinite series. Perhaps such difficulties make us understand why Gregory did not state any connection between his two great results and why Newton, so far as we know, never formulated the Taylor series.

The first to dare to leap over these gaps was Brook Taylor in 1715 [9]. He could do so, since he obviously knew not only Newton's methods but also the concepts and notations introduced in the mean time by Leibniz. He did not use the symbols of Leibniz, but, adapting them to Newton's language, he developed a notation of his own which may, of course, appear a little awkward to us. He applied this symbolism without being influenced by the intrinsic difficulties mentioned above. Thus he came automatically from the interpolation formula to his general series by this purely formal procedure which later on was often performed unscrupulously with the help of the suggestive notation of Leibniz.

3. The binomial series. In an enclosure [10] with the letter to Collins of November 23, 1670, Gregory deals with the problem of finding the "number" of a given logarithm x; that is to say, if we denote the base by $1+d$, of finding $y=(1+d)^x$. For the sake of brevity, we again use modern notations without changing anything else. Gregory gives the solution as follows:

$$(1) \qquad (1+d)^x = 1 + xd + \frac{x(x-1)}{1\cdot 2}d^2 + \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}d^3 + \cdots,$$

which is of course the binomial series. The comparison of Gregory's formula and notation with the statement of the interpolation theorem in the principal part of the same letter [5] shows clearly that he found his result by applying the theorem to the function $f(x)=(1+d)^x$ using the known values at $x=0, 1, 2, \cdots$. Indeed, since the first difference of this function turns out to be

$$(2) \qquad \Delta f(x) = f(x+1) - f(x) = (1+d)^{x+1} - (1+d)^x = d\cdot f(x),$$

the values of its successive differences at $x=0$ become

$$f(0) = 1,\ \Delta f(0) = d,\ \Delta^2 f(0) = d^2,\ \Delta^3 f(0) = d^3, \cdots .$$

Thus, the interpolation formula **2,** (1) yields immediately the binomial series (1).

The correspondence of Gregory and Collins gives full evidence that this discovery of Gregory was entirely independent of Newton's investigations in the binomial theory. Gregory knew at this time only a single one of Newton's results, namely the series for the "zone of the circle," *i.e.* the series for the function $\int_0^x (R^2-x^2)^{1/2}dx$. Collins had communicated the mere statement of the latter to him seven months previously [11]. In fact, Newton had found this series by integrating term by term the expansion of the binomial $(R^2-x^2)^{1/2}$. Having Collins' communication, Gregory tried hard but without success to prove the result directly. Obviously, his discovery of the general binomial theorem was in no way influenced by this knowledge and he did not guess any connection. Afterwards, he recognized suddenly that Newton's series was a simple consequence of his own theorem and, in a letter of December 19 [12], complains much of "his own dullness," not to have noticed the fact before. Besides, Newton's binomial theorem did not become generally known before 1676, when, about ten years after he had found it, he communicated it to Oldenburg in the two famous letters [13] (June 6 and October 4).

It is interesting to compare the way in which Newton had discovered his theorem, as he describes it in the second of these letters, with Gregory's deduction. We mention only the most important points, simplifying the notation as before. Newton computes first the powers $(1+d)^n$ for the lowest integers $n=2, 3, 4, \cdots$, and discusses how to find directly the numerical coefficients of $d, d^2, d^3, \cdots$ in each of these expressions. He then makes the important remark that these coefficients in the expansion of $(1+d)^n$ can be generated by *multiplication* of the numbers $(n-0)/1$, $(n-1)/2$, $(n-2)/3$, $\cdots$, that is to say, that the coefficient of d^m in the expansion of $(1+d)^n$ is equal to

$$\frac{n(n-1)\cdots(n-m+1)}{1\cdot 2\cdot \cdots \cdot m}. \tag{3}$$

Of course, equivalent multiplicative relations for actually the same integers had been discovered a few years before by Pascal who defines them as elements of his "arithmetical triangle," without reference to the binomials.

From this statement Newton proceeds in an extremely audacious way. He got the idea from the procedure by which J. Wallis had developed his famous product formula for π by considering the successive integrals $\int_0^1 (1-x^2)^{n/2}dx$ for $n=0, 1, 2, \cdots$. (As a matter of fact, Newton starts in that letter with the consideration of these integrals instead of with the binomial itself.) He applies the same formula (3) also for the intermediate values $n=1/2, 3/2, 5/2, \cdots$ in order to obtain expressions for $(1+d)^n$ with these fractional values of the exponent, although he now has to write infinite series instead of finite sums. Further generalizations enable him to state the theorem for arbitrary values of the exponent.

To be sure, neither Gregory's nor Newton's deduction is an exact proof in the modern sense. In some respects, Gregory's way may seem to us more satisfactory: he deduces the result from a general theorem, the interpolation formula, and from a characteristic property of the function $(1+d)^x$, namely the difference equation (2). On the other hand, Newton makes this almost adventurous generalization of a finite algebraic identity, deduced for integral exponents only, into an infinite series for fractional exponents. Nevertheless, there is some internal connection between the two procedures. In his investigation, Newton considers the powers of a binomial as a function of the *exponent* as does Gregory, and not as a function of the second term d of the binomial. Thus, the procedures are not so different in their essence as they are in their execution. If one compares them with the usual modern proofs of the binomial theorem, one may remark that the latter are based on the consideration of $(1+d)^x$ as a function of d and that they use the successive *derivatives* with respect to d and the Taylor series instead of the successive *differences* with respect to x and the interpolation formula.

Newton realized the necessity of showing the way in which his consideration may be completed by a proper proof. As an example, he verifies by direct multiplication that the square of his series for $(1+d)^{1/2}$ is equal to $1+d$. Neither Gregory nor Newton tried to prove the convergence of the series. Such a proof was not, at this time, believed to be necessary; but certainly they had the feeling that these infinite sums determined definite numbers.

In this connection, it is interesting to find in a somewhat later letter of Gregory, dated April 9, 1672 [**14**] an early attempt to estimate the remainder of an infinite series by comparing it with the geometrical series. Here, he approximates the logarithmic series $x+x^3/3+x^5/5+x^7/7+\cdots$ by expressions such as

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{9x^7}{7\cdot 9-7\cdot 7x^2}$$

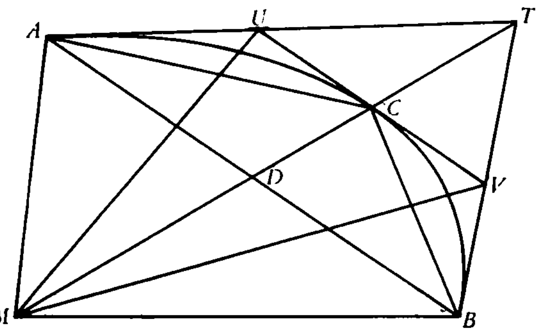
and emphasizes that the analogous expressions formed by using more terms of the original series will give a better approximation. Obviously, this estimate is obtained by comparison with the geometric series

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}\left(1+\frac{7x^2}{9}+\left(\frac{7x^2}{9}\right)^2+\cdots\right).$$

Thus, we see here the first step on the way which, more than a century later, led Cauchy to his convergence tests.

4. Gregory's "Vera Quadratura." Gregory's "*Vera Circuli et Hyperbolae Quadratura*" [**2**], a small volume, contains extremely interesting and original ideas which are, to be sure, a little remote from the mathematics of his time. Even if his mathematical technique was not always sufficient to get a complete solution of the problems he saw, even if he sometimes makes incomplete deductions and wrong conclusions, the investigations show an immense creative power.

He follows in some way the classical procedure of Archimedes, but reveals the algebraic content of the method. Besides, instead of calculating the perimeter of the circle as Archimedes did, he operates on areas. This enables him to deal simultaneously with the sectors of the circle, ellipse and hyperbola.



Let M be the center of a conic ACB, let AT and BT be the tangent lines at A and B, respectively, and let the straight line MT intersect AB at D and the conic at C. Gregory concludes first from fundamental properties of the conics the relations [15]:

$$AD = DB, \qquad MC^2 = MD \cdot MT. \tag{1}$$

Now he draws the tangent line at C which intersects AT at U and BT at V, and compares the following pairs of polygonal areas which are inscribed in or circumscribed about the sector $MACB$: on the one hand he compares the inscribed triangle $i_0 = MAB$ with the circumscribed quadrangle $I_0 = MATB$, on the other hand the inscribed polygon $i_1 = MACB$ with the circumscribed polygon $I_1 = MAUCVB$. The polygon i_1 is composed of two equal triangles MAC and MCB; the polygon I_1 of two equal quadrangles $MAUC$ and $MCVB$. Then, elementary properties of the conics, especially the relations (1), enable him to deduce easily two equations between these four areas as follows:

$$i_1 = \sqrt{i_0 I_0}, \qquad I_1 = \frac{2 i_1 I_0}{i_1 + I_0}.$$

Now, operating on the triangles MAC and MCB, and on the quadrangles $MAUC$ and $MCVB$ in the same way as he had operated on the triangles MAB and the quadrangle $MATB$, he gets four triangles of equal areas $i_2/4$, inscribed in the sector $MACB$, and four quadrangles of equal areas $I_2/4$ circumscribed about the same sector. Obviously, he obtains:

$$i_2 = \sqrt{i_1 I_1}, \qquad I_2 = \frac{2 i_2 I_1}{i_2 + I_1}.$$

Repeating the same operation n times, he constructs for each successive $n = 3, 4, \cdots$ an inscribed polygonal area i_n, composed of 2^n equal triangles, and a circumscribed one I_n, composed of 2^n equal quadrangles. The successive areas are given by:

$$(2) \qquad i_{n+1} = \sqrt{i_n I_n}, \qquad I_{n+1} = \frac{2i_{n+1}I_n}{i_{n+1} + I_n} = \frac{2i_n I_n}{i_n + \sqrt{i_n I_n}} \qquad (n = 0, 1, 2, \cdots).$$

Geometrically it is obvious that the area S of the sector $MACB$ lies between each pair i_n, I_n, and that, if n increases indefinitely, these areas will approach S as closely as one desires, one sequence increasing from below, the other decreasing from above. But Gregory is not satisfied with this visual evidence. He recognizes in the successive construction of the i_n, I_n a new arithmetic operation which yields the value S, and therefore he feels a necessity to *prove* what we call the convergence of the limiting processes

$$(3) \qquad \lim_{n\to\infty} i_n = \lim_{n\to\infty} I_n = S.$$

In fact, with that high degree of exactness which we find in the classical Greek mathematics, he first shows that

$$| I_{n+1} - i_{n+1} | < \tfrac{1}{2} | I_n - i_n |$$

and then concludes that $|I_n - i_n|$ becomes smaller than any given number if n is sufficiently large.

To realize the mathematical importance of Gregory's method we may state that, for the circle and ellipse where $I_0 > i_0$, the area S can be expressed as follows:

$$(4) \qquad S = I_0 \sqrt{\frac{i_0}{I_0 - i_0}} \text{ arc tan } \sqrt{\frac{I_0 - i_0}{i_0}}.$$

For the circle, the first factor is simply $\frac{1}{2}MA^2$, the second the angle $\theta = BMA$. For the hyperbola where $I_0 < i_0$, we have only to interchange I_0 and i_0 and to replace the arc tangent function by the inverse of the hyperbolic tangent function. If we use imaginary numbers, we recognize that we have the same analytic function, since $\tanh ix = i \tan x$. But Gregory has discovered, without applying imaginary numbers, that the same analytical process—the approximation by the formulas (2), (3)—yields the area of the hyperbola as well as the area of the ellipse. In other words, he has found, for the first time in history, the analytical connection between the quadrature of sectors of the ellipse (or of the circle) and the quadrature of sectors of the hyperbola.

The history of these quadratures is interesting. We may assume that astronomical practice originally suggested the introduction of the arc of a circle as independent variable and the coordinates of the point on the circumference as dependent variables, that is to say, the introduction of the circular functions sine, cosine, and so on. This development may be connected with the fact that Archimedes investigated primarily the rectification of the circle instead of the quadrature. But the rectification of the general conics is an entirely different and much more difficult problem. In considering the *area* of the circular sectors Gregory was able to find one single analytical process for the quadrature of all conics.

Now, it has been known since the middle of the 17th century that the quadrature of the hyperbola is connected with the logarithmic function. Therefore, it was obvious to Gregory himself that he had found *one* analytical process for getting from algebraic expressions to logarithmic functions as well as to inverses of the circular functions.

This discovery is generally ascribed to Euler who, some seventy years later, arrived at the connection between the exponential function and the circular functions by using formal operations in the field of complex numbers. It is doubtful whether Euler considered hyperbolic functions as analogous to circular functions and whether he used, in this respect, the analytical analogy between the processes of quadrature of circular and hyperbolic sectors.

The comparison of Euler's and Gregory's achievements may enhance our admiration for Gregory's genius. Indeed, it is not easy to connect in the field of real numbers the two integrals

$$\int \sqrt{1 - x^2}\, dx \quad \text{and} \quad \int \sqrt{1 + x^2}\, dx, \quad \text{or} \quad \int \frac{dx}{1 + x^2} \quad \text{and} \quad \int \frac{dx}{1 - x^2}\cdot$$

As we have seen, this was achieved by Gregory.

In his "*appendicula ad veram circuli et hyperbolae quadraturam*" of 1668 [16] Gregory gives an array of linear combinations of the first i_n and I_n with definite numerical coefficients which yield much better approximations to the area S than do i_n and I_n themselves. Gregory was extremely offended that Huygens did not acknowledge his work to be an essential improvement over his older methods. Therefore he tried to make obvious the strength of the new theory by stating numerous new and surprising results without revealing how he had found them. Turnbull [17] has verified that, for the circle, one gets exactly Gregory's approximations if one first expresses i_n and I_n in terms of trigonometric functions of the angle θ, then expands these expressions in power series in θ, and finally forms such linear combinations of them which begin with the term θ and contain afterwards as many vanishing coefficients as possible. Analogous considerations are valid for the hyperbola. If Gregory operated in this manner he must have known the first terms of the power series for trigonometric and hyperbolic functions as early as 1668. Indeed, it is possible that he got this knowledge without using differentiation, but the published material does not seem to contain anything to corroborate this.

There are two other points in Gregory's speculations which particularly reveal the range of his mathematical ideas with respect to the actual later development of our science. First, the recurrent construction of the areas i_n, I_n is with him only *one* example of a very general, new analytic process which he coordinates as the "sixth" operation along with the five traditional operations (addition, subtraction, multiplication, division, and extraction of roots). In the introduction, he proudly states "ut haec nostra inventio addat arithmeticae aliam operationem et geometriae aliam rationis speciem, ante incognitam orbi geometrico." This operation is, as a matter of fact, our modern limiting process. Clearly, his idea is, if we formulate it in modern language without changing the notions, to investigate two sequences of quantities $a_1a_2, \cdots$ and $b_1, b_2, \cdots$, defined by the recurrent equations

$$a_{n+1} = \phi(a_n, b_n), \qquad b_{n+1} = \chi(a_n, b_n) \qquad (n = 1, 2, 3, \cdots). \tag{5}$$

He uses the word "convergent" for these sequences, very probably for the first time in history, if for each n

$$0 < b_{n+1} - a_{n+1} < b_n - a_n.$$

Of course, this definition does not conform completely to our precise notion of convergence; but in applying his notion he proves in most cases the correct and sufficient inequality

$$0 < b_{n+1} - a_{n+1} < \rho(b_n - a_n)$$

where $\rho < 1$ is independent of n. (In his original problem, he has, as seen previously, $\rho = \frac{1}{2}$.) Then he concludes that the "last convergent terms" of the sequences a_n and b_n are equal, and he calls them *terminatio* of the sequences. In his original problem this terminatio is the area S.

From his further examples we may mention the following ones:

$$a_{n+1} = a_n + \alpha(b_n - a_n), \qquad b_{n+1} = b_n - \beta(b_n - a_n) \tag{6}$$

and

$$a_{n+1} = \frac{2a_nb_n}{a_n + b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}. \tag{7}$$

Here he succeeds in finding the terminatio by an ingenious and simple idea: he determines an invariant expression $F(a_n, b_n)$ such that

$$F(a_{n+1}, b_{n+1}) = F(a_n, b_n); \tag{8}$$

then, the terminatio t will satisfy the equation

$$F(a_1, b_1) = F(t, t), \tag{9}$$

which gives the value t in terms of a_1 and b_1. For the examples (6), (7) he can state immediately the invariant expressions $F(a_n, b_n) = \beta a_n + \alpha b_n$ and $F(a_n, b_n) = a_n \cdot b_n$, respectively, and he finds as the termatio, using (9):

$$t = \frac{\beta a_1 + \alpha b_1}{\beta + \alpha} \quad \text{and} \quad t = \sqrt{a_1 b_1},$$

respectively.

One may remark that Gregory investigated in (2) and (7) different combinations of arithmetical, geometrical and harmonical means. One could imagine that he tried to treat other combinations of these means, but that he could not find out an algebraic expression or a geometric interpretation. In the following century the relation between the arithmetical-geometrical mean and the elliptic integrals was discovered by Lagrange, Legendre and Gauss. We know especially that Gauss studied these means in his early youth before he had any knowledge of the calculus, and that these means, later on, showed him the way to the elliptic integrals [**18**]. We know moreover that Pfaff, the teacher of Gauss, investigated sequences closely related to Gregory's sequence (2) [**19**]. Thus, we could guess that we have here an influence of Gregory's work on one of the most important theories of modern analysis, but we have no definite evidence of such connections.

The second point may be still more momentous. Gregory attempts to prove that the terminatio S of the polygons i_n, I_n cannot be expressed by using the traditional five "elementary" operations on i_0 and I_0. In the preface he puts particular emphasis on this phenomenon. From his exposition we may suppose that he first had tried to "square the circle," *i.e.* to find such an "elementary" expression for S. But he was critical enough to recognize that the difficulties in this search could not be overcome. And realizing that the task of algebra and analysis consists as well in solving a problem as in proving, if necessary, the "impossibility" of a certain solution, he dared to try such a proof, although he did not find any pattern for doing it. He emphasizes that since Euclid's classification of the usual irrationalities in his tenth book, nothing of this kind has even been attempted. Of course, Leonardo Pisano had shown [**20**] about 1200 A.D., that a certain cubic equation cannot be solved by Euclid's irrationalities. However, Gregory could not have had any knowledge of this investigation since it was not published before the nineteenth century. It is a testimony to Gregory's surprising intuition that he mentions further as problems impossible in the same sense just these two: to solve the general algebraic equation and to get an nth root by solving quadratic equations.

To be sure, Gregory does not prove that it is impossible to square the circle, although this is in his mind. He approaches only a much easier problem: to prove that the area of an arbitrary circular sector S cannot be expressed in terms of the areas i_0 and I_0 by the five elementary operations—or, in modern language, that the arc tangent function as given by (4) and defined by the limiting process (2), (3), is not a combination of such algebraic functions. The foundation of his proof is the remark that two sequences (2) yield the same terminatio S whether we begin the process with i_0, I_0 or with i_1, I_1; therefore S depends upon i_0 and I_0 in the same way as upon i_1 and I_1. To put it in modern language, the function satisfies the algebraic functional equation:

$$S(i_0, I_0) = S(i_1, I_1) = S\left(\sqrt{i_0 I_0}, \frac{2i_0 I_0}{i_0 + \sqrt{i_0 I_0}}\right), \tag{10}$$

i.e. $S(i_0, I_0)$ can be transformed algebraically into itself. He tries to prove that the identity (10) is impossible for any function formed only by the five elementary operations. First he removes the irrationality, introducing two suitable new variables u, v by the equations

$$i_0 = u^2(u + v), \qquad I_0 = v^2(u + v).$$

Then (2) shows that

$$i_1 = uv(u + v), \qquad I_1 = 2uv^2,$$

and the identity (10) becomes

$$S(u^2(u + v), v^2(u + v)) = S(uv(u + v), 2uv^2). \tag{11}$$

Now he states two properties of this identity from which he is going to deduce its impossibility for functions S of the above described algebraic type: 1) The arguments of S on the left side contain u up to the third power, while those on the right side contain u only up to the second power. 2) On the left side, both arguments are binomial, while on the right side the second one is only monomial.

Of course, Gregory is able to prove correctly by this procedure that the identity (11) cannot be satisfied by a rational integral function S of its two arguments, and even, with slightly more difficulty, that it cannot be satisfied by any rational function. However, we do not believe that the facts he offers are sufficient to furnish the proof that S is not an irrational function built up in using extraction of roots. Indeed, the algebraic factor $I_0\sqrt{i_0}/\sqrt{I_0 - i_0}$ of (4) satisfies, itself, an identity which differs from (10) only by a factor 2 in the left member, and Gregory's considerations could be applied equally well to the modified identity. The point is that the identity (10), used as basis for his proof implies an intrinsic difficulty: it is equivalent to the algebraic relation between $\tan\theta$ and $\tan 2\theta$ and, moreover, Gregory thinks of it only as valid in the restricted interval $0 < \theta < \frac{1}{2}\pi$.

Today, we would conclude the transcendental character of $\tan\theta$ (and, simultaneously, of the inverse function arc tangent) immediately from the periodicity of that function ($\tan\theta = \tan(\theta + \pi)$). Although such a conclusion seems to us extremely simple, it may have been difficult and remote at Gregory's time.

A modern mathematician will highly admire Gregory's daring attempt of a "proof of impossibility" even if Gregory could not attain his aim. He will consider it a first step into a new group of mathematical questions which became extremely important in the 19th century. However, the contemporary echoes of Gregory's undertaking were in no way favorable. First of all, Huygens criticized [21] the "*Vera Quadratura*" in an extremely unfavorable manner. Gregory had sent him one of the first copies. He expected his discoveries to be fully appreciated by this great mathematician who himself had done very important work on the problem of the quadrature of conics and the circle. But, unfortunately, Huygens was apparently angry that those earlier investigations were not mentioned. Thus, he put more emphasis on some claims of priority and on some objections against Gregory's deductions than on the importance of Gregory's new ideas and results. There is no need to report here on the unpleasant discussion which arose from this criticism [22]. We mention only the single point of importance where Huygens showed a profounder insight. He says: even if the area of an arbitrary circular sector cannot be expressed algebraically in terms of the areas i_0, I_0, one can still imagine such an expression to be possible for particular sectors, for example, for the whole circle itself. Gregory, obviously, had overlooked this possibility in his original publication. In his answer he tried to deduce the result for the "particular case" from that for the arbitrary sector. These endeavors could not but fail; it took more than two centuries before mathematics had developed the necessary means to prove the transcendency of π.

5. Conclusion. Surveying the importance of all these discoveries and ideas of Gregory, and realizing that the total range of his scientific work is by no means covered by our report, one may wonder why this great man did not exert more influence on the actual development of mathematics. The reason can be found in some unfortunate, almost tragical facts in Gregory's life which hampered his activity as well as the effectiveness of his work. After some short sojourns in London (1663 and 1668), and several years of inspiring studies in Italy (1664–1668), mostly in Padua, he was appointed Professor of Mathematics at the Scottish University of St. Andrews. At this old school, still living entirely in medieval traditions, the young scholar was rather isolated. There he was the only one who knew of the new development of mathematics. He himself abounded with new ideas, but there was no possibility to discuss or to teach them. Moreover, hardly any literature was available. Only through his correspondence with Collins whom he had met in London and who had become his close friend, could he learn what the great mathematicians in England and abroad were planning and completing.

Thus, his ideas could not find the response they deserved and he himself did not develop them as far as it might have been possible in closer contact with mathematicians of equal rank. Still worse consequences may have been involved in the lack of appreciation of his first important publication, the *Vera Quadratura*, and especially in the unkind and unjust criticism of Huygens which we have mentioned above.

Apparently, these experiences impressed the proud young Scotchman so deeply that he abandoned entirely the trend of ideas he had started so successfully. We can imagine that otherwise he might have applied his "convergent" pairs of sequences, as defined by recurrence formulas, to various problems and that he might have brought this important process to greater prominence in the early analysis. In fact, he afterwards used the infinite series, probably influenced by the reports he got, scantily, on Newton's work. Yet, also here, fate did not favor him. For he was not given time and opportunity to complete and publish his investigations; and his great merits were darkened by Newton's glory who, meanwhile, could finish his work.

Besides, Gregory had inaugurated research on differential and integral calculus without knowing what his eminent competitors were doing simultaneously in this field. He was even the first to publish, as early as 1668, a proof [23] of the "fundamental theorem," that the two characteristic problems of the calculus, namely, to determine the slopes and to determine the areas, are inverse to one another. Also here he met misfortune; immediately afterwards there appeared Barrow's great work "lectiones geometricae," which went much farther and won all fame. A few years later, Newton's and Leibniz's momentous results on the calculus became known and made obsolete the work of all their predecessors.

Gregory did not live to see this development. He had eventually taken over a professorship at the University of Edinburgh, which granted him better working opportunities. But only one year later, in the fall of 1675, he suddenly fell ill and died in his thirty-seventh year. Most of his discoveries and ideas were buried in his letters and notes or lost through his death.

References

1. Published for the Royal Society of Edinburgh, London, 1939.

2. Pataviae, 1667; reprinted as appendix to J. Gregory's Geometria pars Universalis, Venetiae, 1668, and again in Chr. Huygens, Opera varia, vol. II, Lugduni Batavorum, 1724, pp. 407–462. Our report in no. 4 is based on our essay in the Memorial [1], pp. 468–478.

3. N. Mercator, Logarithmotechnica, Londini, 1668.

4. Published with a comprehensive commentary of H. W. Turnbull in the Memorial [1], pp. 350–359.

5. Memorial [1], pp. 118–122, especially p. 119 f.; *cf.*, Turnbull's commentary, *ibid.*, p. 124. With regard to the earlier publications of that letter, one may compare *ibid.*, pp. 25 and 29.

6. It is mentioned first, but not formulated, in Newton's letter to Oldenburg of October 24, 1676 (Newton, Opuscula Mathematica I, Lausannae et Genevae, 1744, pp. 328–357; see particularly p. 340) and completely stated in his Philosophiae Naturalis Principia, 1687, book III, lemma V, and in his Methodus Differentialis, 1711 (Opuscula [6], p. 271 ff.)—at both places for equidistant and non-equidistant ordinates.

7. See Memorial [1], p. 58, and Turnbull's note, p. 59.

8. In the letter quoted in [6], p. 341.

9. Methodus Incrementorum, Londini, 1715.

10. See Memorial [1], p. 131 f., and Turnbull's commentary, p. 132 f.

11. Letter of March 24, 1670, Memorial [1], p. 88; *cf.*; Gregory's answer, *ibid.*, p. 92.

12. Memorial [1], p. 148; *cf.*, Turnbull's commentary, p. 150 f.

13. Newton, Opuscula I [6], pp. 307–322, especially p. 307 f., and pp. 328–357, especially pp. 329 ff.

14. Memorial [1], p. 230.

15. In our essay in the Memorial [2], p. 469, the second of these formulas is misprinted. We may correct here some other minor misprints in that essay: p. 469, last line, read I_2 instead of $\sqrt{I_2 . i_2}$; p. 471, last formula, read $b_{n+1} = \chi(a_n, b_n)$ instead of $b_{n+1} = \phi(a_n, b_n)$; p. 478, note 7, read tanh instead of tan on one side of the formula.

16. First part of Gregory's Exercitationes Geometricae, Londini, 1668.

17. See Memorial [1], p. 461 ff.

18. *Cf.*, L. Schlesinger, Gauss' Fragmente zur Theorie des arithmetisch-geometrischen Mittels, Nachrichten der Goettinger Gesellschaft der Wissenschaften, 1912, and his essay in Gauss, Werke, vol. X, part 2, Berlin, 1933, Abhandlung 2.

19. *Cf.*, Pfaff's letters to Gauss in Gauss, Werke, vol. X, part 1, Leipzig, 1917, p. 234 ff., and H. Geppert, Mathematische Annalen, vol. 108, 1933, p. 205 ff.

20. Published first by B. Boncompagni, tre scritti inediti di Leonardo Pisano, Firenze, 1854. The proof to which we refer is reviewed comprehensivly by F. Woepke, Journal de Mathématiques pures et appliquées, vol. 19, 1854, p. 401 ff.

21. First in a review in the Journal des Sçavans, Paris, July, 1668; *cf.*, the references in [22].

22. One may compare, also for further literature, our essay [2] and the comprehensive report of E. J. Dijksterhuis, Memorial [1], pp. 478–486. The most important parts of the discussion are reprinted in Chr. Huygens, Opera varia II [2], pp. 463 ff., and again in the Oeuvres complètes de Christiaan Huygens, vol. VI, La Haye, 1895, pp. 228 ff.

23. Contained in the Geometriae pars universalis, Venetiae, 1668. *Cf.*, the essay of A. Prag on this work in the Memorial [1], pp. 487 ff.

THE CLOSURE OF SYSTEMS OF ORTHOGONAL FUNCTIONS

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1. Introduction. A system of functions $\phi_n(x)$, $n=0, 1, 2, \cdots$ is said to be *orthonormal* on the finite interval (a, b) provided that

$$\int_a^b \phi_n(x)\phi_m(x)dx = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m. \end{cases}$$

With any integrable function $g(x)$ there is an associated generalized Fourier series

$$g(x) \sim \sum_{n=0}^{\infty} a_n\phi_n(x), \qquad a_n = \int_a^b g(x)\phi_n(x)dx. \tag{1}$$

An orthonormal system is said to be *closed in the class H* of functions if the series (1) associated with an arbitrary function $g(x)$ in H converges in the mean to $g(x)$; that is, if

$$\lim_{n\to\infty} \int_a^b \{g(x) - s_n(x)\}^2 dx = 0 \tag{2}$$

where $s_n(x)$ denotes the sum of the first n terms of (1).

The importance of the concept of closure in teaching courses involving orthogonal series is quite generally recognized. Various conditions for the validity of the property are known. Unfortunately, the application of those conditions to specific orthogonal systems is, even in the simplest cases, somewhat abstruse for presentation to a class composed of, say, seniors and first year graduate students in physics and engineering. In the second section of this paper a new criterion for closure is given which can be applied directly to verify the property for a number of classical orthogonal systems. In the third section the application of the condition is indicated for the system of Legendre polynomials and the system of trigonometric functions. The entire procedure may be shown to students having no more preparation than a course in Advanced Calculus.