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A Visual Proof of Gregory's Theorem

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The Scottish mathematician James Gregory published the short book *Vera Circuli et Hyperbolae Quadratura* (*The True Squaring of the Circle and the Hyperbola*) in 1667. He continued the tradition started by Archimedes of using regular polygons to approximate a circle. Whereas Archimedes used the perimeters of the polygons to bound the circumference of the circle (to obtain his famous bounds, $223/71 < \pi < 22/7$), Gregory used the areas of the polygons to obtain ever-tightening bounds on the area of the circle. He proved the following recursive formulas for obtaining these bounds.

Theorem. *Let I_k and C_k denote the areas of regular k-gons inscribed in and circumscribed around a given circle. Then I_{2n} is the geometric mean of I_n and C_n, and C_{2n} is the harmonic mean of I_{2n} and C_n; that is,*

$$I_{2n} = \sqrt{I_n C_n} \qquad \text{and} \qquad C_{2n} = \frac{2C_n I_{2n}}{C_n + I_{2n}} = \frac{2}{\frac{1}{I_{2n}} + \frac{1}{C_n}}.$$

Alsina and Nelsen [1] provide a visual proof of the nearly identical theorem about the perimeters of inscribed and circumscribed regular polygons. Here, we give a short, visual proof of the following lemma from which the theorem follows.

Lemma. *For all n,*

$$\frac{I_{2n}}{I_n} = \frac{C_n}{I_{2n}} = \frac{C_n - C_{2n}}{C_{2n} - I_{2n}}.$$

Proof. Suppose we have a circle of radius r with inscribed and circumscribed regular n- and $2n$-gons. Let a be the length of the apothem of the inscribed n-gon, b be the radius of the circumscribed n-gon, and c and d be half the side lengths of the inscribed n-gon and circumscribed $2n$-gon, respectively. Then, as we see in Figure 1,

$$\frac{I_{2n}}{I_n} = \frac{2n \cdot \frac{1}{2}rc}{2n \cdot \frac{1}{2}ac} = \frac{r}{a}, \quad \frac{C_n}{I_{2n}} = \frac{2n \cdot \frac{1}{2}bc}{2n \cdot \frac{1}{2}rc} = \frac{b}{r}, \quad \text{and}$$

$$\frac{C_n - C_{2n}}{C_{2n} - I_{2n}} = \frac{2n \cdot \frac{1}{2}(b-r)d}{2n \cdot \frac{1}{2}(r-a)d} = \frac{b-r}{r-a}.$$

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And, by similar triangles (see Figure 2),

$$\frac{r}{a} = \frac{b}{r} = \frac{b-r}{r-a}. \qquad \blacksquare$$

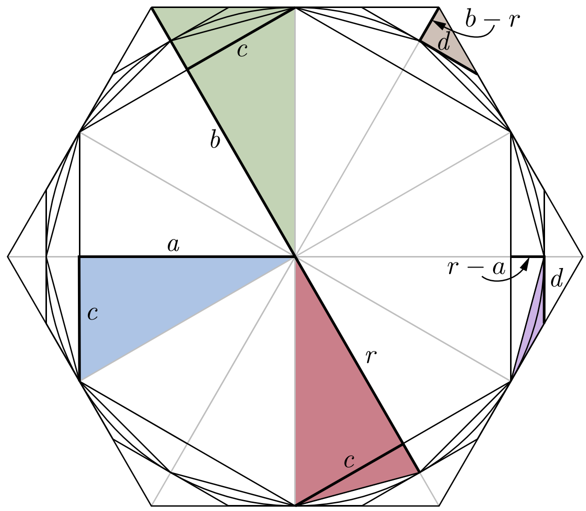

Figure 1 A circle with inscribed and circumscribed regular n- and $2n$-gons.

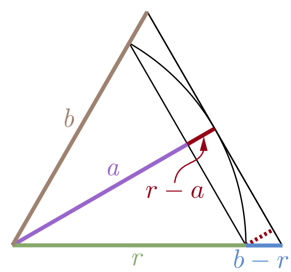

Figure 2 A sector of the circle and sides of the inscribed and circumscribed regular n-gons.

In fact, *Vera Circuli* contained more than this result. Gregory proved that for all n, $|C_{2n} - I_{2n}| < \frac{1}{2}|C_n - I_n|$, and thus as $k \to \infty$, C_k and I_k converge to the area of the circle (in fact, Gregory coined the term *convergent*). So, if the circle has radius 1, we can use these values to approximate π. For instance, inscribed and circumscribed squares yield $I_4 = 2$ and $C_4 = 4$. Applying the formulas produces the tighter bounds, $I_8 = 2\sqrt{2} = 2.8284\ldots$ and $C_8 = 8\sqrt{2} - 8 = 3.3137\ldots$, and Table 1 shows the next several bounds. Moreover, Gregory proved a more general version of this theorem that applies to ellipses and hyperbolas.

Vera Circuli also contained Gregory's proof that the ancient Greek problem of squaring the circle is impossible. In particular, he claimed that the circumference of the circle cannot be obtained from the radius using addition, subtraction, multiplication, division, and the extraction of roots.

n	I_n	C_n	$C_n - I_n$
4	2	4	2
8	2.8284	3.3137	0.4852
16	3.0614	3.1825	0.1211
32	3.1214	3.1517	0.0302
64	3.1365	3.1441	0.0075
128	3.1403	3.1422	0.0018
256	3.1412	3.1417	0.0004

TABLE 1: Bounds for π from inscribed and circumscribed n-gons.

Gregory sent his manuscript to Christian Huygens who was 10 years his senior and a leading mathematician of the day. Rather than replying to Gregory directly, Huygens published a review of *Vera Circuli* identifying a flaw in Gregory's argument and asserting that some of Gregory's results had previously appeared his own work. This review initiated an unpleasant dispute between the two mathematicians.

Although Gregory was correct that it is impossible to square the circle, the mathematical community had to wait over two centuries for a rigorous proof—Lindemann's 1882 proof that π is transcendental. Despite the flaw in Gregory's work, twentieth century mathematicians Max Dehn and E. D. Hellinger wrote, "A modern mathematician will highly admire Gregory's daring attempt of a 'proof of impossibility' even if Gregory could not attain his aim." [**2**]

The disagreement between Gregory and Huygens reveals more than just the issue of the correctness of Gregory's proof and Huygens's accusation of plagiarism. Huygens was at heart a mathematical traditionalist—a geometer. Whereas Gregory was one of the new breed—an algebraist and a pioneer of the new field of calculus. As Scriba noted, Gregory "was one of the wild young men who wanted to tear down the barriers of traditional mathematics at almost any price, who wanted to view hitherto uncultivated areas. Inspired by hopes for as yet unheard-of results, he freely introduced new methods while at times he neglected necessary care for details and exactness." [**3**]

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Summary. We visually demonstrate recursive formulas for areas of certain regular polygons.

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